Tentamen
MMA110/TMV100 Integration Theory

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This exam together with the hand-in exercises provides the grounds for grading. A total score of 18 is needed for the grade 3 or G, 28 for 4 and 38 for 5 or VG. For 3 or G one also needs at least 6 points at this exam and for the grade 4 one also needs at least 10 points on this exam.

Solutions will be published on the course url, the day efter the exam.

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1. (4p) Let f and g be measurable functions on the measurable space $(X, \mathcal{M})$. Prove that fg is measurable.

Solution. Assume first that $f, g \geq 0$. If $f(x)g(x) > c$ for a nonnegative number c, then there must exist a nonnegative rational number q such that $f(x) > q$ and $g(x) > 1/q$. Hence

$$\{x : f(x)g(x) > c\} = \bigcup_{q \in \mathbb{Q} \cap [0,\infty)} (\{x : f(x) > q\} \cap \{x : g(x) > c/q\}).$$

This proves that fg is measurable. In the general case, write $f = f^+ - f^-$ and $g = g^+ - g^-$. then $fg = f^+g^+ + f^-f^- - f^+g^- - f^-g^+$ and since sums of measurable functions are measurable, we are done.

2. (4p) Let $(X, \mathcal{M}, \mu)$ be a semifinite measure space; recall that μ semifinite by definition means that whenever $\mu(E) = \infty$, then there exists $F \subset E$ such that $0 < \mu(F) < \infty$. Prove that more generally, for any $0 < c < \infty$, there exists a set $F \subseteq E$ such that $c < \mu(F) < \infty$.

Solution. Suppose to the contrary that $s := \sup\{c : \exists F \subset E : c < \mu(F) < \infty\} < \infty$. Then for each $n = 1, 2, \ldots$, there is an $F_n \subset E$ with $s - 1/n < \mu(F_n) \leq s$. Taking $F = \bigcup_n F_n$, we get $\mu(F) = s$. However since $\mu(E \setminus F) = \infty$, there is a $G \subset E \setminus F$ such that $0 < \mu(G) < \infty$, so that $F \cup G \subset E$ and $s < \mu(F \cup G) < \infty$, a contradiction to the definition of s.

3. (4p) Find the limit

$$\lim_{n\to\infty} n \int_{-1}^{1} (1 - t^2)^{n^2} (1 + n^2 \sin(t^2)) dt.$$

Solution. By substituting $t = s/n$, we want to calculate the limit

$$\lim_n \int_{-n}^{n} \left(1 - \frac{s^2}{n^2}\right)^{n^2} (1 + n^2 \sin(s^2/n^2)) ds = \lim_n \int f_n(s) ds$$

where $f_n(s) = \chi_{[-n,n]}(s)(1 + s^2/n^2)^{n^2}(1 + n^2 \sin(s^2/n^2))$ which is positive and approaches the limit $g(s) = 2e^{-s^2}$, since $n^2 \sin(s^2/n^2) = \sin(s^2/n^2)/(1/n^2) \to 1$. Since $|f_n(s)| \leq e^{-s^2}(1 + s^2)$ and the right hand side defines an integrable function, dominated convergence gives that the desired limit is

$$\int_{\mathbb{R}} 2e^{-s^2} ds = 2\sqrt{\pi}.$$

4. (4p) Let μ_n be the law of the standard normal probability distribution on $\mathbb{R}^n$, i.e.

$$\mu_n(B) = \int_B \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2} dx$$

for all $B \in \mathcal{B}(\mathbb{R}^n)$. Compute

$$\lim_{k\to\infty} \int \prod_{i=1}^{n} \left(1 + \frac{x_i + x_i^2}{4k}\right)^k d\mu_n(x).$$

Solution. Note that

$$0 \le \left(1 + \frac{x_i + x_i^2}{4k}\right)^k \to e^{(x_i+x_i+2)/4}.$$

Therefore

$$\prod_1^n \left(1 + \frac{x_i + x_i^2}{4k}\right)^k \to g(x) := \prod_1^n e^{(x_i+x_i^2)/4}.$$

Observe now that $g \in L^1(\mu_n)$ since by Tonelli,

$$\int g\, d\mu_n = \prod_1^n \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{(x-x^2)/4} dx = e^{n/16} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-(x-1/2)^2/4} dx = e^{n/16} 2^{n/2}.$$

Hence by dominated convergence, the limit we were asked for is $e^{n/16}2^{n/2}$.

5. (4p) Let $(X, \mathcal{M}, \mathbb{P})$ be a probability space and let $\xi_1, \xi_2, \ldots$ be iid random variables that are not in L^1. Prove that

$$\limsup_n \left| \frac{1}{n} \sum_{i=1}^{n} \xi_i \right| = \infty$$

almost surely.

Solution. Note that for any nonnegative random variable η, $\mathbb{E}[\eta] = \sum_{n=1}^{\infty} \mathbb{P}(\eta \ge n)$. Hence $\sum_n \mathbb{P}(|\xi_n| \ge n) = \sum_n \mathbb{P}(|\xi_1| \ge n) = \infty$. By Borel-Cantelli, $|\xi_n(x)| \ge n$ for infinitely many n for almost every x. Pick such an x and assume without loss of generality that $\xi_n(x) \ge n$ for infinitely many n. Assume now for contradiction that $(1/m)|\sum_1^m \xi_i(x)| \le K$ for all m for some finite K. Pick an $n \ge 3K$ such that $\xi_{n+1} \ge n+1$. Then it is not possible to have the average of the n first and of the $n+1$ first ξ_i:s to both be bounded by K in absolute value, a contradiction.

6. (4p) Construct a Lebesgue measurable set $E \subset [0,1]$ such that for every open interval $I \subseteq [0,1]$, one has $0 < m(E \cap I) < m(I)$.

Solution. Use the following variant of the Cantor set. Let $A_1 = [0,1]$. Let $A_2 = A_1 \setminus (1/4, 3/4)$. Let A_3 be A_2 with the open middle fourth of the two intervals $[0, 1/4]$ and $[3/4, 1]$ removed. Let A_4 be A_3 with the middle 8'th of the four pieces removed, etc. Let $B_1 = \bigcap_n A_n$. Then $1/2 > m(B_1) = (1/2)(3/4)(7/8)\ldots > 0$ and B_1 is closed, so that we can write B_1^c as the countable union of disjoint open intervals $I_1, I_2, \ldots$. Next we start a second stage, where we on each of these I_n's, we construct a new Cantor type set of measure between 0 and 1/4 and add these to B_1 to get B_2. Next in the third stage, on each disjoint open interval making up B_2^c construct a new Cantor type set of measure between 0 and 1/8, etc and add these to B_2 to make up B_3 and so on. Finally let $B = \bigcup_n B_n$. If I is an arbitrary open interval, then for some stage, one of the I_n's will be contained in I and hence B contains the Cantor type set on I_n on in this stage and has hence positive measure. On the other hand if I_n is on stage k, then $\mu(B \cap I_n) \le \mu(I_n)/2^k$ and hence $\mu(B \cap I) < \mu(I)$.

Lycka till!
Johan Jonasson