Tentamen
MMA110/TMV100 Integration Theory

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This exam together with the hand-in exercises provides the grounds for grading. A total score of 18 is needed for the grade 3 or G, 28 for 4 and 38 for 5 or VG. For 3 or G one also needs at least 6 points at this exam and for the grade 4 one also needs at least 10 points on this exam.

Solutions will be published on the course url, the day efter the exam.

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1. (4p) Assume that $(X, \mathcal{P}(X), \mu)$ is a measure space such that $\mu\{x\} > 0$ for all $x \in X$. Define $d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+$ by

$$d(A, B) = \mu(A\Delta B).$$

Prove that d is a metric on $\mathcal{P}(X)$, i.e. that

- $d(A, B) = 0 \Leftrightarrow A = B$,
- $d(A, B) = d(B, A)$ for all A, B,
- $d(A, C) \leq d(A, B) + d(B, C)$ for all A, B, C.

Solution. The left implication in the first part is obvious. Also if $A \neq B$, then $A\Delta B$ is nonempty and has thus nonzero measure since all singleton sets are assumed to have nonzero measure. The second part is immediate since Δ is commutative. For the third part, observe that $A\Delta C \subseteq (A\Delta B) \cup (B\Delta C)$ and that the right hand side has measure at most $\mu(A\Delta B) + \mu(B\Delta C)$.

2. (4p) Let ν be a signed measure on $(X, \mathcal{M})$ and let λ and μ be two measures such that $\nu = \lambda - \mu$. Show that then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.

Solution. Let $P \cup N$ be a Hahn decomposition of X with respect to ν. Then for any measurable set $E \subset P$, $\lambda(E) \geq \lambda(E) - \mu(E) = \nu(E) = \nu^+(E)$. For general E we then get $\lambda(E) \geq \lambda(E \cap P) \geq \nu^+(E \cap P) = \nu^+(E)$. The other part is analogous.

3. (4p) Let $(X, \mathcal{M}, \mu)$ be a measure space. Let

$$\overline{\mathcal{M}} = \{A \cup N : A \in \mathcal{M}, \exists B \in \mathcal{M} : \mu(B) = 0, B \supseteq N\}.$$

Prove that $\overline{\mathcal{M}}$ is a σ-algebra. Prove also that $\overline{\mu} : \overline{\mathcal{M}} \to \mathbb{R}_+$ given by

$$\overline{\mu}(A \cup N) = \mu(A)$$

is a measure.

Solution. It is obvious that $\overline{M}$ is closed under countable unions and contains X. For closedness under complements: let $A \cup N \in \overline{M}$ and let B be a null set that contains N. Then $(A \cup N)^c = (A \cup B)^c \cup (B \setminus N)$ which is also a set in $\overline{M}$. To show that $\overline{\mu}$ is a measure is easy.

4. (4p) Let $(X, \mathcal{M}, \mathbb{P})$ be a probability space and let $\xi_1, \xi_2, \ldots$ be iid random variables that are not in L^1. Prove that

$$\limsup_n \left| \frac{1}{n} \sum_{i=1}^{n} \xi_i \right| = \infty$$

almost surely.

Solution. Note that for any nonnegative random variable η, $\mathbb{E}[\eta] = \sum_{n=1}^{\infty} \mathbb{P}(\eta \geq n)$. Hence $\sum_n \mathbb{P}(|\xi_n| \geq n) = \sum_n \mathbb{P}(|\xi_1| \geq n) = \infty$. By Borel-Cantelli, $|\xi_n(x)| \geq n$ for infinitely many n for almost every x. Pick such an x and assume without loss of generality that $\xi_n(x) \geq n$ for infinitely many n. Assume now for contradiction that $(1/m)|\sum_1^m \xi_i(x)| \leq K$ for all m for some finite K. Pick an $n \geq 3K$ such that $\xi_{n+1} \geq n+1$. Then it is not possible to have the average of the n first and of the $n+1$ first ξ_i:s to both be bounded by K in absolute value, a contradiction.

5. (4p) Let $(X, \mathcal{M}, \mu)$ be a measure space and $f_1, f_2, \ldots$ real-valued measurable functions such that

$$\limsup_n n^2 \mu\{x : |f_n(x)| \geq n^{-2}\} < \infty.$$

Prove that $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely for almost every x.

Solution. Let $C = \limsup_n n^2 \mu\{|f_n| \geq n^{-2}\}$. Then $\mu\{|f_n| \geq n^{-2}\} \leq C/n^2$ for all but finitely many n. Hence by the MCT,

$$\int \sum_n \chi_{\{|f_n| \geq n\}} d\mu < \infty.$$

Thus $\sum_n \chi_{\{|f_n(x)| \geq n\}}$ converges for a.e. x, i.e., for a.e. x, $|f_n(x)| \leq 1/n^2$ for all but finitely many n. From this it follows that $\sum_n |f_n(x)| < \infty$ for such x.

6. (4p) Let m be the Lebesgue measure on $\mathbb{R}$ and let $E \in \mathcal{L}(\mathbb{R})$ be such that $m(E) > 0$. Let $\hat{E} = \{x - y : x, y \in E\}$. Prove that $\hat{E}$ must contain an open interval that contains 0.

Solution. Recall that in general for a Lebesgue set A, $m(A) = \inf \sum_{k=1}^{\infty} m(I_k)$, where the infimum is taken over all sets of disjoint intervals I_k such that $\bigcup_k I_k \supseteq A$. Hence for any $a < 1$, there is a set of I_k:s covering A such that

$$m(A) = \sum_k m(A \cap I_k) \leq \sum_k m(I_k) < a^{-1} m(A)$$

so for at least one of the I_k:s, we have $m(A \cap I_k) > am(I_k)$. In particular this means that for our given E, there is an interval I such that $m(E \cap I) > (3/4)m(I)$. Since $\hat{E}$ is invariant under translation, we may assume, possibly after translating and rescaling, that $I = [-1, 1]$. Now for any $x \in [-1, 1]$, we must have that $E \cap (x + E) \neq \emptyset$, since otherwise we would have $m(E \cup (x + E)) > (3/2)m[-1, 1]$, a contradiction since $E \cup (x + E)$ is contained in either $[-2, 1]$ or $[-1, 2]$. Hence there is a point $y \in E \cap (x + E)$, i.e. $y = a = x + b$ for some $a, b \in E$. However then $x = a - b$, so $x \in \hat{E}$. Consequently $\hat{E} \supseteq [-1, 1]$ as desired.

Lycka till!
Johan Jonasson