Solutions:
INTEGRATION THEORY (7.5 hp)
(GU$[MMA110]$,**CTH**$[tmv100]$**)**
August 15, 2011, morning, v.
No aids.
Questions on the exam: Adam Andersson, 0703 - 08 83 04
Each problem is worth 3 points.

Notation: Lebesgue measure on $\mathbf{R}^d$ is denoted by m_d.

1. Compute

$$\lim_{n\to\infty}\int_0^n \frac{(1-\frac{x}{n})^n}{\sqrt{x}}dx.$$

Solution. Suppose $n \in \mathbf{N}_+$. Since $1+t \leq e^t$ for every real t,

$$\chi_{[0,n]}(x)(1-\frac{x}{n})^n \leq e^{-x} \text{ if } x \geq 0.$$

From this

$$f_n(x) =_{def} \chi_{[0,n]}(x)\frac{(1-\frac{x}{n})^n}{\sqrt{x}} \leq \frac{e^{-x}}{\sqrt{x}},\ x \geq 0$$

and, in addition,

$$\lim_{n\to\infty} f_n(x) = \frac{e^{-x}}{\sqrt{x}}.$$

Here $\frac{e^{-x}}{\sqrt{x}} \in L^1(m_1$ on $[0,\infty[)$ since $\frac{e^{-x}}{\sqrt{x}} \geq 0$ and

$$\int_0^\infty \frac{e^{-x}}{\sqrt{x}}dx = 2\int_0^\infty e^{-x^2}dx = \sqrt{\pi}.$$

Moreover $f_n \geq 0$ for every $n \in \mathbf{N}_+$ and by using dominated convergence we get

$$\lim_{n\to\infty}\int_0^n \frac{(1-\frac{x}{n})^n}{\sqrt{x}}dx = \lim_{n\to\infty}\int_0^\infty f_n(x)dx =$$

$$\int_0^\infty \lim_{n\to\infty} f_n(x)dx = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}dx = \sqrt{\pi}.$$

2. Let $(X, \mathcal{M})$ be a measurable space and suppose $\mu{:}\mathcal{M} \to]-\infty, \infty]$ and $\nu{:}\mathcal{M} \to]-\infty, \infty]$ are signed measures. Prove that

$$(\mu + \nu)^+ \leq \mu^+ + \nu^+.$$

Solution. If θ is a signed measure defined on the σ-algebra $\mathcal{M}$, let $P_\theta, N_\theta \in \mathcal{M}$ be disjoint with $P_\theta \cup N_\theta = X$ and such that θ is positive on P_θ and negative on N_θ. Then if $A \in \mathcal{M}$,

$$(\mu + \nu)^+(A) = (\mu + \nu)(A \cap P_{\mu+\nu}) =$$

$$\mu(A \cap P_{\mu+\nu}) + \nu(A \cap P_{\mu+\nu}) \leq$$

$$\mu(A \cap P_{\mu+\nu} \cap P_\mu) + \nu(A \cap P_{\mu+\nu} \cap P_\nu) \leq$$

$$\mu(A \cap P_\mu) + \nu(A \cap P_\nu) = \mu^+(A) + \nu^+(A).$$

3. Suppose $f \in L^1(m_2)$. Show that $\lim_{n\to\infty} f(nx) = 0$ for m_2-almost all $x \in \mathbf{R}^2$.

Solution. Writing $dm_2 = dx$, we have

$$\int_{\mathbf{R}^2} |\, f(nx) \,|\, dx = \frac{1}{n^2} \int_{\mathbf{R}^2} |\, f(x) \,|\, dx$$

and, hence, by the Beppo Levi theorem

$$\int_{\mathbf{R}^2} \sum_{n=1}^{\infty} |\, f(nx) \,|\, dx = \sum_{n=1}^{\infty} \int_{\mathbf{R}^2} |\, f(nx) \,|\, dx =$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\mathbf{R}^2} |\, f(x) \,|\, dx < \infty.$$

From this it follows that

$$\sum_{n=1}^{\infty} |\, f(nx) \,| < \infty \text{ a.e } [m_2]$$

and the series

$$\sum_{n=1}^{\infty} f(nx)$$

converges for m_2-almost all $x \in \mathbf{R}^2$. Since the general term a_n in a convergent series $\Sigma_1^\infty a_n$ converges to zero as $n \to \infty$ we are done.

4. Suppose $(X, \mathcal{M}, \mu)$ is a positive measure space and $w : X \to [0, \infty]$ a measurable function. Define

$$\nu(A) = \int_A w d\mu, \ A \in \mathcal{M}.$$

Prove that ν is a positive measure and

$$\int_X f d\nu = \int_X f w d\mu$$

for every measurable function $f : X \to [0, \infty]$.

5. Suppose θ is an outer measure on X and let $\mathcal{M}(\theta)$ be the set of all $A \subseteq X$ such that

$$\theta(E) = \theta(E \cap A) + \theta(E \cap A^c) \text{ for all } E \subseteq X.$$

Prove that $\mathcal{M}(\theta)$ is a σ-algebra and that the restriction of θ to $\mathcal{M}(\theta)$ is a complete measure.