Solutions:
INTEGRATION THEORY (7.5 hp)
(GU$[MMA110]$,**CTH**$[tmv100]$**)**
January 08, 2011, morning, v.
No aids.
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Each problem is worth 3 points.

1. Let $(X, \mathcal{M}, \mu)$ be a positive measure space and $f_n : X \to \mathbf{R}$, $n \in \mathbf{N}_+$, a sequence of measurable functions such that

$$\limsup_{n\to\infty} n^2 \mu(\mid f_n \mid \geq n^{-2}) < \infty.$$

Prove that the series $\sum_{n=1}^{\infty} f_n(x)$ converges for μ-almost all $x \in X$.

Solution. There exist a $C \in [0, \infty[$ such that

$$\mu(\mid f_n \mid \geq n^{-2}) \leq Cn^{-2} \text{ if } \; n \in \mathbf{N}_+.$$

Hence

$$\sum_{n=1}^{\infty} \int_X \chi_{\{|f_n| \geq n^{-2}\}} d\mu < \infty$$

and the Beppo Levi theorem yields

$$\int_X \sum_{n=1}^{\infty} \chi_{\{|f_n| \geq n^{-2}\}} d\mu < \infty.$$

Thus

$$\sum_{n=1}^{\infty} \chi_{\{|f_n| \geq n^{-2}\}}(x) < \infty$$

for μ-almost all $x \in X$ and it follows that there exists a function $N : X \to \mathbf{N}_+$ such that

$$\mid f_n(x) \mid < n^{-2} \text{ if } n \geq N(x)$$

for μ-almost all $x \in X$. Accordingly, from this the series $\sum_{n=1}^{\infty} \mid f_n(x) \mid$ converges for μ-almost all $x \in X$. Finally, since an absolutely convergent real series converges, the series $\sum_{n=1}^{\infty} f_n(x)$ must converge for μ-almost all $x \in X$.

2. Compute the n-dimensional Lebesgue integral

$$\int_{|x|<1} \ln(1- \mid x \mid)dx$$

where $\mid x \mid$ denotes the Euclidean norm of the vector $x \in \mathbf{R}^n$. (Hint: $\sigma(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$.)

Solution. We have

$$\int_{|x|<1} \ln(1- \mid x \mid)dx = \sigma(S^{n-1}) \int_0^1 r^{n-1} \ln(1-r)dr$$

$$= -\sigma(S^{n-1}) \int_0^1 \sum_{k=1}^{\infty} \frac{r^{k+n-1}}{k} dr.$$

Moreover, the Beppo Levi theorem implies that

$$\int_0^1 \sum_{k=1}^{\infty} \frac{r^{k+n-1}}{k} dr = \sum_{k=1}^{\infty} \int_0^1 \frac{r^{k+n-1}}{k} dr$$

$$= \sum_{k=1}^{\infty} \frac{1}{k(k+n)} = \frac{1}{n} \sum_{k=1}^{\infty} (\frac{1}{k} - \frac{1}{k+n}) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k}.$$

Thus

$$\int_{|x|<1} \ln(1- \mid x \mid)dx = -\frac{2\pi^{n/2}}{n\Gamma(n/2)} \sum_{k=1}^{n} \frac{1}{k}.$$

3. The set $A \subseteq \mathbf{R}$ has positive Lebesgue measure and

$$A + \mathbf{Q} = \{x + y;\ x \in A \text{ and } y \in \mathbf{Q}\}$$

where $\mathbf{Q}$ stands for the set of all rational numbers. Show that the set

$$\mathbf{R} \backslash (A + \mathbf{Q})$$

is a Lebesgue null set. (Hint: The function $m(A\Delta(A-x))$, $x \in \mathbf{R}$, is continuous.)

Solution. Without loss of generality we may assume A is compact. Suppose $m(\mathbf{R}\backslash(A+\mathbf{Q})) > 0$ and pick a compact set $K \subseteq \mathbf{R}\backslash(A+\mathbf{Q})$ of positive Lebesgue measure. We first claim that

$$m(K \cap (A+x)) > 0 \text{ for some } x \in \mathbf{R}.$$

In fact, if not,

$$\int_{\mathbf{R}} \chi_K(y)\chi_A(y-x)dy = 0 \text{ if } x \in \mathbf{R}$$

and, hence,

$$\int_{\mathbf{R}} e^{-x^2} \left(\int_{\mathbf{R}} \chi_K(y)\chi_A(y-x)dy \right) dx = 0.$$

Now by the Tonelli theorem

$$0 = \int_{\mathbf{R}} \chi_K(y) \left(\int_{\mathbf{R}} e^{-x^2} \chi_A(y-x)dx \right) dy$$

$$= \int_{\mathbf{R}} \chi_K(y) \left(\int_{\mathbf{R}} e^{-(x-y)^2} \chi_A(x)dx \right) dy$$

and as

$$\int_{\mathbf{R}} e^{-(x-y)^2} \chi_A(x)dx > 0 \text{ if } y \in \mathbf{R}$$

it follows that $\chi_K = 0$ a.e. $[m]$, which is a contradiction. Accordingly from this, there is an $x_0 \in \mathbf{R}$ such that

$$m(K \cap (A+x_0)) > 0.$$

But, if $q \in \mathbf{Q}$,

$$\mid m(K \cap (A+x_0)) - m(K \cap (A+q)) \mid$$

$$= \mid \int_K (\chi_{A+x_0} - \chi_{A+q})dm \mid \leq \int_{\mathbf{R}} \mid \chi_{A+x_0} - \chi_{A+q} \mid dm$$

$$= m((A+x_0)\Delta(A+q)) = m(A\Delta(A+q-x_0)).$$

Hence $m(K \cap (A + q)) > 0$ if q is sufficiently close to x_0 and therefore $K \cap (A + q) \neq \phi$ if q is sufficiently close to x_0, which contradicts the relation $K \subseteq \mathbf{R} \backslash (A + q)$. From this contradiction we conclude that

$$m(\mathbf{R} \backslash (A + \mathbf{Q})) = 0.$$

4. Let $(X, \mathcal{M}, \mu)$ be a positive measure space. (a) Suppose $f_n \to f$ in measure and $f_n \to g$ in measure. Show that $f = g$ a.e. $[\mu]$. (b) Suppose $f_n \to f$ in L^1. Show that $f_n \to f$ in measure.

5. Suppose the function $F{:}\mathbf{R} \to \mathbf{R}$ is of bounded variation. (a) Define the total variation T_F of F. (b) Show that the functions $T_F + F$ and $T_F - F$ are increasing. (c) Show that T_F is right continuous if F is.