Solutions:
INTEGRATION THEORY (7.5 hp)
(GU$[MMA110]$,**CTH**$[TMV100]$**)**
January 11, 2010, morning (5 hours), v
No aids.
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Each problem is worth 3 points.

1. Suppose $p \in \mathbf{N}_+$ and define $f_n(x) = n^p x^{p-1}(1-x)^n$, $0 \leq x \leq 1$, for every $n \in \mathbf{N}_+$. Show that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = (p-1)!.$$

Solution. We have

$$\int_0^1 f_n(x)dx = \left\{ x = \frac{t}{n} \right\} = \int_0^n t^{p-1}(1-\frac{t}{n})^n dt$$

$$= \int_0^\infty \chi_{[0,n]}(t) t^{p-1}(1-\frac{t}{n})^n dt.$$

Set $g_n(t) = \chi_{[0,n]}(t)t^{p-1}(1-\frac{t}{n})^n, t \geq 0$. Then

$$\lim_{t\to\infty} g_n(t) = t^{p-1}e^{-t} =_{def} h(t)$$

and, as $e^y \geq 1+y$, $y \in \mathbf{R}$, it follows that

$$\mid g_n(t) \mid \leq h(t),\ t \geq 0,\ n \in \mathbf{N}_+.$$

Here $h \in \mathcal{L}^1(m \text{ on } [0,\infty[)$, and by using the dominated convergence theorem we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \lim_{n\to\infty} \int_0^\infty g_n(t)dt$$

$$= \int_0^\infty t^{p-1}e^{-t}dt = \Gamma(p) = (p-1)!.$$

2. Let $(X, \mathcal{M}, \mu)$ be a probability space and suppose the sets $A_1, ..., A_n \in \mathcal{M}$ satisfy the inequality $\sum_1^n \mu(A_i) > n-1$. Show that $\mu(\cap_1^n A_i) > 0$.

Solution. We have

$$\sum_1^n \mu(A_i^c) = \sum_1^n (1-\mu(A_i)) = n - \sum_1^n \mu(A_i) < n-(n-1) = 1.$$

Hence

$$\mu(\bigcup_1^n A_i^c) \leq \sum_1^n \mu(A_i^c) < 1$$

and

$$\mu(\bigcap_1^n A_i) = \mu((\bigcup_1^n A_i^c)^c) = 1 - \mu(\bigcup_1^n A_i^c) > 0.$$

3. Let μ and ν be probability measures on $(X, \mathcal{M})$ such that $\mid \mu - \nu \mid (X) = 2$. Show that $\mu \perp \nu$.

Solution. Set $\sigma = (\mu + \nu)/2$ and note that μ and ν are absolutely continuous with respect to the probability mesure σ. By applying the Radon-Nykodym theorem we get non-negative measurable functions f and g such that $d\mu = f d\sigma$ and $d\nu = g d\sigma$. Here

$$\int_X f d\sigma = \int_X g d\sigma = 1,$$

$$d(\mu - \nu) = (f-g) d\sigma$$

and

$$d \mid \mu - \nu \mid = \mid f - g \mid d\sigma.$$

Now, since $\mid f - g \mid \leq f + g$,

$$2 = \int_X \mid f - g \mid d\sigma \leq \int_X (f+g) d\sigma = 2$$

and we conclude there exists a set $A \in \mathcal{M}$ with $\sigma(A) = 1$ such $f + g = \mid f - g \mid$ on A or, stated otherwise, $(f+g)^2 = \mid f - g \mid^2$ on A. Thus $fg = 0$ on A. Now set $P = \{x \in A;\ f(x) > 0\}$ and $N = P^c$. Then $\mu(P) = 1$ and $\nu(N) = 1$ as $\mu(A^c) = \nu(A^c) = 0$. This proves that $\mu \perp \nu$.

4. State and prove Fatou's Lemma.

5. Let ν be a finite signed measure and μ a positive measure on $(X, \mathcal{M})$. Show that $\nu << \mu$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mid \nu(E) \mid < \varepsilon$ whenever $\mu(E) < \delta$.