Solutions:
INTEGRATION THEORY (7.5 hp)
(GU$[MMA110GU]$ **,CTH**$[TMV100]$**)**
October 22, 2009, morning (5 hours), H
No aids.
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Each problem is worth 3 points.

1. Let $n \in \mathbf{N}_+$ and define $f_n(x) = e^x(1 - \frac{x^2}{2n})^n$, $x \in \mathbf{R}$. Compute

$$\lim_{n\to\infty} \int_{-\sqrt{2n}}^{\sqrt{2n}} f_n(x)dx.$$

Solution. We have

$$I_n =_{def} \int_{-\sqrt{2n}}^{\sqrt{2n}} f_n(x)dx = \int_{-\infty}^{\infty} g_n(x)dx$$

where $g_n(x) = \chi_{\left[-\sqrt{2n},\sqrt{2n}\right]}(x)e^x(1 - \frac{x^2}{2n})^n$, $x \in \mathbf{R}$. Now

$$\lim_{t\to\infty} g_n(x) = e^{x-\frac{x^2}{2}} =_{def} h(x)$$

and, as $e^y \geq 1 + y$, $y \in \mathbf{R}$,

$$(1 - \frac{x^2}{2n})^n \leq e^{-\frac{x^2}{2}} \text{ if } -\sqrt{2n} \leq x \leq \sqrt{2n}.$$

Hence,

$$\mid g_n(x) \mid \leq h(x),\ x \in \mathbf{R},\ n \in \mathbf{N}_+$$

where $h \in \mathcal{L}^1(m\ )$ and by using the dominated convergence theorem,

$$\lim_{n\to\infty} I_n = \lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)dt$$

$$= \int_{-\infty}^{\infty} e^{x-\frac{x^2}{2}}dx = e^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{(x-1)^2}{2}}dx = e^{\frac{1}{2}}\sqrt{2\pi}.$$

2. Let $(X, \mathcal{M}, \mu)$ be a positive measure space and $f : X \to \mathbf{R}$ an $(\mathcal{M},\mathcal{R})$-measurable function. Moreover, for each $t > 1$, let

$$a(t) = \sum_{n=-\infty}^{\infty} t^n \mu(t^n \leq \mid f \mid < t^{n+1}).$$

Show that

$$\lim_{t \to 1^+} a(t) = \int_X \mid f \mid d\mu.$$

Solution. Define

$$g_t = \sum_{n=-\infty}^{\infty} t^n \chi_{\{t^n \leq |f| < t^{n+1}\}} \text{ if } t > 1$$

and note that the Beppo Levi theorem implies that

$$\int_X g_t d\mu = a(t).$$

If $\mid f(x) \mid = 0$, then $g_t(x) = 0$. Moreover, if $t^n \leq \mid f(x) \mid < t^{n+1}$ for some integer n, then $g_t(x) = t^n$ and$\mid f(x) \mid \geq g_t(x)$. Thus

$$\mid f \mid \geq g_t$$

and we get

$$\int_X \mid f \mid d\mu \geq \int_X g_t d\mu = a(t).$$

Next suppose $\mid f(x) \mid > 0$ and choose n such that $t^n \leq \mid f(x) \mid < t^{n+1}$. Then

$$tg_t(x) = \sum_{n=-\infty}^{\infty} t^{n+1} \chi_{\{t^n \leq |f| < t^{n+1}\}}(x) = t^{n+1} > \mid f(x) \mid$$

and, hence,

$$tg_t \geq \mid f \mid .$$

Now, by integration,

$$ta(t) \geq \int_X \mid f \mid d\mu.$$

Thus

$$t^{-1} \int_X \mid f \mid d\mu \leq a(t) \leq \int_X \mid f \mid d\mu$$

and

$$\lim_{t \to 1^+} a(t) = \int_X \mid f \mid d\mu.$$

3. Suppose $(X, \mathcal{M}, \mu)$ is a finite positive measure space and $f \in L^1(\mu)$. Define

$$g(t) = \int_X \mid f(x) - t \mid d\mu(x), \ \ t \in \mathbf{R}.$$

Prove that g is absolutely continuous and

$$g(t) = g(a) + \int_a^t (\mu(f \leq s) - \mu(f \geq s))ds \ \ \text{if } a, t \in \mathbf{R}.$$

Solution. Suppose $\varepsilon > 0$ is given and let $]a_k, b_k[\,,\ k = 1, ..., n$, be disjoint open intervals such that $\Sigma_1^n \mid b_k - a_k \mid < \varepsilon/(1 + \mu(X))$. Then

$$\mid g(a_k) - g(b_k) \mid = \mid \int_X \mid f(x) - a_k \mid - \mid f(x) - b_k \mid d\mu(x) \mid$$

$$\leq \int_X \mid\mid f(x) - a_k \mid - \mid f(x) - b_k \mid\mid d\mu(x)$$

$$\leq \int_X \mid (f(x) - a_k) - (f(x) - b_k) \mid d\mu(x) = \mu(X) \mid b_k - a_k \mid$$

and, consequently,

$$\sum_1^n \mid g(a_k) - g(b_k) \mid \leq \varepsilon.$$

This proves that g is absolutely continuous and therefore g' exists a.e. with respect to Lebesgue measure on $\mathbf{R}$ and

$$g(t) = g(a) + \int_a^t g'(s)ds \text{ for all } t \in \mathbf{R}.$$

Let $A = \{t \in \mathbf{R};\ \mu(f = t) > 0\}$ and note that A is at most denumerable. To compute $g'(s)$ for fixed $s \notin A$, let $(h_n)_0^\infty$ be a sequence of non-zero real numbers which converges to zero. Then

$$\frac{g(s + h_n) - g(s)}{h_n} = \int_X \frac{\mid s + h_n - f(x) \mid - \mid s - f(x) \mid}{h_n} d\mu(x)$$

$$= \int_{\{f \neq s\}} \frac{\mid s + h_n - f(x) \mid - \mid s - f(x) \mid}{h_n} d\mu(x).$$

Here

$$\mid \frac{\mid s + h_n - f(x) \mid - \mid s - f(x) \mid}{h_n} \mid \leq 1$$

and

$$\lim_{n \to \infty} \frac{\mid s + h_n - f(x) \mid - \mid s - f(x) \mid}{h_n} = \begin{cases} 1 \text{ if } s > f(x) \\ -1 \text{ if } s < f(x). \end{cases}$$

Now the dominated convergence theorem gives

$$g'(s) = \int_{\{f \neq s\}} (\chi_{\{f<s\}} - \chi_{\{f>s\}}) d\mu = \int_X (\chi_{\{f<s\}} - \chi_{\{f>s\}}) d\mu$$
$$= \mu(f < s) - \mu(f > s) = \mu(f \leq s) - \mu(f \geq s).$$

In particular,

$$g'(s) = \mu(f \leq s) - \mu(f \geq s)$$

a.e. with respect to Lebesgue measure on $\mathbf{R}$ and since g is absolutely continuous we have

$$g(t) = g(a) + \int_a^t (\mu(f \leq s) - \mu(f \geq s)) ds \quad \text{if } a, t \in \mathbf{R}.$$

4. Suppose $(X, \mathcal{M}, \mu)$ is a positive measure space and $A_n \in \mathcal{M}$, $n \in \mathbf{N}_+$. Set

$$E = \bigcup_{n \in \mathbf{N}_+} A_n \text{ and } F = \bigcap_{n \in \mathbf{N}_+} A_n.$$

(a) Show that

$$\lim_{n \to \infty} \mu(A_n) = \mu(E)$$

if $A_1 \subseteq A_2 \subseteq A_3 \subseteq ...$.

(b) Show that

$$\lim_{n \to \infty} \mu(A_n) = \mu(F)$$

if $\mu(A_1) < \infty$ and $A_1 \supseteq A_2 \supseteq A_3 \supseteq ...$.

5. State and prove the monotone convergence theorem.