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INTEGRATIONSTEORI (5p)
(**GU**[$MAF440$] ,**CTH**[$TMV100$])
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Notation: Lebesgue measure on **R** is denoted by m

1. Evaluate

$$\lim_{n\to\infty}\int_0^n (1-\frac{x}{n})^n \frac{1+nx}{n+x}\cos x dx.$$

Solution. Set

$$f_n(x) = \chi_{[0,n]}(x)(1-\frac{x}{n})^n\frac{1+nx}{n+x}\cos x,\ x \geq 0$$

where $n \in \mathbf{N}_+$. Then

$$\mid f_n(x) \mid \leq \chi_{[0,n]}(x)(1-\frac{x}{n})^n\frac{1+nx}{n+x} \leq \chi_{[0,n]}(x)(1-\frac{x}{n})^n(1+x)$$

and since $e^t \geq 1+t$ for all real t, we get

$$\mid f_n(x) \mid \leq e^{-x}(1+x) \in L^1(m_{0,\infty})$$

where $m_{0,\infty}$ denotes Lebesgue measure on $[0,\infty[$. Moreover,

$$f_n(x) \to e^{-x}x\cos x \text{ as } n \to \infty$$

and the Lebesgue Dominated Convergence Theorem gives

$$\lim_{n\to\infty}\int_0^n (1-\frac{x}{n})^n\frac{1+nx}{n+x}\cos x dx = \int_0^\infty e^{-x}x\cos x dx.$$

To evaluate this integral we first use partial integration to obtain

$$\int_0^\infty e^{-tx}\cos x dx = \frac{t}{1+t^2},\ t > 0.$$

Here for every $t \geq \frac{1}{2}$,

$$\mid \frac{\partial}{\partial t} e^{-tx} \cos x \mid = \mid -xe^{-tx} \cos x \mid \leq xe^{-\frac{1}{2}x} \in L^1(m_{0,\infty})$$

and the theorem about interchanging a derivative and an integral yields

$$\int_0^\infty \frac{\partial}{\partial t} e^{-tx} \cos x dx = \frac{\partial}{\partial t} \frac{t}{1+t^2}, \ t \geq \frac{1}{2}.$$

In particular,

$$\int_0^\infty xe^{-x} \cos x dx = \left[\frac{t^2-1}{(t^2+1)^2} \right]_{|t=1} = 0$$

and we get

$$\lim_{n \to \infty} \int_0^n (1 - \frac{x}{n})^n \frac{1+nx}{n+x} \cos x dx = 0.$$

2. Suppose μ is a finite positive Borel measure on $\mathbf{R}^n$ and f: $\mathbf{R}^n \to \mathbf{R}$ a Borel measurable function. Set $g(x, y) = f(x) - f(y)$, $x, y \in \mathbf{R}^n$. Prove that $f \in L^1(\mu)$ if and only if $g \in L^1(\mu \times \mu)$.

Solution. The function g is a Borel function. Moreover,

$$\int_{\mathbf{R}^n \times \mathbf{R}^n} \mid g \mid d(\mu \times \mu) = \int_{\mathbf{R}^n \times \mathbf{R}^n} \mid f(x) - f(y) \mid d\mu(x) d\mu(y)$$

$$\leq \int_{\mathbf{R}^n \times \mathbf{R}^n} (\mid f(x) \mid + \mid f(y) \mid) d\mu(x) d\mu(y)$$

$$\leq \int_{\mathbf{R}^n \times \mathbf{R}^n} \mid f(x) \mid d\mu(x) d\mu(y) + \int_{\mathbf{R}^n \times \mathbf{R}^n} \mid f(y) \mid d\mu(x) d\mu(y)$$

and the Tonelli Theorem gives

$$\int_{\mathbf{R}^n \times \mathbf{R}^n} \mid g \mid d(\mu \times \mu) \leq 2\mu(\mathbf{R}^n) \int_{\mathbf{R}^n} \mid f \mid d\mu.$$

Thus $g \in L^1(\mu \times \mu)$ if $f \in L^1(\mu)$.

Tonelli's Theorem also shows that

$$\int_{\mathbf{R}^n \times \mathbf{R}^n} | g | \, d(\mu \times \mu) = \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} | f(x) - f(y) | \, d\mu(x) \right) d\mu(y).$$

Therefore, if $g \in L^1(\mu \times \mu)$, then

$$\int_{\mathbf{R}^n} | f(x) - f(y) | \, d\mu(x) < \infty$$

for μ-almost all $y \in \mathbf{R}^n$. In particular,

$$\int_{\mathbf{R}^n} | f(x) - f(y_0) | \, d\mu(x) < \infty$$

for an appropriate $y_0 \in \mathbf{R}^n$. Now

$$\int_{\mathbf{R}^n} | f(x) | \, d\mu(x) \leq \int_{\mathbf{R}^n} (| f(x) - f(y_0) | + | f(y_0) |) d\mu(x)$$

$$\leq \int_{\mathbf{R}^n} | f(x) - f(y_0) | \, d\mu(x) + | f(y_0) | \, \mu(\mathbf{R}^n) < \infty$$

and we conclude that $f \in L^1(\mu)$.

3. a) Suppose $f{:}\mathbf{R} \to [0, \infty[$ is Lebesgue measurable and $\int_{\mathbf{R}} f dm < \infty$. Prove that

$$\lim_{\alpha \to \infty} \alpha m(f \geq \alpha) = 0.$$

b) Find a Lebesgue measurable function $f{:}\mathbf{R} \to [0, \infty[$ such that $f \notin L^1(m)$, $m(f > 0) < \infty$, and

$$\lim_{\alpha \to \infty} \alpha m(f \geq \alpha) = 0.$$

Solution. a) Suppose $\alpha > 0$. Then $f\chi_{\{f \geq \alpha\}} \geq \alpha \chi_{\{f \geq \alpha\}}$ and integration yields

$$\int_{\mathbf{R}} f\chi_{\{f \geq \alpha\}} dm \geq \alpha m(f \geq \alpha).$$

If $(\alpha_n)_{n=1}^{\infty}$ is an arbitrary increasing sequence such that $\alpha_n \to \infty$ as $n \to \infty$,

$$f\chi_{\{f \geq \alpha_n\}} \leq f \in L^1(m), \; n \in \mathbf{N}_+$$

and

$$f\chi_{\{f\geq\alpha_n\}} \to 0$$

as $n \to \infty$. Therefore by using the Lebesgue Dominated Convergence Theorem it follows that

$$\lim_{n\to\infty} \int_{\mathbf{R}} f\chi_{\{f\geq\alpha_n\}} dm = 0$$

and we get

$$\lim_{n\to\infty} \alpha_n m(f \geq \alpha_n) = 0.$$

Accordingly from this,

$$\lim_{\alpha\to\infty} \alpha m(f \geq \alpha) = 0$$

and we are done.

b) Set $x = g(y) = \frac{1}{y\ln y}$, $y \geq 2$. Define $f(x) = g^{-1}(x)$, $0 < x \leq \frac{1}{2\ln 2}$, and $f(x) = 0$, $x \notin \left]0, \frac{1}{2\ln 2}\right]$. Then

$$\int_{\mathbf{R}} f dm = \int_{0\leq y\leq f(x)} dx dy = \int_0^\infty m(f \geq y) dy$$

$$= \int_0^2 m(f \geq y) dy + \int_2^\infty m(f \geq y) dy$$

$$= \frac{1}{\ln 2} + \int_2^\infty \frac{dy}{y\ln y} = \frac{1}{\ln 2} + [\ln\ln y]_2^\infty = \infty.$$

Thus $f \notin L^1(m)$ and

$$\lim_{y\to\infty} y m(f \geq y) = \lim_{y\to\infty} \frac{1}{\ln y} = 0.$$

4. Let λ be a complex measure and μ a positive measure on the σ-algebra $\mathcal{M}$ such that $\lambda << \mu$. Prove that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mid \lambda(E) \mid < \varepsilon$ whenever $E \in \mathcal{M}$ and $\mu(E) < \delta$.

5. Formulate and prove the Lebesgue Monotone Convergence Theorem.