LÖSNINGAR
INTEGRATIONSTEORI (5p)
(**GU**$[MAF440]$,**CTH**$[TMV100]$)
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Hjälpmedel: Inga.
Notation: m_n denotes Lebesgue measure on $\mathbf{R}^n$, m_1 is written m, and $m_{a,b}$ denotes Lebesgue measure on the compact interval $[a, b]$

1. The function $f : \mathbf{R} \to [0, \infty[$ is Lebesgue measurable and $\int_{\mathbf{R}} f dm = 1$. Determine all non-zero real numbers α such that $\int_{\mathbf{R}} h dm < \infty$, where

$$h(x) = \Sigma_{n=0}^{\infty} f(\alpha^n x + n),\ x \in \mathbf{R}.$$

Solution. Since the map $x \to f(\alpha^n x + n)$ is Lebesgue measurable, the function h is $(\mathcal{R}^-, \mathcal{R}_{0,\infty})$-measurable and the Beppo-Levi Theorem yields

$$\int_{\mathbf{R}} h(x)dx = \Sigma_{n=0}^{\infty} \int_{\mathbf{R}} f(\alpha^n x + n)dx.$$

Here

$$\int_{\mathbf{R}} f(\alpha^n x + n)dx = \mid \alpha \mid^{-n} \int_{\mathbf{R}} f(y)dy = \mid \alpha \mid^{-n}$$

and

$$\int_{\mathbf{R}} h(x)dx = \Sigma_{n=0}^{\infty} \mid \alpha \mid^{-n} .$$

Thus $\int_{\mathbf{R}} h(x)dx < \infty$ if and only if $\mid \alpha \mid > 1$.

2. If $k = (k_1, ..., k_n) \in \mathbf{N}_+^n$, set

$$e_k(x) = \prod_{i=1}^{n} \sin k_i x_i,\ x = (x_1, ..., x_n) \in \mathbf{R}^n$$

and $\mid k \mid = (\sum_{i=1}^n k_i^2)^{\frac{1}{2}}$. Prove that

$$\lim_{|k| \to \infty} \int_{\mathbf{R}^n} f e_k dm_n = 0$$

for every $f \in L^1(m_n)$.

Solution. Let S denote the set of all $f \in L^1(m_n)$ such that

$$\lim_{|k|\to\infty} \int_{\mathbf{R}^n} f e_k dm_n = 0.$$

Clearly, S is a vector subspace of $L^1(m_n)$. Moreover, S is closed in $L^1(m_n)$. To see this let $f \in L^1(m_n)$, $f_j \in S$, and $f_j \to f$ in $L^1(m_n)$ as $j \to \infty$. Then

$$\left| \int_{\mathbf{R}^n} f e_k dm_n \right| \leq \int_{\mathbf{R}^n} \mid f - f_j) e_k \mid dm_n + \left| \int_{\mathbf{R}^n} f_j e_k dm_n \right|$$

and

$$\left| \int_{\mathbf{R}^n} f e_k dm_n \right| \leq \| f - f_j \|_1 + \left| \int_{\mathbf{R}^n} f_j e_k dm_n \right| .$$

Thus

$$\limsup_{|k|\to\infty} \left| \int_{\mathbf{R}^n} f e_k dm_n \right| \leq \| f - f_j \|_1$$

and by letting $j \to \infty$ we have

$$\lim_{|k|\to\infty} \int_{\mathbf{R}^n} f e_k dm_n = 0.$$

Accordingly from this, $f \in S$ and it follows that S is closed.

Next let $I =]a_1, b_1[\times ... \times]a_n, b_n[$ be a bounded open n-cell in $\mathbf{R}^n$. Then

$$\left| \int_{\mathbf{R}^n} \chi_I e_k dm_n \right| = \left| \prod_{i=1}^{n} \frac{\cos k_i a_i - \cos k_i b_i}{k_i} \right|$$

$$\leq \frac{2^n}{k_1 \cdot ... \cdot k_n} \leq \frac{2^n}{\max_{1\leq i\leq n} k_i} \leq \frac{n 2^n}{\mid k \mid} \to 0$$

as $\mid k \mid \to \infty$ and we conclude that $\chi_I \in S$. Since S is closed and the linear span of the functions χ_I, where I is an open bounded n-cell, is dense in $L^1(m_n)$ it follows that $S = L^1(m_n)$.

3. Let $(X, \mathcal{M}, \mu)$ be a σ-finite positive measure space and suppose λ and τ are two probability measures defined on the σ-algebra $\mathcal{M}$ such that $\lambda << \mu$ and $\tau << \mu$. Prove that

$$\sup_{A \in \mathcal{M}} \mid \lambda(A) - \tau(A) \mid = \frac{1}{2} \int_X \mid \frac{d\lambda}{d\mu} - \frac{d\tau}{d\mu} \mid d\mu.$$

Solution. Set

$$f = \frac{d\lambda}{d\mu} - \frac{d\tau}{d\mu}$$

and note that $f \in L^1(\mu)$. Now

$$\int_X f d\mu = \lambda(X) - \tau(X) = 0$$

and we get

$$\int_X f^+ d\mu = \int_X f^- d\mu.$$

Thus

$$\int_X \mid f \mid d\mu = \int_X f^+ d\mu + \int_X f^- d\mu = 2 \int_X f^+ d\mu.$$

Furthermore,

$$\lambda(A) - \tau(A) = \int_A f^+ d\mu - \int_A f^- d\mu \leq \int_A f^+ d\mu \leq \int_X f^+ d\mu = \frac{1}{2} \int_X \mid f \mid d\mu$$

where equality occurs if

$$A = \left\{ f^+ > 0 \right\}.$$

In a similar way,

$$\tau(A) - \lambda(A) = \int_A f^- d\mu - \int_A f^+ d\mu \leq \int_A f^- d\mu \leq \int_X f^- d\mu = \frac{1}{2} \int_X \mid f \mid d\mu$$

(where equality occurs if $A = \{f^- > 0\}$). Thus

$$\sup_{A \in \mathcal{M}} \mid \lambda(A) - \tau(A) \mid = \frac{1}{2} \int_X \mid \frac{d\lambda}{d\mu} - \frac{d\tau}{d\mu} \mid d\mu.$$

4. Suppose $(X, \mathcal{M}, \mu)$ is a positive measure space and $w : X \to [0, \infty]$ a measurable function.

a) Set

$$\nu(A) = \int_A w d\mu, \ A \in \mathcal{M}.$$

Prove that ν is a positive measure.

b) Prove that

$$\mu(w \geq \alpha) \leq \frac{1}{\alpha} \int_X w d\mu, \ 0 < \alpha < \infty.$$

5. Suppose $a, b \in \mathbf{R}$, $a < b$, and that $F : [a, b] \to \mathbf{R}$ is absolutely continuous. Prove that there exists a function $f \in L^1(m_{a,b})$ such that

$$F(x) = F(a) + \int_a^x f(t) dt, \ a \leq x \leq b.$$