LÖSNINGAR
INTEGRATIONSTEORI (5p)
(GU$[MAF440]$ **,CTH**$[TMV100]$**)**
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Hjälpmedel: Inga.

1. Suppose

$$f(t) = \int_0^\infty e^{-tx} \frac{\ln(1+x)}{1+x} dx, \ t > 0.$$

a) Show that $\int_0^\infty f(t)dt < \infty$.
b) Show that f is infinitely many times differentiable.

Solution. a) The function

$$e^{-tx} \frac{\ln(1+x)}{1+x}, \ t > 0, \ x \geq 0$$

is non-negative and continuous. Thus $f \geq 0$ and, moreover, the Tonelli Theorem yields

$$\int_0^\infty f(t)dt = \int_0^\infty \left\{ \int_0^\infty e^{-tx} \frac{\ln(1+x)}{1+x} dt \right\} dx$$

$$= \int_0^\infty \frac{\ln(1+x)}{x(1+x)} dx.$$

Here

$$\lim_{x \to 0} \frac{\ln(1+x)}{x(1+x)} = 1$$

and

$$0 \leq \frac{\ln(1+x)}{x(1+x)} \leq \frac{1}{x^{3/2}} \text{ if } x \text{ large enough.}$$

Since $\int_1^\infty \frac{dx}{x^{3/2}} < \infty$ Part a) is proved.

b) Define

$$d\mu = \frac{\ln(1+x)}{1+x}dx \text{ on } [0,\infty[$$

and observe that μ is a non-negative measure such that

$$f(t) = \int_0^\infty e^{-tx}d\mu(x),\ t > 0.$$

Now choose $a > 0$. It is enough to prove that f is infinitely many times differentiable restricted to the interval $]a,\infty[$. For any fixed $n \in \mathbf{N}_+$, the function

$$h_n(x) = x^n e^{-ax},\ x \geq 0$$

belongs to $L^1(\mu)$ since

$$0 \leq x^n e^{-ax}\frac{\ln(1+x)}{1+x} \leq e^{-\frac{a}{2}x} \text{ if } x \text{ large enough.}$$

Since

$$\mid \frac{\partial}{\partial t}e^{-tx} \mid \leq h_1(x),\ t > a,\ 0 \leq x < \infty$$

it follows from the theorem about interchanging a derivative with an integral that

$$f'(t) = \int_0^\infty -xe^{-tx}d\mu(x),\ t > a.$$

In a similar way

$$\mid \frac{\partial^2}{\partial t^2}e^{-tx} \mid \leq h_2(x),\ t > a,\ 0 \leq x < \infty$$

and it follows that

$$f''(t) = \int_0^\infty x^2e^{-tx}d\mu(x),\ t > a.$$

By repetition (or mathematical induction), we now conclude that f is infinitely many times differentiable restricted to the interval $]a,\infty[$.

2. Suppose α is a positive real number and f a function on $[0,1]$ such that $f(0) = 0$ and $f(x) = x^\alpha \sin\frac{1}{x}$, $0 < x \leq 1$. Prove that f is absolutely continuous if and only if $\alpha > 1$.

Solution. Recall that f is absolutely continuous if and only if the following properties hold:

(i) $f'(x)$ exists for $m_{0,1}$-almost all $x \in [0,1]$

(ii) $f' \in L^1(m_{0,1})$

(iii) $f(x) = f(0) + \int_0^x f'(t)dt$, $0 \le x \le 1$.

In this case

$$f'(x) = \alpha x^{\alpha-1} \sin \frac{1}{x} + x^{\alpha-2} \cos \frac{1}{x} \text{ if } x > 0.$$

Here $\mid \alpha x^{\alpha-1} \sin \frac{1}{x} \mid \le \alpha x^{\alpha-1}$ and we get

$$\alpha x^{\alpha-1} \sin \frac{1}{x} \in L^1(m_{0,1}).$$

Moreover,

$$\int_0^1 \mid x^{\alpha-2} \cos \frac{1}{x} \mid dx = \left[t = \frac{1}{x} \right]$$

$$=_{def} \int_1^\infty t^{-\alpha} \mid \cos t \mid dt =_{def} I_\alpha.$$

Here $I_\alpha < \infty$ if $\alpha > 1$. Morover, if $\alpha \ge 1$

$$I_\alpha \ge \int_1^\infty t^{-1} \mid \cos t \mid dt \ge \frac{1}{\sqrt{2}} \sum_{n=1}^\infty \int_{n2\pi}^{n2\pi+\frac{\pi}{4}} t^{-1} dt$$

$$= \frac{1}{\sqrt{2}} \sum_{n=1}^\infty \ln(1 + \frac{1}{8n}) = \infty$$

since

$$\ln(1 + \frac{1}{8n}) \ge \frac{1}{16n} \text{ if } n \text{ large.}$$

Thus

$$x^{\alpha-2} \cos \frac{1}{x} \in L^1(m_{0,1}) \text{ iff } \alpha > 1$$

and

$$f' \in L^1(m_{0,1}) \text{ iff } \alpha > 1.$$

It follows that the function f is not absolutely continuous for $\alpha \le 1$. If $\alpha > 1$ and $0 < x \le 1$,

$$f(x) = \delta^\alpha \sin \frac{1}{\delta} + \int_\delta^x f'(t)dt, \text{ all } 0 < \delta \le 1$$

and by letting $\delta \to 0$

$$f(x) = f(0) + \int_0^x f'(t)dt.$$

It follows that f is absolutely continuous for every $\alpha > 1$.

3. Suppose $\mu : \mathcal{M} \to \mathbf{C}$ is a complex measure and $f, g : X \to \mathbf{R}$ measurable functions. Show that

$$\mid \mu(f \in A) - \mu(g \in A) \mid \leq \mid \mu \mid (f \neq g)$$

for every $A \in \mathcal{R}$.

Solution. Below $\{f \in A, g \in B\})$ means $\{f \in A\} \cap \{g \in B\}$. We have

$$\mu(f \in A) = \mu(f \in A,\ g \in A) + \mu(f \in A,\ g \notin A)$$

and

$$\mu(g \in A) = \mu(g \in A,\ f \in A) + \mu(g \in A,\ f \notin A)$$

and, accordingly,

$$\mu(f \in A) - \mu(g \in A) = \mu(f \in A,\ g \notin A) - \mu(g \in A,\ f \notin A).$$

Hence

$$\mid \mu(f \in A) - \mu(g \in A) \mid \leq \mid \mu(f \in A,\ g \notin A) \mid + \mid \mu(g \in A,\ f \notin A) \mid$$

$$\leq \mid \mu \mid (f \in A,\ g \notin A) + \mid \mu \mid (g \in A,\ f \notin A)$$

$$= \mid \mu \mid (\{f \in A,\ g \notin A\} \cup \{g \in A,\ f \notin A\}) \leq \mid \mu \mid (f \neq g).$$

4. Suppose $(X, \mathcal{M}, \mu)$ is a positive measure space. Prove the following results:
   a) If $f_n \to f$ in measure and $f_n \to g$ in measure, then $f = g$ a.e. $[\mu]$.
   b) Convergence in $L^1(\mu)$ implies convergence in measure.

5. Formulate and prove the Lebesgue Monotone Convergence Theorem.