TMS165/MSA350 Stochastic Calculus Part I

Written Exam Friday 17 January 2014 2–6 pm

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Aids: None.

Grades: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

Motivations: All answers/solutions must be motivated.

Good Luck!

Througout this exam $B = \{B(t)\}_{t\geq 0}$ denotes a Brownian motion.

Task 1. Can a random process $\{X(t)\}_{t\geq 0}$ have (a) zero variation and zero quadratic variation over finite intervals? (b) non-zero but finite variation and zero quadratic variation over finite intervals? (c) zero variation and non-zero but finite quadratic variation over finite intervals? (d) non-zero but finite variation and non-zero but finite quadratic variation over finite intervals? For each of these for cases answer the question positive by means of providing an example of a process that has the required properties or negative by means of explaining why a process cannot have both the required properties.

(5 points)

Task 2. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Find the quadratic variation process of the process $\{X(t)\}_{t\geq 0}$ given by $X(t) = \int_0^{B(t)} g(s)\, ds$. **(5 points)**

Task 3. Let $\{X(t)\}_{t\in[0,T]}$ solve the Langevin SDE

$$dX(t) = -\mu\, X(t)\, dt + \sigma\, dB(t) \quad \text{for } \ t \in (0,T], \qquad X(0) = X_0,$$

where $\mu, \sigma > 0$ are constants. What SDE does the process $\{Y(t)\}_{t\in[0,T]}$ given by $Y(t) = X(t)^2$ solve? **(5 points)**

Task 4. Give one example of a diffusion process (/a solution to an SDE) that have a stationary distribution. Also, give one example of a diffusion process (/a solution to an SDE) that does not have a stationary distribution. **(5 points)**

Task 5. Consider a sample space Ω equipped with a σ-field of events $\mathcal{F}$. Given that $\{B(t)\}_{t\in[0,T]}$ is a Brownian motion under the probability measure $\mathbf{P}$ on $(\Omega, \mathcal{F})$, find a new probabilitty measure $\mathbf{Q}$ on $(\Omega, \mathcal{F})$ such that the process $\{W(t)\}_{t\in[0,T]}$ given by

$W(t) = B(t) + \mu t$ (where $\mu \in \mathbb{R}$ is a constant) is a Brownian motion under the probability measure $\mathbf{Q}$. **(5 points)**

Task 6. Let $\mu, \sigma : \mathbb{R} \to \mathbb{R}$ be sufficiently smooth functions. The so called fully implicit Euler method for calculation of an approximative numerical solution $\{\hat{X}(t)\}_{t\in[0,T]}$ [as an alternative to the exact solution $\{X(t)\}_{t\in[0,T]}$] of a time homogeneous SDE

$$dX(t) = \mu(X(t))\,dt + \sigma(X(t))\,dB(t) \quad \text{for } t \in (0,T], \qquad X(0) = X_0,$$

is given by $\hat{X}(0) = X_0$ together with the recursive scheme

$$\begin{aligned}&\hat{X}(t_n)\\ &= \hat{X}(t_{n-1}) + \big(\mu(\hat{X}(t_n)) - \sigma(\hat{X}(t_n))\,\sigma'(\hat{X}(t_n))\big)\,(t_n - t_{n-1}) + \sigma(\hat{X}(t_n))\,(B(t_n) - B(t_{n-1}))\end{aligned}$$

for $n = 1, \ldots, N$, where $0 = t_0 < t_1 < \ldots < t_N = T$. Explain why the recursice scheme does not take the simpler form

$$\hat{X}(t_n) = \hat{X}(t_{n-1}) + \mu(\hat{X}(t_n))\,(t_n - t_{n-1}) + \sigma(\hat{X}(t_n))\,(B(t_n) - B(t_{n-1})). \qquad \textbf{(5 points)}$$

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Solutions to Written Exam Friday 17 January 2014

Task 1. The answer to query a is positive as we may take $X(t)$ to be a constant. The answer to query b is positive as we may take $X(t)$ to be any continuously differentiable function with non-zero derivative – see Example 1.5 and Theorem 1.11 in Klebaner's book. The answer to query c is negative as any process with zero variation must be constant and therefore have zero quadratic variation. The answer to query d is positive as we may take $X(t)$ to be a Poisson process which has both variation process and quadratic variation process equal to the process itself.

Task 2. Writing $G(t)$ for a primitive function of $g(t)$ Itô's formula gives

$$d[X,X](t) = (dX(t))^2 = (d(G(B(t)))^2 = \big(g(B(t))\,dB(t) + \tfrac{1}{2}\,g'(B(t))\,dt\big)^2 = g(B(t))^2\,dt,$$

so that $[X,X](t) = \int_0^t g(B(s))^2\,ds$.

Task 3. By Itô's formula we have

$$dY(t) = d(X(t)^2) = 2\,X(t)\,dX(t) + d[X,X](t) = 2\,X(t)\,(-\mu\,X(t)\,dt + \sigma\,dB(t)) + \sigma^2\,dt,$$

so that

$$dY(t) = (\sigma^2 - 2\,\mu\,Y(t))\,dt + 2\,\sigma\sqrt{Y(t)}\,dB(t) \quad \text{for } t\in(0,T], \qquad Y(0) = X_0^2.$$

Task 4. According to Examples 6.15 and 6.16 in Klebaner's book Brownian motion does not have a stationary distribution while an Ornestein-Uhlenbeck process (/a solution to the Langevin SDE) does have a stationary distribution.

Task 5. See Theorem 10.15 in Klebaner's book.

Task 6. For the solution $X(t)$ to the SDE we have

$$\sum_{n=1}^{N}\Big(\mu(X(t_{n-1}))\,(t_n - t_{n-1}) + \sigma(X(t_{n-1}))\,(B(t_n) - B(t_{n-1}))\Big) \to X(T)$$

as $\max_{1\le n\le N}(t_n - t_{n-1}) \downarrow 0$. Here we have

$$\begin{aligned}\mu(X(t_{n-1}))\,(t_n - t_{n-1}) &\approx \big(\mu(X(t_n)) - \mu'(X(t_n))\,(X(t_n) - X(t_{n-1}))\big)\,(t_n - t_{n-1})\\ &\approx \mu(X(t_n))\,(t_n - t_{n-1})\end{aligned}$$

and

$$
\begin{aligned}
\sigma(X(t_{n-1}))\,(B(t_n)-B(t_{n-1})) \\
&\approx \big(\sigma(X(t_n)) - \sigma'(X(t_n))\,(X(t_n)-X(t_{n-1}))\big)\,(B(t_n)-B(t_{n-1})) \\
&\approx \sigma(X(t_n))\,(B(t_n)-B(t_{n-1})) - \sigma'(X(t_n))\,\sigma(X(t_n))\,(B(t_n)-B(t_{n-1}))^2 \\
&\approx \sigma(X(t_n))\,(B(t_n)-B(t_{n-1})) - \sigma'(X(t_n))\,\sigma(X(t_n))\,(t_n-t_{n-1}),
\end{aligned}
$$

so that

$$
\sum_{n=1}^{N}\Big(\big(\mu(X(t_n)) - \sigma'(X(t_n))\,\sigma(X(t_n))\big)\,(t_n-t_{n-1}) + \sigma(X(t_n))\,(B(t_n)-B(t_{n-1}))\Big) \to X(T).
$$