TMS 165/MSA350 Stochastic Calculus Part I Fall 2010

Written exam Thursday 13 January 2011 8.30 am - 1.30 pm

Teacher and Jour: Patrik Albin.

Aids: None.

Grades: 12000 points (40%) out of the full score 30000 points to pass the exam.

Motivations: All answers/solutions must be motivated.

Througout this exam $B = \{B(t)\}_{t\geq 0}$ is a Brownian motion.

Task 1. Find a random process (or function) with strictly positive variation but zero quadratic variation. Also, does there exist a random process (or function) with strictly positive quadratic variation but zero variation? **(5000 points)**

Task 2. Find the probability that $\mathbf{P}\{\int_0^2 B(t)\,dt > \int_0^1 B(t)\,dt\}$. **(5000 points)**

Task 3. Find the quadratic variation process of $\{\sin(B(t)\}_{t\geq 0}$. **(5000 points)**

Task 4. Find the solution $\{X(t)\}_{t\geq 0}$ to the SDE

$$dX(t) = X(t)\,\sin(B(t))\,dB(t) \quad \text{for } t>0, \quad X(0)=1.$$ **(5000 points)**

Task 5. Let $X : \Omega \to \mathbb{R}$ be a standard normal distributed random variable defined on a sample space Ω with probability measure $\mathbf{P}$ and σ-field of measurable events $\mathcal{F}$. Define a new probability measure $\mathbf{Q}$ on $\mathcal{F}$ under which X is unit-mean exponential distributed. **(5000 points)**

Task 6. Write down the Milstein method for numerical solution of the Ornstein-Uhlenbeck (Langevin) SDE

$$dX(t) = -\alpha\,X(t)\,dt + \sigma\,dB(t) \quad \text{for } t>0, \quad X(0)=x_0,$$

where $\alpha, \sigma > 0$ are constants. Say as much as you can about the derivation of the method. **(5000 points)**

Good Luck!

TMS 165/MSA350 Stochastic Calculus Part I Fall 2010

Solutions to written exam Thursday 13 January 2011

Task 1. Any continuously differentiable function $f:[0,\infty)\rightarrow\mathbb{R}$ such that $f'(0)\neq 0$ will have strictly positive variation $\int_0^t |f'(s)|\,ds$ for $t>0$ but zero quadratic variation. Any random process (or function) that has zero variation must be constant and therefore also has zero quadratic variation.

Task 2. We have $\mathbf{P}\{\int_0^2 B(t)\,dt > \int_0^1 B(t)\,dt\} = \mathbf{P}\{\int_1^2 B(t)\,dt > 0\} = \frac{1}{2}$ as $\int_1^2 B(t)\,dt$ is a zero-mean normal distributed random variable with strictly positive variance.

Task 3. According to Itô's formula we have

$$\sin(B(t)) = \int_0^t \cos(B(s))\,dB(s) - \frac{1}{2}\int_0^t \sin(B(s))\,ds \quad \text{for } t\geq 0.$$

As the second process on the right-hand side is a continuous finite variation process the quadratic variation process of $\sin(B)$ is equal to the quadratic variation process of the first process on the right-hand side, which in turn is $\{\int_0^t \cos(B(s))^2\,ds\}_{t\geq 0}$.

Task 4. The solution X is the stochastic exponential of the martingale $\{\int_0^t \sin(B(s))\,dB(s)\}_{t\geq 0}$, which in turn is given by $\{\exp[\int_0^t \sin(B(s))\,dB(s) - \frac{1}{2}\int_0^t \sin(B(s))^2\,ds]\}_{t\geq 0}$.

Task 5. Clearly X has probability density function f under the probability measure

$$\mathbf{Q}(A) = \int_A f(X)\,\sqrt{2\pi}\,\mathrm{e}^{X^2/2}\,d\mathbf{P} \quad \text{for } A\in\mathcal{F},$$

as this gives

$$\begin{aligned}
\mathbf{Q}\{X\in B\} &= \mathbf{E_Q}\{I_{\{X\in B\}}\} \\
&= \mathbf{E_P}\{I_{\{X\in B\}}\,f(X)\,\sqrt{2\pi}\,\mathrm{e}^{X^2/2}\} \\
&= \int_{\mathbb{R}} I_B(x)\,f(x)\,\sqrt{2\pi}\,\mathrm{e}^{x^2/2}\,\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}\,dx \\
&= \int_B f(x)\,dx \quad \text{for } B\subseteq\mathbb{R}.
\end{aligned}$$

So just take $f(x)=\mathrm{e}^{-x}$ for $x\in\mathbb{R}$.

Task 6. See Stig Larsson's lecture notes on numerical methods.