Fluid Mekanik

TME055

1 Flersvarsfrågor

(*3 poäng för rätt svar, -1 poäng för varje fel svar*)

1. $\boldsymbol{\omega} = \mathbf{0}$ är en konsekvens av

   a Rotationsfritt flöde.

   b Inkompressibelt flöde.

   c Isotermiskt flöde.

   d Tvådimensionellt flöde.

2. Vilket av följande är en egenskap hos en fluid?

   a En fluid kan inte flöda snabbare än ljudets hastighet.

   b En fluid kan endast bestå av idealiska molekyler.

   c En fluid kan inte motstå en långdragen spänning.

   d En fluid i en behållare expanderar alltid så att den fyller upp hela behållarens volym.

3. Det Euleriska betraktelsesättet, jämfört med det Lagrangska betraktelsesättet,

   a Rörelsen hos en partikel är medelvärderad över hela fluiddomänen, så att flödets storheter kan definieras.

   b Varje fluidpartikel i flödet definieras med hjälp av sin ursprungliga position.

   c Storheter så som densitet och hastighet är definierade som kontinuerliga funktioner av position och tid, oberoende av de underliggande fluidpartiklar.

   d Vi kan hanskas med ett oändligt antal fluidpartiklar.

4. Vilket av följnade påstående är *sant*:

   a Vorticiteten är nollskild i rotationsfria flöden.

   b Vorticiteten kan ses som cirkulationen per area enhet för ett ytelement vinkelrätt mot vorticitets vektorn.

- c Vorticitet är endast definerat för viskösfria flöden.
- d Vortexlinjer är per definition vinkelräta mot vorticites vektorerna.

5. Vilket av följnade på ståenden är *inte sant*:

   - a Spänningstensorn beror på positionen av en punkt på en yta, men inte p ytans orientering.
   - b För en inkompressibel fluid, är det termodynaiska trycker lika med det mekaniska trycket.
   - c Spänningstensorn är per definition anti-symmetrisk.
   - d Cauchys rörelseekvation relaterar accelerationen till netto kraften vid en punkt och gäller för alla kontinuum.

6. I fluid dynamik beräkningar (Computational Fluid Dynamics),

   - a Det initiala felet är viktigare än diskretiseringsfelet.
   - b Endast finita differans diskretisering kan tillämpas.
   - c Ekvationerna som beskriver flödet är lineariserade för att approximera flödets beteende.
   - d Randvillkor är vanligtvis inte nödvändiga.

7. Dynamisk likformighet

   - a kan användas för att beräkna två olika och godtyckliga flödesproblem på samma sätt.
   - b kräver bara geometrisk likformighet vid ränderna av fluiden.
   - c kräver bara att egenskaperna hos flödet är skalade.
   - d kräver geometrisk likformighet och att egenskaperna hos flödet är skalde.

2 Öppna frågor

1. (*15 poäng*) Börja från Reynolds transportteorem,

$$\frac{d}{dt}\int_{V(t)} F dV = \int_{V(t)} \frac{\partial F}{\partial t} dV + \int_{A(t)} F u_i n_i dA$$

   - a Härled kontinuitetsekvationen genom att utnyttja att massan hos en fluid partikel inte ändras.

b Härled Reynolds transportteorem for egenskapen f som förhåller sig linjärt till densiteten, $F = \rho f$

c Beakta kroppskraften pga gravitationen, ρg_i, och ytkraften pga spänningarna, $\tau_{ij} n_i$, som verkar på fluidpartikeln. Härled impulsekvationen från resultatet i *b*.

d Härled den mekaniska engergibalansen i följande termer $E = \frac{1}{2}\rho u_i^2$, genom att multiplicera resultatet från c med hastigheten.

e Det totala arbetet som utförs av ytkrafterna är $\frac{\partial}{\partial x_j}(u_i \tau_{ij})$. Vilken del finns inte i den mekaniska energibalansen? Vad representerar denna del?

f Den generella Newtonska spänningstensorn kan skrivas som

$$\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij} - \frac{2}{3}\mu\left(\nabla \cdot \vec{u}\right)\delta_{ij}$$

Använd detta och resultatet från c för att härleda impulsekvationen för en *inkompressibel* fluid.

2. (*10 poäng*) Givet följande hastighetsfält i cylindriska koordinater

$$u_r = -ar$$
$$u_\theta = \frac{\Lambda}{2\pi r}\left[1 - \exp\left(-\frac{ar^2}{2\nu}\right)\right]$$
$$u_z = 2az$$

där a, Λ och ν är konstanter,

a Skriv ut skjuvspänningstensorn i polärcylindriska koordinater.

b Bestäm vorticitetsvektorn i polärcylindriska koordinater.

c Beräkna cirkulationen Γ runt en cirkel med radien $r = 1$ vars centrum är placerat i origo i r-θ planet.

3. (*19 poäng*) I tvådimensinellt, inkompressiblet flöde, kan en så kallad "*strömfunktion*" definieras. Med strömfuntioner kan strömlinjer definieras. Från strömfunktionen kan hastighetsfältet deriveras med hjälp av följande ekvationer

$$u \equiv \frac{\partial \psi}{\partial y}$$
$$v \equiv -\frac{\partial \psi}{\partial x}$$

Den materiella derivatan, i.e. derivatan som följer med partikel, är definierad i det Euleriska betraktelsesättet som

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \vec{u} \cdot \nabla \phi$$

Och de inkompressibla Navier-Stokes ekvationer följer

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \mu \nabla^2 \vec{u}$$

Betrakta följande strömfunktion för ett inkpressibelt flöde:

$$\psi = C\left(x^2 y + \frac{y^3}{3}\right)$$

a Bestäm de tvåhastighets komponenterna u och v.

b Verifiera att detta flöde är inkompressiblet.

c Bestäm fluidpartikelns acceleration $\dfrac{D\vec{u}}{Dt}$.

d Bestäm Laplacen av hastighetsvektorn,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$
$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

e Använd resultatet från c, d och Navier-Stokes ekvationer för att hitta tryckfördelningen $p(x, y)$ genom att integrera $\dfrac{\partial p}{\partial x}$ och $\dfrac{\partial p}{\partial y}$. *Ledtråd:* Jämför och kombinera resultatet av de båda integrationerna så att ett ensamt tryckfält kan skrivas ner.

4. (*20 poäng*) Börja från den inkompressibla Navier-Stokes ekvationer i x-riktningen,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

a Skriv om ekvationen på dimensionslös form för problemet i figur 1.

b Identifiera de dimensionslösa grupperna som uppkommer i den dimensionslösa formen av Navier-Stokes ekvation och förklara vad de betyder.

c En flygplansbyggare vill bygga en vattentunnel för att testa nya former på flygplansvingar. Byggaren har lyckats bygga en 1 : 10 modell av ett *riktig* flygplan. En erfaren pilot berättar att ett riktigt flygplan flyger med en hastighet på 150 km/h när det ska landa. Byggaren vet också att de viskösa effekterna är signifikanta under landningen ($\rho_{luft} \approx 1.183\,\text{kg/m}^3$ och $\mu_{luft} \approx 1.84 \times 10^{-5}\,\text{Pa}\cdot\text{s}$). Så, vad ska tunnelns hastighet vara för att försäkra dynamisk likformighet? Antag $\rho_{vatten} = 988\,\text{kg/m}^3$ och $\mu_{vatten} = 0.548 \times 10^{-3}\,\text{Pa}\cdot\text{s}$ vilket motsvarar en vatten temperatur på 50°C och atmosfärstryck.

d Vad är drag kvoten $D_{prototyp}/D_{modell}$?

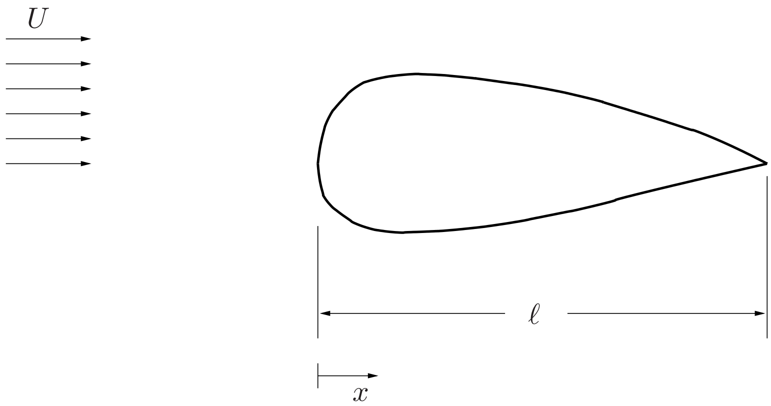

Figure 1: Problem 4: Skalningsparametrar.

5. (*15 poäng*) Luft vid 60°C och 1 atm ($\rho_{luft} = 1.06\,\text{kg/m}^3$, $\nu_{luft} = 1.86 \times 10^{-5}\,\text{m}^2/\text{s}$) flödar i 19 m/s förbi en platta, se figur 2. Ett Pitotrör, placerat 3 mm från väggen, utvecklar en manometerhöjd $\Delta h = 10$ mm av glycerin, $\rho_G = 1260\,\text{kg/m}^3$.

   a Beräkna hastigheten vid positionen x nedströms. *Ledtråd:* Använd Bernoullis ekvation,

   $$\frac{1}{2}q^2 + \int \frac{dp}{\rho} + \text{g}z = \text{constant}$$

   b Uppskatta nedströmspositionen x för Pitotröret. Antag laminärt flöde, så att Blausius lösning given i figur 3 kan användas.

   c Beräkna Re_x. Har vi verkligen laminärt flöde vid denna punkt?

d Skjuvspänningen vid plattans vägg kan bestämas genom

$$\tau_{wall} = \mu \frac{\partial u}{\partial y}\Big|_{y=0}$$

Vad är skjuvspänningen vid väggen? *Ledtråd:* använd figur 3 för att bestämma derivatan.

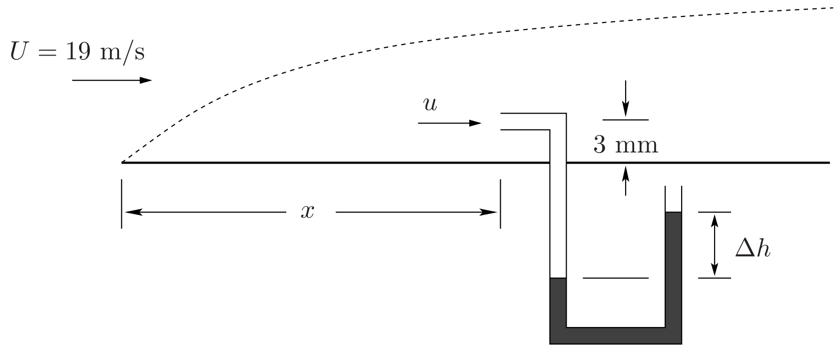

Figure 2: Problem 5: Fldet ver en platta.

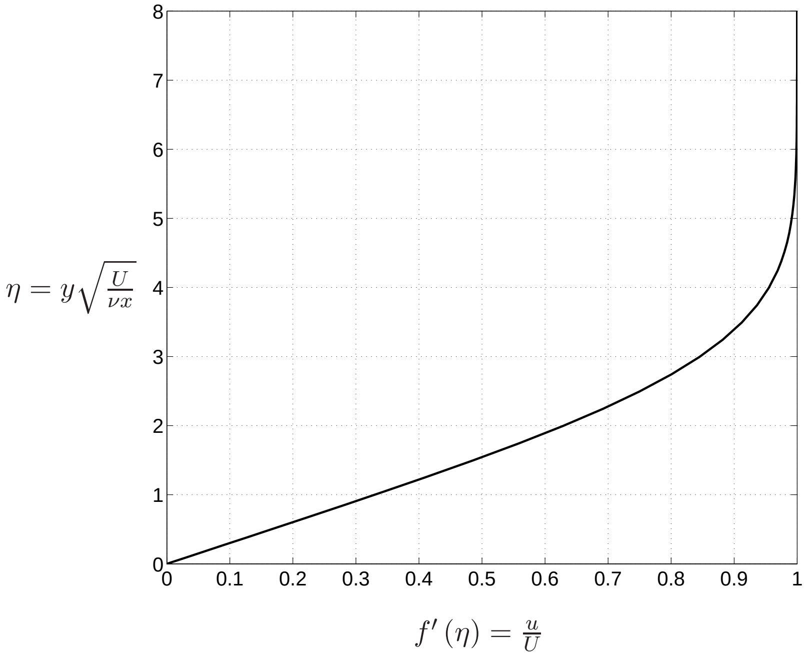

Figure 3: Problem 5: Blasius lösning.

Fluid Mechanics

TME055, Friday April 13, 2007

1 Multiple Choice Questions

(*3 points for each correct answer, -1 point for each incorrect anwser*)

1. $\boldsymbol{\omega} = \mathbf{0}$ is equivalent to

   a Irrotational Flow

   b Incompressible Flow

   c Isothermal Flow

   d Two-dimensional Flow

2. What is a defining property of a fluid?

   a A fluid cannot flow faster than the speed of sound.

   b A fluid can only consist of ideal molecules.

   c A fluid cannot sustain prolonged stress.

   d A fluid always expands to occupy the entire volume of a container.

3. What defines the *Eulerian* framework opposed to the *Lagrangian* framework?

   a The motion of individual fluid particles is averaged over the whole domain, such that mean flow quantities can be defined.

   b Each fluid particle in the flow is identified by its original position.

   c Quantities such as density and velocity are defined as continuous functions of position and time, independent of the underlying fluid particles.

   d We can deal with an infinite number of fluid particles.

4. Which of the following is *true*:

   a Vorticity is non-zero for irrotational flows.

   b Vorticity can be seen as the circulation per unit area for an elemental surface perpendicular to the vorticity vector.

   c Vorticity is only defined for inviscid flow.

   d Vortex lines are by definition perpendicular to the vorticity vectors.

5. Which of the following is *not true*:

   a The stress tensor depends on the position of a point on a surface, but not on the orientation of the surface.

   b For an incompressible fluid, the thermodynamic pressure equals the mechanical pressure.

   c By definition, the stress tensor is anti-symmetric.

   d Cauchy's equation of motion relates acceleration to the net force at a point and holds for any continuum.

6. In Computational Fluid Dynamics,

   a The input error is always more important than the discretisation error.

   b Only finite difference discretisation can be employed.

   c A discretisation is employed to linearize the fluid equations.

   d Boundary conditions are usually not required.

7. Dynamic similarity

   a can be used to treat any two flow problems similar.

   b requires only geometric similarity of the boundaries.

   c requires only that the flow properties are scaled.

   d requires geometric similarity and that the flow properties are scaled.

2 Open Questions

1. (*15 pts*) Starting from the Reynolds Transport Theorem,

$$\frac{d}{dt}\int_{V(t)} F dV = \int_{V(t)} \frac{\partial F}{\partial t} dV + \int_{A(t)} F u_i n_i dA$$

   a Derive the continuity equation by realizing that the mass of a fluid particle does not change.

   b Derive the Reynolds Transport Theorem for the property f linear in density, $F = \rho f$

   c Consider the body force of gravity, ρg_i, and shear stress surface forces, $\tau_{ij} n_i$, working on the fluid particle. Derive the momentum equation from the result obtained under b.

   d Derive the mechanical energy balance in terms of $E = \frac{1}{2}\rho u_i^2$.

   e The total work done by the surface forces is $\frac{\partial}{\partial x_j}(u_i \tau_{ij})$. Which part of this is not present on the mechanical energy balance? What does this part represent?

   f The general Newtonian stress tensor is expressed as

$$\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij} - \frac{2}{3}\mu \left(\nabla \cdot \vec{u}\right) \delta_{ij}$$

   Use this, and the results obtained under c, to derive the momentum equation for an *incompressible* fluid.

2. (*10 pts*) Given the following velocity field in cylindrical coordinates

$$u_r = -ar$$
$$u_\theta = \frac{\Lambda}{2\pi r}\left[1 - \exp\left(-\frac{ar^2}{2\nu}\right)\right]$$
$$u_z = 2az$$

   where a, Λ and ν are constants.

a Write out the strain rate tensor in cylindrical coordinates.

b Determine the vorticity vector in cylindrical coordinates.

c Compute the circulation Γ around a circle of radius $r = 1$ centered at the origin of the r-θ plane.

3. (*19 pts*) In two dimensional, incompressible flow, a so-called "*stream-function*" can be defined. With stream-functions, streamlines can be defined. From the stream-function, the velocity field can be derived from the equations

$$u \equiv \frac{\partial \psi}{\partial y}$$
$$v \equiv -\frac{\partial \psi}{\partial x}$$

The material derivative, i.e. the derivative following a fluid particle, is defined in the Eulerian framework as

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \vec{u} \cdot \nabla \phi$$

And the incompressible Navier-Stokes equation is

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \mu \nabla^2 \vec{u}$$

Consider the following stream function for incompressible flow:

$$\psi = C \left(x^2 y + \frac{y^3}{3} \right)$$

a Determine the two velocity components u and v.

b Verify whether this flow is incompressible.

c Determine the acceleration of a fluid particle $\frac{D\vec{u}}{Dt}$.

d Determine the Laplacian of the velocity vector,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$
$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

e Use the results of c, d and Navier Stokes equation to find out the pressure distribution $p(x, y)$ by integrating $\frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial y}$. *Hint:* Compare and combine the result of both integrations such that a single pressure field can be written down.

4. (*20 pts*) Starting from the incompressible Navier-Stokes equation in the x-direction,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$

a Nondimensionalize this equation for the problem sketched in Figure 1.

b Identify the dimensionless group(s) appearing in the dimensionless Navier-Stokes equation, and explain their meaning.

c An airplain builder wants to build a water tunnel to test new wing airfoil shapes. The builder has successfully made a 1 : 10 model of the airfoil used by a *real* airplane. An experienced pilot tells the builder that the real airplane travels at 150 km/h when it is about to land. The builder also knowns that viscous effects are significant during the landing process ($\rho_{air} \approx 1.183\,\mathrm{kg/m^3}$ and $\mu_{air} \approx 1.84 \times 10^{-5}\,\mathrm{Pa \cdot s}$). So, what should the water tunnel speed be in order to ensure dynamic similarity? Assume $\rho_{water} = 988\,\mathrm{kg/m^3}$ and $\mu_{water} = 0.548 \times 10^{-3}\,\mathrm{Pa \cdot s}$ which correspond to a temperature of 50°C and atmospheric pressure.

d What is the drag ratio $D_{prototype}/D_{model}$?

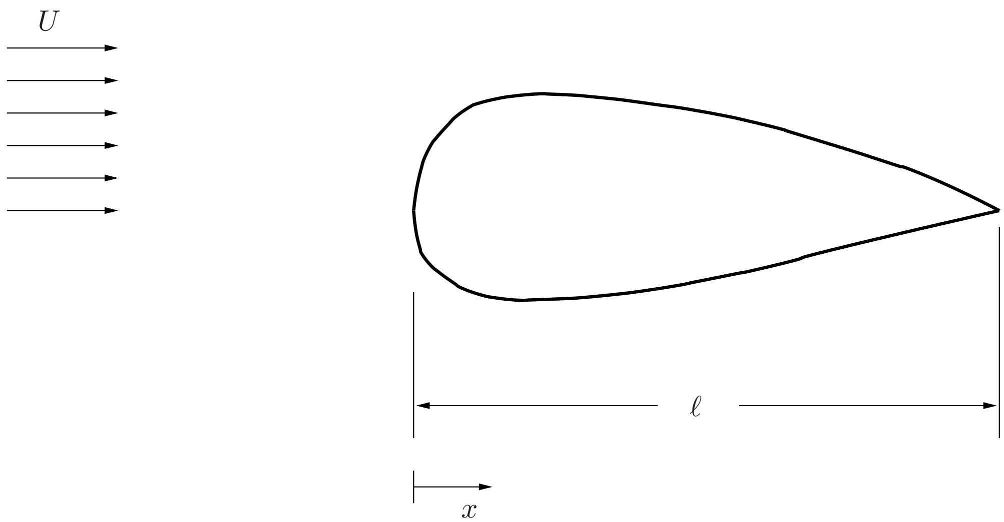

Figure 1: Problem 4: Scaling parameters.

5. (*15 pts*) Air at 60°C and 1 atm ($\rho_{air} = 1.06\,\mathrm{kg/m^3}$, $\nu_{air} = 1.86 \times 10^{-5}\,\mathrm{m^2/s}$) flows at 19 m/s past the flat plate as shown in Figure 2. A Pitot tube, placed 3 mm from the wall, develops a manometer height $\Delta h = 10$ mm of Glycerine, $\rho_G = 1260\,\mathrm{kg/m^3}$.

   a Calculate the velocity at the downstream position x. *Hint:* Use Bernoulli's equation,

$$\frac{1}{2}q^2 + \int \frac{dp}{\rho} + \mathrm{g}z = \text{constant}$$

   b Estimate the downstream position x of the pitot tube. Assume laminar flow, so you can use the Blasius solution given in Figure 3.

   c Calculate Re_x. Do we really have laminar flow at this point?

   d The shear stress at the wall of the plate can be determined as

$$\tau_{wall} = \mu \frac{\partial u}{\partial y}\bigg|_{y=0}$$

   What is the shear stress at the wall? *Hint:* use Figure 3 to determine the derivative.

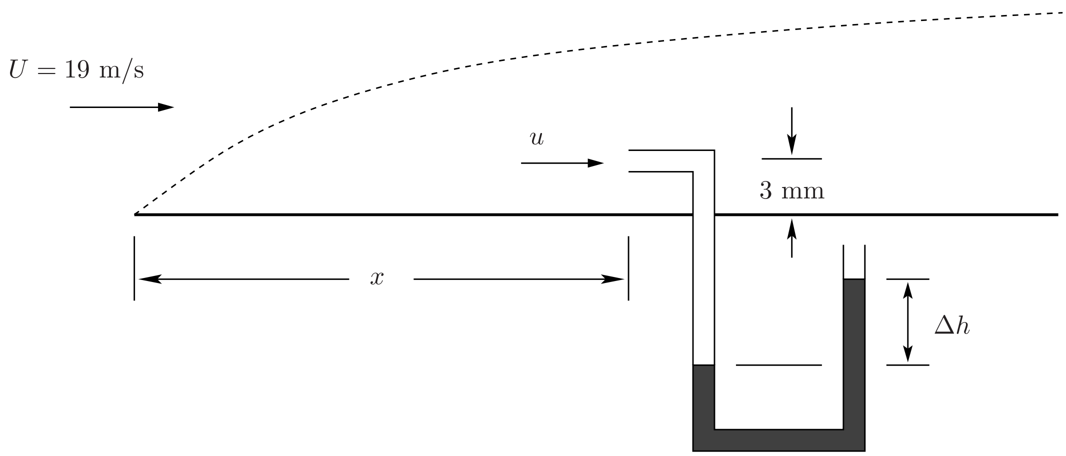

Figure 2: Problem 5: Flow past a flat plate.

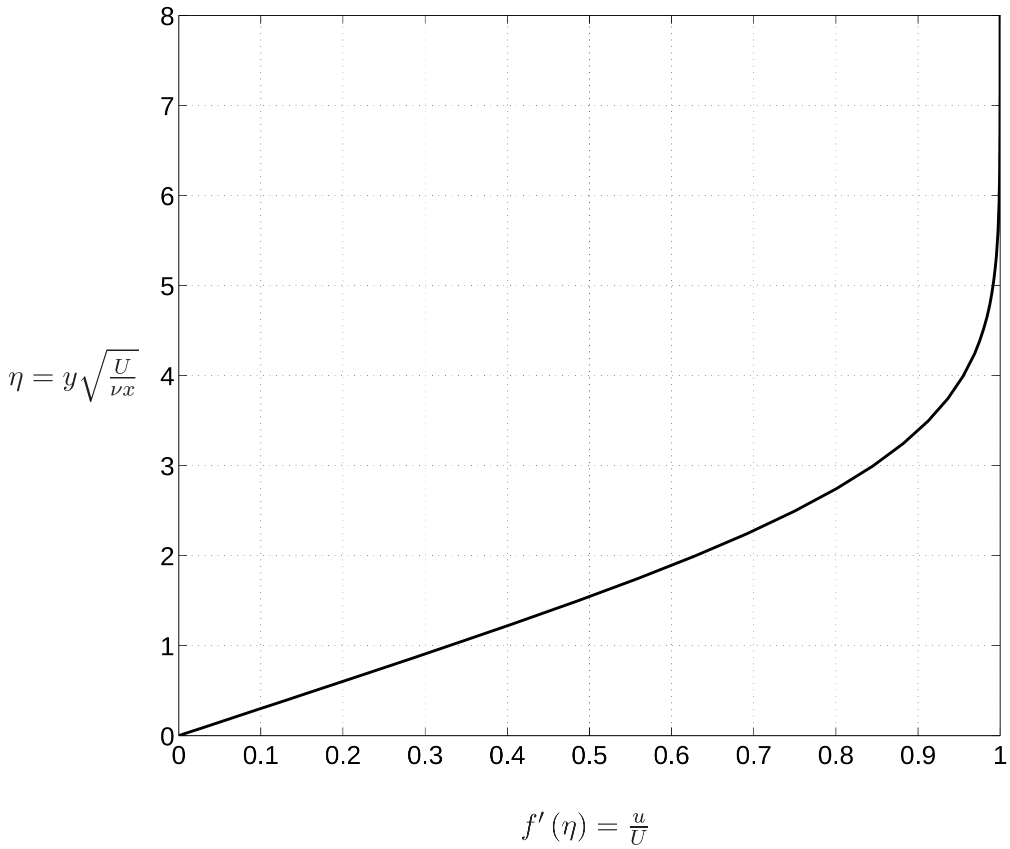

Figure 3: Problem 5: Blasius solution.

Fluid Mechanics
TME055

1 Multiple Choice Questions

1. a
2. c
3. c
4. b
5. c
6. c
7. d

2 Open Questions

1. Reynolds Transport Theorem

$$\frac{d}{dt}\int_{V(t)} F dV = \int_{V(t)} \frac{\partial F}{\partial t} dV + \int_{A(t)} F u_i n_i dA$$

a Setting $F = \rho$, we have

$$\frac{d}{dt}\int_{V(t)} \rho dV = 0$$

due to mass conservation. Using Reynolds transport theorem and Gauss theorem we get

$$\int_V \frac{\partial \rho}{\partial t} dV + \int_A \rho u_i n_i \, dA = 0$$

$$\int_V \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i)\right] dV = 0$$

As the volume V is arbitrary, we finally get

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0}$$

b Setting $F = \rho f$ we have

$$\frac{d}{dt}\int_V \rho f dV = \int_V \frac{\partial}{\partial t}(\rho f) dV + \int_A \rho f u_i n_i dA$$

$$= \int_V \left[\frac{\partial}{\partial t}(\rho f) + \frac{\partial}{\partial x_i}\left(\rho f u_i\right)\right] dV$$

Using continuity equation we can rewritte the right hand side

$$\frac{d}{dt}\int_V \rho f dV = \int_V \left[\rho\frac{\partial f}{\partial t} + \rho u_i \frac{\partial f}{\partial x_i}\right] dV$$

Now we can identify the material derivative of f times ρ as the integrand on the right hand side, so

$$\frac{d}{dt}\int_V \rho f dV = \int_V \rho \frac{Df}{Dt} dV$$

c The momentum equation can be derived from the integral relation for f found in b if we set $f = u_i$:

$$\int_V \rho \frac{Du_i}{Dt} dV = \underbrace{\int_V \rho g_i dV + \int_A \tau_{ij} n_j dA}_{\text{Forces acting on a fluid particle}}$$

$$= \int_V \left[\rho g_i + \frac{\partial \tau_{ij}}{\partial x_j}\right] dV$$

We can group all terms into a single volume integral on the left hand side

$$\int_V \left[\rho\frac{Du_i}{Dt} - \rho g_i - \frac{\partial \tau_{ij}}{\partial x_j}\right] dV = 0$$

As the integration volume is arbitrary, we finally get

$$\boxed{\rho\frac{Du_i}{Dt} = \rho g_i + \frac{\partial \tau_{ij}}{\partial x_j}}$$

d Mechanical energy balance $E = \frac{1}{2}\rho u_i u_i$. We need to multiply the momentum equation derived in c by u_i

$$\rho u_i \frac{Du_i}{Dt} = \rho u_i g_i + u_i \frac{\partial \tau_{ij}}{\partial x_j}$$

Using continuity equation, we can rewrite the left hand side as

$$\rho \frac{D}{Dt}\left(\tfrac{1}{2}u_i u_i\right) = \rho \frac{\partial}{\partial t}\left(\tfrac{1}{2}u_i u_i\right) + \rho u_j \frac{\partial}{\partial x_j}\left(\tfrac{1}{2}u_i u_i\right) + \left(\tfrac{1}{2}u_i u_i\right)\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\right]$$

$$= \frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u_i u_i\right) + \frac{\partial}{\partial x_j}\left(\tfrac{1}{2}\rho u_i u_i u_j\right)$$

or in terms of E we get

$$\boxed{\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j}(u_j E) = \rho u_i \mathrm{g}_i + u_i \frac{\partial \tau_{ij}}{\partial x_j}}$$

e Total work done by the surface forces

$$\frac{\partial}{\partial x_j}(u_i \tau_{ij}) = \underbrace{\tau_{ij}\frac{\partial u_i}{\partial x_j}}_{\substack{\text{Not present on } E\text{-equation} \\ \text{Deformation work} \\ \text{Increase internal energy}}} + u_i \frac{\partial \tau_{ij}}{\partial x_j}$$

f Newtonian stress tensor becomes

$$\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij} - \frac{2}{3}\mu \underbrace{(\nabla \cdot \vec{u})}_{\substack{=0 \\ \text{Incompresible} \\ \text{fluid}}} \delta_{ij}$$

Substituting in the momentum equation c, we get

$$\rho \frac{Du_i}{Dt} = \rho \mathrm{g}_i + \frac{\partial}{\partial x_j}(-p\delta_{ij} + 2\mu e_{ij})$$

If we assume that temperature differences are small within the fluid, $\mu \approx constant$, so

$$\rho \frac{Du_i}{Dt} = \rho \mathrm{g}_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

As $\frac{\partial u_j}{\partial x_j} = 0$ for an incompressible fluid, we get

$$\boxed{\rho \frac{Du_i}{Dt} = \rho \mathrm{g}_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}}$$

2. Velocity field in cylindrical coordinates

$$u_r = -ar$$
$$u_\theta = \frac{\Lambda}{2\pi r}\left[1 - \exp\left(-\frac{ar^2}{2\nu}\right)\right]$$
$$u_z = 2az$$

a From Appendix B we have:

$$\boldsymbol{e} = \begin{bmatrix} e_{rr} & e_{r\theta} & e_{rz} \\ e_{\theta r} & e_{\theta\theta} & e_{\theta z} \\ e_{zr} & e_{z\theta} & e_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} -a & \frac{\Lambda}{2\pi}\left[\left(\frac{a}{2\nu} + \frac{1}{r^2}\right)e^{-ar^2/2\nu} - \frac{1}{r^2}\right] & 0 \\ \frac{\Lambda}{2\pi}\left[\left(\frac{a}{2\nu} + \frac{1}{r^2}\right)e^{-ar^2/2\nu} - \frac{1}{r^2}\right] & -a & 0 \\ 0 & 0 & 2a \end{bmatrix}$$

b The vorticity vector is:

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_r \\ \omega_\theta \\ \omega_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{a\Lambda}{2\pi\nu}\exp\left(-\frac{ar^2}{2\nu}\right) \end{pmatrix}$$

c The circulation can easily be computed using Stokes' theorem:

$$\Gamma = \oint \boldsymbol{u} \cdot d\boldsymbol{l} = \int_A \boldsymbol{\omega} \cdot \boldsymbol{n}\, dA$$

$$= \int_o^1 \omega_z\, 2\pi r dr$$

$$= \int_o^1 \Lambda \frac{ar}{\nu} \exp\left(-\frac{ar^2}{2\nu}\right) dr$$

$$= \Lambda\left[1 - \exp\left(-\frac{a}{2\nu}\right)\right]$$

3. Stream function:

   a The velocity components are

$$u = C\left(x^2 - y^2\right)$$

$$v = -2Cxy$$

   b We calculate the derivatives

$$\frac{\partial u}{\partial x} = 2Cx$$

$$\frac{\partial v}{\partial y} = -2Cx$$

   and we see that incompressible flow is satisfied $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$.

   c The components of $\frac{D\vec{u}}{Dt}$ are

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

$$= 0 + 2C^2\left(x^3 + xy^2\right)$$

   and

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}$$

$$= 0 + 2C^2\left(y^3 + x^2 y\right)$$

   d Taking the derivative of u and v in the x and y directions yields to

$$\nabla^2 u = 2C - 2C = 0$$
$$\nabla^2 v = 0 - 0 = 0$$

e The Navier-Stokes equation simplifies to

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

and using the results of b we can integrate the pressure gradien on each direction to get the pressure field,

$$p(x,y) = \int -\frac{1}{\rho}\frac{\partial p}{\partial x}dx + f(y) = C^2x^2\left(\frac{1}{2}x^2 + y^2\right) + f(y)$$
$$p(x,y) = \int -\frac{1}{\rho}\frac{\partial p}{\partial y}dy + g(x) = C^2y^2\left(\frac{1}{2}y^2 + x^2\right) + g(x)$$

and now we can combine both result by looking at the missing part on each expression for the pressure field. We can finally write

$$p(x,y) = C^2\left[x^2y^2 + \frac{1}{2}\left(x^4 + y^4\right)\right]$$

4. Dimensional analysis

a Using the scaling parameters of Figure 1, we can nondimensionalize the velocity components, pressure, coordinate directions and time as follows:

$$u' = \frac{u}{U} \qquad v' = \frac{v}{U} \qquad w' = \frac{w}{U}$$

$$x' = \frac{x}{\ell} \qquad y' = \frac{y}{\ell} \qquad z' = \frac{z}{\ell}$$

$$t' = \frac{t}{\ell/U} \qquad p' = \frac{p - p_\infty}{\rho U^2}$$

Where p_∞ is a reference pressure (constant). Then we have,

$$\frac{U^2}{\ell}\frac{\partial u'}{\partial t'} + \frac{U^2}{\ell}\left(u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} + w'\frac{\partial u'}{\partial z'}\right) = -\frac{U^2}{\ell}\frac{\partial p'}{\partial x'} + \frac{\nu U}{\ell^2}\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^2 w'}{\partial z'^2}\right)$$

or

$$\frac{\partial u'}{\partial t'} + \left(u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} + w'\frac{\partial u'}{\partial z'}\right) = -\frac{\partial p'}{\partial x'} + \frac{\nu}{U\ell}\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^2 w'}{\partial z'^2}\right)$$

b The term on the right hand side multiplying the three terms between brackets can be identified as the reciprocal of Reynolds number

$$Re = \frac{U\,\ell}{\nu}$$

Re is the ratio of inertial to viscous forces and provides a criterion for determining dynamic similarity.

c Dynamic similarity requires that

$$\left(\frac{\rho U \ell}{\mu}\right)_{model} = \left(\frac{\rho U \ell}{\mu}\right)_{prototype}$$

so, the free-stream velocity in the water tunnel is

$$U_m = \frac{\rho_p}{\rho_m}\frac{\mu_m}{\mu_p}\frac{\ell_p}{\ell_m}U_p = \frac{1.183}{988}\frac{0.548 \times 10^{-3}}{1.84 \times 10^{-5}} \times 10 \times \frac{150}{3.6}$$

$$\boxed{U_m = 14.9\,\mathrm{m/s}}$$

d The drag experienced by the airfoil (model and prototype) is of the form $D \sim \rho U^2 \ell^2$, so the drag ratio becomes

$$\frac{D_p}{D_m} = \frac{\rho_{air}}{\rho_{water}}\left(\frac{U_p \ell_p}{U_m \ell_m}\right)^2 = \frac{1.183}{988} \times \left(\frac{150/3.6}{14.9} \times 10\right)^2$$

$$\boxed{\frac{D_p}{D_m} = 0.94}$$

5. Flow past a flat plate

a Assuming constant stream pressure, the manometer can be used to estimate the local velocity u at the position of the Pitot inlet:

$$\Delta p = (\rho_G - \rho_{air})\,\mathrm{g}\Delta h = (1260 - 1.06) \times 9.81 \times 0.010 = 124\,\mathrm{Pa}$$

$$u \approx \sqrt{\frac{2\Delta p}{\rho_{air}}} = \sqrt{\frac{2 \times 124}{1.06}} = \boxed{15.3\,\mathrm{m/s}}$$

b The Blasius solution uses $f'(\eta) = \frac{u}{U}$ to determine the position η (see Figure 3)

$$\frac{u}{U} = \frac{15.3}{19} = 0.81$$

which corresponds to $\eta \approx 2.8$ from Figure 3. The downstream position is

$$x = \left(\frac{y}{\eta}\right)^2 \frac{U}{\nu} = \left(\frac{3 \times 10^{-3}}{2.8}\right)^2 \frac{19}{1.86 \times 10^{-5}}$$

$$\boxed{x \approx 1.17\,\mathrm{m}}$$

c We check: $Re_x = 19 \times 1.17/(1.86 \times 10^{-5}) \approx 1.2 \times 10^6$ which can be considered laminar if the flow is very smooth and free of fluctuations that can lead to the growth of instabilities.

d The wall shear stress can be rewritten in terms of boundary layer variables:

$$\tau_{wall} = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu \frac{U}{\delta} \frac{df'}{d\eta}\bigg|_{\eta=0}$$

where the boundary layer grows as $\delta \sim \sqrt{\nu x/U}$, and from Figure 3 we can estimate the slope to be $f''(0) \approx 0.37$, so we get

$$\tau_{wall} = \mu \frac{\sqrt{Re_x}}{x} U f''(0) = 1.97 \times 10^{-5} \times \frac{\sqrt{1.2 \times 10^6}}{1.17} \times 19 \times 0.37$$

$$\boxed{\tau_{wall} \approx 0.13\,\mathrm{Pa}}$$