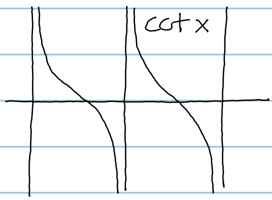
Föreläsning 12/11-13

16.39 ? a, b s.a.

$$\exists \lim_{x \to 0} x^{-3}\left(\cot x + \frac{a}{x} + bx\right)$$

Lösning: $\dfrac{\cot x + \frac{a}{x} + bx}{x^3} \xrightarrow[x \to 0]{} \text{"}\dfrac{\pm\infty \mp \infty + 0}{0}\text{"}$



$a < 0$

$$\frac{\frac{\cos x}{\sin x} + \frac{a}{x} + bx}{x^3} =$$

$$= \frac{x\cos x + a\sin x + bx^2 \sin x}{x^4 \sin x}$$

(nämnaren beter sig som x^5, om L'Hospitals regel används kommer man behöva derivera ca 5 ggr)

Taylors formel: (t.o.m x^5)

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)$$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7)$$

$$\frac{\cancel{x} - \cancel{\frac{x^3}{2}} + \frac{x^5}{24} + \cancel{ax} - \cancel{\frac{ax^3}{6}} + \frac{ax^5}{120} + \cancel{bx^3} - b\frac{x^5}{6} + O(x^7)}{x^5 + O(x^7)}$$

$x + ax = 0 \Rightarrow a = -1 \quad (x \neq 0, x \to 0)$

$-\frac{1}{2} + \frac{1}{6} + b = 0 \Rightarrow b = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

$\longrightarrow$

$$\frac{\frac{1}{24}-\frac{1}{120}-\frac{1}{18}+O(x^2)}{1+O(x^2)} \longrightarrow \frac{1}{24}-\frac{1}{120}-\frac{1}{18} =$$

$$= \frac{1}{30}-\frac{1}{18} = \frac{3-5}{90} = -\frac{1}{45}$$

16.40 a) $\lim_{x\to 0} x^{-3}\left(\int_0^1 \frac{\sin xt}{t}\,dt - x\right)$

$$\frac{\int_0^1 \frac{\sin xt}{t}\,dt - x}{x^3} = \frac{\int_0^1 \frac{\cancel{x}\cancel{t} - \frac{x^3 t^{\cancel{3}\,2}}{6} + O(t^{\cancel{5}\,4} x^5)}{\cancel{t}}\,dt - x}{x^3} =$$

begränsad

$$= \frac{\cancel{x}\cdot 1 - x^3 \cdot \frac{1}{6}\cdot\left[\frac{t^3}{3}\right]_0^1 + O(x^{\cancel{5}\,2}) - \cancel{x}}{x^3}$$

$$\longrightarrow -\frac{1}{6}\cdot\frac{1}{3} = -\frac{1}{18}$$

alternativt: $\int_0^1 \frac{\sin xt}{t}\,dt = \left[\begin{array}{l|l} xt=u & t=0:\ u=0 \\ t=\frac{u}{x};\ dt=\frac{1}{x}du & t=1:\ u=x \end{array}\right] =$

$$= \int_0^x \frac{\sin u}{\frac{u}{x}}\cdot\frac{1}{x}\,du = \int_0^x \frac{\sin u}{u}\,du$$

bra form om man ska använda L'Hospitals regel

$$\int_0^\infty \frac{\sin ax}{x}\,dx$$

$a>0$ oberoende av $\underline{\underline{a}}$

Varningsexempel för L'Hospitals regel:

$$\lim_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{\sin x}$$

$$\frac{f'(x)}{g'(x)} = \frac{\overbrace{\underbrace{2x}_{\to 0}\ \sin\frac{1}{x}}^{\text{begränsad}} + x^2 \cos\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right)}{\cos x} \xrightarrow[x\to 0]{} \ ?$$

$\to 0$

Lösning m.h.a standardgränsvärden:

$$\frac{x^2 \sin\frac{1}{x}}{\sin x} = \frac{x \sin\frac{1}{x}}{\underbrace{\frac{\sin x}{x}}_{\to 1}} \xrightarrow[x\to 0]{} 0$$

$$\lim_{x\to\infty} \frac{e^{x^2}}{e^x} = \lim_{x\to\infty} e^{x^2-x} = \text{"}e^{\infty}\text{"} = \infty$$

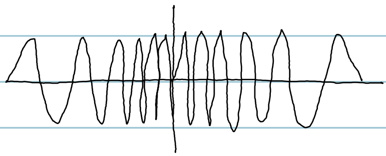
m.h.a L'Hospitals regel:

$$\frac{f'(x)}{g'(x)} = \frac{2x e^{x^2}}{e^x}$$

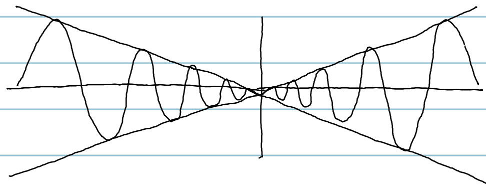
L'Hospitals regel ger inget!
Svårigheten kommer bara att reproduceras

$f_p(x) = x^p \sin \frac{1}{x}$

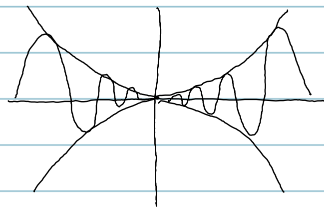
$p=0$ $\quad \sin \frac{1}{x}$ $\quad$ saknar $\lim\limits_{x\to 0}$



$p=1$ $\quad x \sin \frac{1}{x} \xrightarrow[x\to 0]{} 0$



$p=2$ $\quad x^2 \sin \frac{1}{x}$ $\quad$ har derivata
derivatan ej
kont. i 0



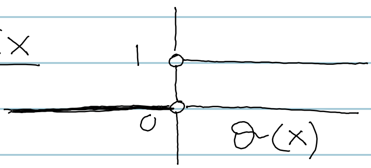
<u>Darboux sats</u>

f deriverbar i I; $a, b \in I$
$f'(a) \cdot f'(b) < 0$ $\quad$ ($f'(a)$ och $f'(b)$ har olika tecken)

$\Rightarrow \exists\, \xi \in (a,b) : f'(\xi) = 0$

(f' uppfyller påståendet i satsen om mellanliggande värden utan att f' behöver vara kontinuerlig)

<u>Ex</u>

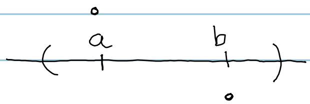


$\nexists$ deriverbar funktion s.a.
$f' = \theta$

BEVIS av sats: $f'(a) > 0$

$f'(b) < 0$ WLOG

$a < b$



$$f'(a) = \lim_{h \to 0+} \frac{f(a+h) - f(a)}{h} > 0$$

$\Rightarrow$ Välj $\varepsilon = \frac{1}{2} f'(a)$; $\exists\, \delta > 0 : \forall h : 0 < h < \delta$

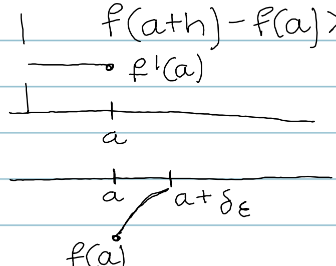
$$\left| \frac{f(a+h) - f(a)}{h} - f'(a) \right| < \frac{1}{2} f'(a)$$

$$\Rightarrow -\frac{1}{2} f'(a) < \frac{f(a+h) - f(a)}{h} - f'(a) < \frac{1}{2} f'(a)$$

$$\Rightarrow 0 < \underbrace{\frac{1}{2} f'(a) < \frac{f(a+h) - f(a)}{h}}_{} \left(< \frac{3}{2} f'(a) \right)$$

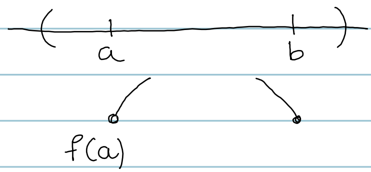
$$\Rightarrow \frac{f(a+h) - f(a)}{h} > 0 \quad \text{för } 0 < h < \delta_\varepsilon$$

$f(a+h) - f(a) > 0 \Rightarrow f(a+h) > f(a) \quad \forall h : 0 < h < \delta_\varepsilon$



$\longrightarrow$

$f'(b) < 0 \Rightarrow$ p.s.s. $\dfrac{f(b-h) - f(b)}{-h} < 0 \quad \forall h : 0 < h < \mu_\varepsilon$



f deriverbar i $I \Rightarrow f$ kontinuerlig i $[a,b]$

$\Rightarrow f$ har ett största värde i $[a,b]$
f_{max} kan inte antas i a eller b

$\Rightarrow f_{max}$ antas i en inre pkt ξ $(\in (a,b))$

$\Rightarrow f'(\xi) = 0$

<u>Alternativt</u>: $f'(a) \neq f'(b)$
μ mellan $f'(a)$ och $f'(b)$

$\Rightarrow \exists \, \xi \in (a,b) : f'(\xi) = \mu$

$g(x) = f(x) - \mu x$
g uppfyller satsen innan

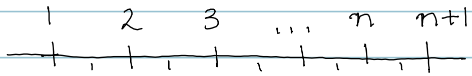
Differensekvationer

Diskret analog till differentialekvationer

$$\frac{\Delta f}{\Delta x} \xrightarrow[\Delta x \to 0]{} f'$$

enhet i x-led: 1

det är inte naturligt att låta den gå mot noll



$$\frac{f(n+1)-f(n)}{1} = \frac{\overbrace{y_{n+1}-y_n}^{\Delta y}}{\underbrace{(n+1)-n}_{\Delta x}}$$

Differensekvation

$$F(y_k, y_{k+1}, \ldots, y_n, n) = 0$$

Linjära differensekvationer:

$$y_{n+k} + a y_{n+k-1} + \ldots + b y_n = f(n)$$

differensekvationens ordning