Tentamen i Matematisk analys, fortsättningskurs F/TM, TMA976
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OBS: Motivera dina svar väl. Det är i huvudsak beräkningarna och motiveringarna som ger poäng, inte svaret. SKRIV TYDLIGT OCH REDOVISA VILKA RESULTAT FRÅN KURSEN SOM DU ANVÄNDER.

Totalt 50 poäng, fördelade på 4 uppgifter (2 sidor). Inga hjälpmedel!

Betygsgränser: 3 (20-34 poäng), 4 (35-43 poäng), 5 (44-50 poäng).

1. Bestäm lösningen $y : \mathbb{R} \to \mathbb{R}$ till initialvärdesproblemet

$$y(0) = y'(0) = 0, \quad y''(x) + 3y'(x) + 2y(x) = \frac{1}{1+e^x}, \quad x \in \mathbb{R}.$$

Du kan utan härledning använda att

$$\int \ln x \, dx = x(\ln(x) - 1) + C.$$

(10p)

2. Bestäm lösningen och dess definitionsmängd till initialvärdesproblemet:

$$y(0) = 1, \quad e^{y(x)}y'(x) = (2 + 3e^{y(x)} + e^{2y(x)})x.$$

Ledtråd: Använd partialbråksuppdelningen:

$$\frac{1}{2+3u+u^2} = \frac{1}{u+1} - \frac{1}{u+2}.$$

(10p)

3. Beräkna gränsvardet:

$$\lim_{x\to 0} \frac{\sin(\sin(\sin(x))) - x\cos(x)}{(1-\cos(x))^2 \tan(x)}$$

Du behöver inte ange ett numeriskt värde. Det räcker med ett tydligt uttryck som involverar summor av kvoter av fakulteter. (10p)

4. a) Visa att för alla $p > 0$:

$$\lim_{x \to 1^-} (1-x)^{1/p} \sum_{n=0}^{\infty} x^{n^p} = \int_0^{\infty} e^{-u^p}\, du.$$

(8p)

b) För vilka $x \in \mathbb{R}$ konvergerar potensserien

$$f(x) = \sum_{n=0}^{\infty} 3^n x^{2^n}.$$

Ange även huruvida konvergensen är betingad eller absolut. (4p)

d) Konvergerar serien

$$f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}, \quad x \in [0,1],$$

likformigt på det slutna intervallet $[0,1]$? (8p)

SOLUTIONS EXAM 3, 2023/24

Problem 1

The characteristic equation $r^2 + 3r + 2 = (r+2)(r+1) = 0$ has the roots $r_1 = -2$ and $r_1 = -1$. Since $y(0) = y'(0) = 0$, the homogeneous solution is identically zero. From the course we then know that

$$y(x) = y_p(x) = e^{-2x} \int_0^x e^{-(-2-(-1))t} \Big(\int_0^t \frac{e^\tau}{1+e^\tau}\, d\tau \Big)\, dt,$$

for all $x \geq 0$. Note that

$$\int_0^t \frac{e^\tau}{1+e^\tau}\, d\tau = \int_1^{e^t} \frac{ds}{1+s} = \ln(1+e^t) - \ln(2),$$

for all $t \geq 0$. Hence,

$$\begin{aligned} y(x) &= e^{-2x} \int_0^x e^t \big(\ln(1+e^t) - \ln(2) \big)\, dt \\ &= e^{-2x} \Big(\int_1^{e^x} \ln(1+s)\, ds - (e^x - 1)\ln(2) \Big) \\ &= e^{-2x} \Big(\int_2^{e^x+1} \ln(s)\, dx - (e^x - 1)\ln(2) \Big) \\ &= e^{-2x} \Big[s(\ln s - 1) \Big]_2^{e^x+1} - (e^{-x} - e^{-2x})\ln(2) \\ &= e^{-2x} \Big((e^x+1)(\ln(e^x+1) - 1) - 2(\ln(2) - 1) \Big) - (e^{-x} - e^{-2x})\ln(2) \\ &= \boxed{e^{-2x}(e^x+1)\ln(e^x+1) - e^{-x}(1+\ln(2)) + e^{-2x}(1-\ln(2))}. \end{aligned}$$

Problem 2

The differential equation is separable. After the partial fraction trick, alluded to on the exam, we can write

$$\Big(\frac{e^y}{1+e^y} - \frac{e^y}{2+e^y}\Big)y' = x.$$

Upon integrating both sides, we get

$$\ln(1+e^y) - \ln(2+e^y) = x^2/2 + C,$$

where C is determined from the initial condition $y(0) = 1$:

$$C = \ln\Big(\frac{1+e}{2+e}\Big).$$

Hence,

$$\frac{1+e^{y(x)}}{2+e^{y(x)}} = e^C \cdot e^{x^2/2},$$

or equivalently,

$$e^{y(x)} = \frac{2e^C e^{x^2/2} - 1}{1 - 2e^C e^{x^2/2}}, \quad x > 0.$$

We conclude that if $|x| < \sqrt{2(C + \ln(2))}$), then

$$\boxed{y(x) = \ln\Big(\frac{2e^C e^{x^2/2} - 1}{1 - 2e^C e^{x^2/2}}\Big)},$$

is a well-defined solution to the initial value problem.

Problem 3

We begin by analysing the denominator. Using that the denominator is an even function, combined with standard limits for sin and tan, we get

$$(1-\cos(x))^2\tan(x) = 4\sin^4(x/2)\tan(x) = 4(x/2)^4 \cdot \Big(\frac{\sin(x/2)}{x/2}\Big)^4 \cdot \Big(\frac{\tan(x)}{x}\Big) \cdot x$$

$$= \frac{x^5}{4}\Big(1 + \mathcal{O}(x^2)\Big) = \frac{x^5}{4} + \mathcal{O}(x^7).$$

For the nominator, we first note that

$$\sin(\sin(x)) = \sin(x) - \frac{\sin^3(x)}{3!} + \frac{\sin^5(x)}{5!} + \mathcal{O}(x^7),$$

and thus

$$\begin{aligned}
\sin(\sin(\sin(x)) &= \sin\Big(\sin(x) - \frac{\sin^3(x)}{3!} + \frac{\sin^5(x)}{5!} + \mathcal{O}(x^7)\Big) \\
&= \sin(x) - \frac{\sin^3(x)}{3!} + \frac{\sin^5(x)}{5!} + \mathcal{O}(x^7) \\
&- \frac{1}{3!}\Big(\sin(x) - \frac{\sin^3(x)}{3!} + \frac{\sin^5(x)}{5!} + \mathcal{O}(x^7)\Big)^3 \\
&+ \frac{1}{5!}\Big(\sin(x) - \frac{\sin^3(x)}{3!} + \frac{\sin^5(x)}{5!} + \mathcal{O}(x^7)\Big)^5 + \mathcal{O}(x^7) \\
&= \sin(x) - \frac{2}{3!}\sin^3(x) + \frac{1}{5!}\Big(2 + \frac{3\cdot 5!}{3!\cdot 3!}\Big)\cdot \sin^5(x) + \mathcal{O}(x^7) \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \mathcal{O}(x^7) - \frac{1}{3}\Big(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \mathcal{O}(x^7)\Big)^3 \\
&+ \frac{1}{5!}\Big(2 + \frac{3\cdot 5!}{3!\cdot 3!}\Big)\cdot\Big(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \mathcal{O}(x^7)\Big)^5 + \mathcal{O}(x^7) \\
&= x - \Big(\frac{1}{3!} + \frac{1}{3}\Big)x^3 + \Big(\frac{1}{5!} + \frac{3}{3\cdot 3!} + \frac{1}{5!}\Big(2 + \frac{3\cdot 5!}{3!\cdot 3!}\Big)\Big)x^5 + \mathcal{O}(x^7) \\
&= x\Big(1 - \frac{x^2}{2}\Big) + A\cdot x^5,
\end{aligned}$$

where

$$A = \frac{1}{5!} + \frac{3}{3\cdot 3!} + \frac{1}{5!}\Big(2 + \frac{3\cdot 5!}{3!\cdot 3!}\Big) = \frac{33}{120}.$$

Furthermore,

$$x\cos(x) = x\Big(1 - \frac{x^2}{2} + \frac{x^4}{4!} + \mathcal{O}(x^6)) = x\Big(1 - \frac{x^2}{2}) + \frac{x^5}{4!} + \mathcal{O}(x^7).$$

We conclude that

$$\sin(\sin(\sin(x))) - x\cos(x) = \Big(A - \frac{1}{4!}\Big)x^5 + \mathcal{O}(x^7),$$

and thus

$$\begin{aligned}
\frac{\sin(\sin(\sin(x))) - x\cos(x)}{(1-\cos(x))^2\tan(x)} &= \frac{\Big(A - \frac{1}{4!}\Big)x^5 + \mathcal{O}(x^7)}{\frac{x^5}{4} + \mathcal{O}(x^7)} \\
&= \frac{\Big(A - \frac{1}{4!}\Big) + \mathcal{O}(x^2)}{\frac{1}{4} + \mathcal{O}(x^2)}
\end{aligned}$$

Hence,

$$\lim_{x\to 0} \frac{\sin(\sin(\sin(x))) - x\cos(x)}{(1-\cos(x))^2 \tan(x)} = 4 \cdot \Big(A - \frac{1}{4!}\Big) = \boxed{\frac{14}{15}}.$$

Problem 4

a) Note that if $0 < x < 1$, then $f_x(t) = x^{t^p}$ is a decreasing function of $t \in [0, \infty)$, and thus

$$f_x(n) \geq \int_n^{n+1} f_x(t)\, dt \geq f_x(n+1),$$

for all integers $n \geq 0$. Hence, if we define

$$S(x) = \sum_{n=1}^{\infty} f_x(n),$$

then

$$S(x) \geq \sum_{n=1}^{\infty} \int_n^{n+1} f_x(t)\, dt = \int_1^\infty f_x(t)\, dt = \int_1^\infty e^{t^p \ln(x)}\, dt = \int_1^\infty e^{-((-\ln(x))^{1/p} t)^p}\, dt$$

$$= \frac{1}{(-\ln(x))^{1/p}} \int_{(-\ln(x))^{1/p}}^\infty e^{-t^p}\, dt = (1-x)^{-1/p} \Big(\frac{1-x}{-\ln(1-x)}\Big)^{1/p} \int_{(-\ln(x))^{1/p}}^\infty e^{-t^p}\, dt,$$

and, similarly,

$$S(x) = \sum_{n=0}^{\infty} f_x(n+1) \leq \sum_{n=0}^{\infty} \int_n^{n+1} f_x(t)\, dt = \int_0^\infty f_x(t)\, dt$$

$$= (1-x)^{-1/p} \Big(\frac{1-x}{-\ln(1-x)}\Big)^{1/p} \int_0^\infty e^{-t^p}\, dt.$$

We conclude that if

$$T(x) = (1-x)^{1/p} \sum_{n=0}^{\infty} x^{n^p} = (1-x)^{1/p} + (1-x)^{1/p} S(x), \quad 0 < x < 1,$$

then

$$(1-x)^{1/p} + \Big(\frac{1-x}{-\ln(1-x)}\Big)^{1/p} \int_{(-\ln(x))^{1/p}}^\infty e^{-t^p}\, dt \leq T(x)$$

and

$$T(x) \leq (1-x)^{1/p} + \Big(\frac{1-x}{-\ln(1-x)}\Big)^{1/p} \int_0^\infty e^{-t^p}\, dt.$$

Clearly the lower and upper bounds both converge to

$$\int_0^\infty e^{-t^p}\, dt, \quad \text{as } x \to 1^-,$$

which finishes the proof.

b) We write

$$f(x) = \sum_{n=0}^{\infty} 3^n x^{2^n} = \sum_{k=0}^{\infty} a_k x^k,$$

where

$$a_k = \begin{cases} 3^n & \text{if } k = 2^n \\ 0 & \text{otherwise} \end{cases},$$

and thus the radius of convergence equals

$$R = \frac{1}{\limsup_{k\to\infty} |a_k|^{1/k}} = \frac{1}{\limsup_{n\to\infty} |3^n|^{1/2^n}} = 1,$$

so $f(x)$ converges absolutely if $|x| < 1$ and diverges if $|x| > 1$. For $x = \pm 1$, we have

$$f(x) = \sum_{n=0}^{\infty} 3^n$$

which clearly diverges.

c) We recall that

$$f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}, \quad x \in [0,1],$$

which is clearly well-defined (convergent). Note that in particular, $f(0) = 0$. By geometric summation, for $0 < x \leq 1$,

$$f(x) = x^2 \cdot \sum_{n=0}^{\infty} (1+x^2)^{-n} = \frac{x^2}{1 - 1/(1+x^2)} = 1 + x^2.$$

Hence,

$$\lim_{x\to 0^+} f(x) = 1 \neq f(0) = 0,$$

so f is not continuous at $x = 0$. If the convergence were indeed uniform on $[0,1]$, the limit would be continuous, so we reach a contradiction (so the convergence is *not* uniform) on $[0,1]$).