Tentamen i Matematisk analys, fortsättningskurs F/TM, TMA976
4 april 2024 kl. 8.30-12.30

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OBS: Motivera dina svar väl. Det är i huvudsak beräkningarna och motiveringarna som ger poäng, inte svaret. SKRIV TYDLIGT OCH REDOVISA VILKA RESULTAT FRÅN KURSEN SOM DU ANVÄNDER.

Totalt 50 poäng, fördelade på 5 uppgifter (2 sidor). Inga hjälpmedel!

Betygsgränser: 3 (20-34 poäng), 4 (35-43 poäng), 5 (44-50 poäng).

1. Låt $y : \mathbb{R} \to \mathbb{R}$ beteckna lösningen till initialvärdesproblemet

$$y(0) = 0, \;\; y'(0) = 1, \quad y''(x) + 2y'(x) + y(x) = (1+x)e^{-x}, \quad x \in \mathbb{R}.$$

Lös ekvationen $y(x) = 0$, $x \in \mathbb{R}$. (8p)

2. Bestäm lösningen y, med tillhörande maximala lösningsintervall (innehållande startpunkten $x = 0$), för det separabla initalvärdesproblemet

$$y(0) = 0, \quad (1 + x^2)y'(x) = (1 + y^2)x.$$

(7p)

3. Beräkna gränsvardena:

a)

$$I_a := \lim_{x\to 0} \frac{2\sin(\sin^2(x)) - (1 - \cos(2x))}{(\ln(1+\sqrt{x}) - \sqrt{x})^6}$$

(8p)

b)

$$I_b := \lim_{n\to\infty} \frac{e^{-n}}{n} \cdot \sum_{k=1}^{n^2} e^{\sqrt{k}}.$$

(8p)

c)

$$I_c := \lim_{(x,y)\to(0,0)} \frac{x^4 \sin(xy)}{x^4 + y^4}$$

(4p)

4. För vilka $x \in \mathbb{R}$ konvergerar potensserien

$$\sum_{n=2}^{\infty} \tan\left(\pi\left(\frac{1}{2} - \frac{1}{n}\right)\right) x^{n^2} \ ?$$

För varje sådant x, ange även om konvergensen är absolut eller betingad. (5p)

5. Givet en 1-periodisk kontinuerlig function $f : \mathbb{R} \to \mathbb{C}$, definiera dess Fourier-transform $\widehat{f} : \mathbb{Z} \to \mathbb{C}$ enligt

$$\widehat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} \, dx, \quad n \in \mathbb{Z}.$$

a) Visa att om f är två gånger kontinuerligt deriverbar, så gäller

$$|\widehat{f}(n)| \leq \frac{\max_{x\in[0,1]} |f''(x)|}{4\pi^2 n^2}, \quad \text{för alla } n \neq 0.$$

LEDTRÅD: Partialintegrera två gånger. (4p)

b) Visa att om f är två gånger kontinuerligt deriverbar så konvergerar serien

$$g(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} \widehat{f}(n) e^{2\pi i n x}$$

likformigt på $\mathbb{R}$, och $g : \mathbb{R} \to \mathbb{C}$ är en kontinuerlig 1-periodisk funktion som uppfyller $\widehat{g}(n) = \widehat{f}(n)$ för alla $n \in \mathbb{Z}$.

DU FÅR ANVÄNDA DIG AV RESULTATET I A) ÄVEN OM DU INTE HAR BEVISAT DET. (6p)

SOLUTIONS EXAM 2, 2023/24

Problem 1

The characteristic equation $r^2+2r+1 = (r+1)^2 = 0$ has a double root $r_1 = r_2 = -1$. We make the Ansatz:

$$y_p(x) = u(x)e^{-x}, \quad \text{for some function } u : \mathbb{R} \to \mathbb{R}.$$

Note that

$$y''(x) + 2y'(x) + y(x) = u''(x)e^{-x} = (1+x)e^{-x},$$

and thus $u''(x) = 1 + x$. We conclude that

$$u(x) = A + Bx + \frac{x^2}{2} + \frac{x^3}{6},$$

for some constants A and B. Since $y(0) = 0$, we have $u(0) = 0$, and thus $A = 0$. Furthermore,

$$y'(x) = (u'(x) - u(x))e^{-x} = \Big(B + x + \frac{x^2}{2} - Bx + \frac{x^2}{2} + \frac{x^3}{6}\Big)e^{-x},$$

so $y'(0) = B = 1$. Hence,

$$y(x) = \frac{x}{6}\Big(6 + 3x + x^2\Big)e^{-x}.$$

Note that the only real solution to $y(x) = 0$ is $x = 0$.

Problem 2

Integrating both sides of the equation

$$\frac{y'(x)}{1+y^2(x)} = \frac{x}{1+x^2}$$

gives us that

$$\arctan(y(x)) = C + \frac{1}{2}\ln(1+x^2), \quad \text{for some constant } C.$$

Since $y(0) = 0$, we have $C = 0$, and thus

$$y(x) = \tan\Big(\frac{1}{2}\ln(1+x^2)\Big),$$

which is well-defined for all $-\infty < x < (e^{\pi} - 1)^{1/2}$.

Problem 3

a) Note that $(1-\cos(2x)) = 2\sin^2(x)$, so if we set $y(x) = \sin^2(x)$, then

$$\lim_{x\to 0} \frac{\sin(y(x)) - y(x)}{y(x)^3} = -1/6 \quad \text{and} \quad \lim_{x\to 0} \frac{y(x)^3}{x^6} = 1,$$

and

$$\frac{2\sin(\sin^2(x)) - (1-\cos(2x))}{(\ln(1+\sqrt{x}) - \sqrt{x})^6} = \frac{2(\sin(y(x)) - y(x))}{y(x)^3} \cdot \frac{y(x)^3}{x^6} \cdot \frac{x^6}{(\ln(1+\sqrt{x}) - \sqrt{x})^6},$$

whence

$$\lim_{x\to 0} \frac{2\sin(\sin^2(x)) - (1-\cos(2x))}{(\ln(1+\sqrt{x}) - \sqrt{x})^2} = -\frac{1}{3} \cdot \Big(\lim_{x\to 0} \frac{x}{\ln(1+\sqrt{x}) - \sqrt{x}}\Big)^6.$$

Now,

$$\ln(1+\sqrt{x}) = \sqrt{x} - \frac{x}{2} + \mathscr{O}(x^{3/2}),$$

and thus

$$\lim_{x\to 0} \frac{x}{\ln(1+\sqrt{x}) - \sqrt{x}} = -2,$$

and

$$\boxed{\lim_{x\to 0} \frac{2\sin(\sin^2(x)) - (1-\cos(2x))}{(\ln(1+\sqrt{x}) - \sqrt{x})^6} = -\frac{(-2)^6}{3} = -\frac{64}{3}}.$$

b) The function $f(x) = e^{\sqrt{x}}$ is an increasing function, so the sequence

$$s_n = \sum_{k=1}^{n} f(k)$$

satisfies

$$s_n \leq \int_1^{n+1} f(x)\,dx \leq s_{n+1} - e, \quad \text{for all } n \geq 1.$$

Since

$$\int_1^{n+1} e^{\sqrt{x}}\,dx = 2\int_1^{\sqrt{n+1}} ue^u\,du = 2\Big(\sqrt{n+1}e^{\sqrt{n+1}} - e - (e^{\sqrt{n+1}} - 1)\Big),$$

we see that

$$s_{n^2} \leq 2\Big(\sqrt{n^2+1}e^{\sqrt{n^2+1}} - e - e^{\sqrt{n^2+1}} + 1\Big) \leq s_{n^2+1} - e = s_{n^2} + e^{\sqrt{n^2+1}} - e,$$

for all n. Note that since

$$\sqrt{n^2+1} - n = n\Big(\sqrt{1+\frac{1}{n^2}} - 1\Big) \to 0, \quad n\to\infty,$$

we have

$$\lim_{n\to\infty} \frac{e^{-n}}{n}\sqrt{n^2+1}e^{\sqrt{n^2+1}} = 1,$$

and thus

$$\boxed{\lim_{n\to\infty} \frac{e^{-n}}{n} \cdot s_{n^2} = 2}.$$

c) Note that

$$\frac{x^4}{x^4+y^4} \quad \text{for all } (x,y) \neq (0,0),$$

and thus

$$\Big|\frac{x^4 \sin(xy)}{x^4+y^4}\Big| \leq |\sin(xy)| \to 0, \quad \text{as } (x,y) \to (0,0).$$

Problem 4

Note that

$$\sum_{n=2}^{\infty} \tan\left(\pi\Big(\frac{1}{2} - \frac{1}{n}\Big)\right) x^{n^2} = \sum_{n=0}^{\infty} a_m x^m,$$

where

$$a_m = \begin{cases} 0 & \text{if } m = 0, 1 \text{ or } m \neq n^2 \text{ for some } n \geq 2 \\ \tan\left(\pi\Big(\frac{1}{2} - \frac{1}{n}\Big)\right) & \text{if } m = n^2 \text{ for some } n \geq 2. \end{cases}$$

Since

$$\tan\left(\pi\Big(\frac{1}{2} - \frac{1}{n}\Big)\right) \to \infty, \quad n \to \infty,$$

the radius of convergence is at most 1 and the series diverges at $x = \pm 1$, but since

$$\Big|\tan\left(\pi\Big(\frac{1}{2} - \frac{1}{n}\Big)\right)\Big| \leq \frac{1}{\Big|\cos\left(\pi\Big(\frac{1}{2} - \frac{1}{n}\Big)\right)\Big|} = \frac{1}{\sin(\pi/n)} \leq \frac{2n}{\pi},$$

we have

$$\overline{\lim_{n\to\infty}} |a_m|^{1/m} \leq 1,$$

and thus the radius of convergence is in fact equal to one (so for $|x| < 1$, the series converges absolutely).

Problem 5

a) Note that for $n \neq 0$,

$$\begin{aligned}\widehat{f}(n) &= \int_0^1 f(x)e^{-2\pi inx}\,dx = \Big[\frac{f(x)e^{-2\pi inx}}{-2\pi in}\Big]_0^1 + \frac{1}{2\pi in}\int_0^1 f'(x)e^{-2\pi inx}\,dx \\ &= \frac{1}{2\pi in}\Big(\Big[\frac{f'(x)e^{-2\pi inx}}{-2\pi in}\Big]_0^1 + \frac{1}{2\pi in}\int_0^1 f''(x)e^{-2\pi inx}\,dx\Big) \\ &= -\frac{1}{4\pi^2n^2}\int_0^1 f''(x)e^{-2\pi inx}\,dx,\end{aligned}$$

and thus,

$$|\widehat{f}(n)| \leq \frac{\max_{x\in[0,1]}|f''(x)|}{4\pi^2n^2}, \quad \text{för alla } n \neq 0.$$

b) It follows from a) that $\sum_{n=-\infty}^{\infty}|\widehat{f}(n)| < \infty$, so by Weierstrass M-test, the function series

$$g(x) = \lim_{N\to\infty}\sum_{n=-N}^{N}\widehat{f}(n)e^{2\pi inx}$$

converges uniformly on $\mathbb{R}$ (hence g is continuous on $\mathbb{R}$). In particular, we can exchange the order of limit and integration, so

$$\widehat{g}(m) = \int_0^1 g(x)e^{-2\pi imx}\,dx = \lim_{N\to\infty}\sum_{n=-N}^{N}\widehat{f}(n)\underbrace{\int_0^1 e^{2\pi i(n-m)x}\,dx}_{=0 \text{ if } m\neq n} = \widehat{f}(m),$$

for all m.
