Tentamen i Matematisk analys, fortsättningskurs F/TM, TMA976
12 januari 2024 kl. 14.00-18.00

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OBS: Motivera dina svar väl. Det är i huvudsak beräkningarna och motiveringarna som ger poäng, inte svaret. SKRIV TYDLIGT OCH REDOVISA VILKA RESULTAT FRÅN KURSEN SOM DU ANVÄNDER.

Totalt 50 poäng, fördelade på 5 uppgifter (2 sidor). Inga hjälpmedel!

Betygsgränser: 3 (20-34 poäng), 4 (35-43 poäng), 5 (44-50 poäng).

1. Givet ett reellt tal $p > 0$, låt $y_p : [0, \infty) \to \mathbb{R}$ beteckna lösningen till initialvärdesproblemet

$$y_p''(x) - 2y_p'(x) + y_p(x) = x^p\, e^x, \quad y_p(0) = y_p'(0) = 0.$$

Definiera $z_p(x) = p^2\, y_p(x)$ för $x \in [0, 1]$.

a) Beräkna gränsvärdet $z_\infty(x) := \lim_{p\to\infty} z_p(x)$ för alla $x \in [0, 1]$. (7p)

b) Konvergerar (z_p) likformigt mot z_∞ på $[0, 1]$? . (3p)

2. Bestäm alla reella a för vilka gränsvärdet

$$I(a) := \lim_{x\to 0} \frac{\sin(x + \ln(1 + x)) - 2x(1 + \sin(ax))}{\arctan(x) - x}$$

existerar och beräkna $I(a)$ för dessa a. Standardutvecklingar för sin, arctan och ln behöver ej härledas.

(10p)

3. a) Avgör för vilka reella x som serien

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(\pi(x + k))}{\ln(1 + k)} x^k$$

konvergerar (och ange huruvida konvergensen är absolut eller betingad).

(7p)

b) Existerar gränsvärdet

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+2xy+y^2)}{x^2+y^2}?$$

(3p)

4. Beräkna gränsvärdet

$$I = \lim_{n\to\infty} \frac{1}{(\ln n)^2} \sum_{k=1}^{n} \frac{\ln(1+k^2)}{k}.$$

(10p)

5. Givet ett reellt tal $L > 0$, låt (f_n) beteckna en följd av reellvärda funktioner på $[0,1]$ som uppfyller

$$|f_n(x) - f_n(y)| \leq L|x-y|, \quad \text{för alla } x, y \in [0,1] \text{ och för alla } n. \qquad (\bullet)$$

Antag att gränsvärdet

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \text{existerar för alla } x \in [0,1].$$

Målet är att visa att (f_n) konvergerar likformigt mot f på $[0,1]$, d.v.s. att det existerar, för varje $\epsilon > 0$, ett heltal n_ϵ sådant att

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| < \epsilon, \quad \text{för alla } n \geq n_\epsilon. \qquad (\bullet\bullet)$$

Låt oss hädanefter fixera $\epsilon > 0$ och ett heltal N sådant att $L/N < \epsilon/3$.

a) Visa att f uppfyller (1p)

$$|f(x) - f(y)| \leq L|x-y|, \quad \text{för alla } x, y \in [0,1].$$

b) Visa att det existerar ett heltal n_ϵ sådant att, för alla $n \geq n_\epsilon$, (3p)

$$|f_n(m/N) - f(m/N)| < \epsilon/3, \quad \text{för alla } m = 0, 1, \ldots, N.$$

c) Antag att $x \in [m/N, (m+1)/N)$ för något $m = 0, 1 \ldots, N-1$. Visa att $(\bullet)$ och a) medför att (4p)

$$\max(|f_n(x) - f_n(m/N)|, |f(x) - f(m/N)|) < \epsilon/3.$$

d) Visa nu att b) och c) medför $(\bullet\bullet)$. (2p)

I c): du får använda dig av a) även om du inte har löst denna uppgift.
I d): du får använda dig av b) och c) även om du inte har löst dessa uppgifter.

SOLUTIONS EXAM 1, 2023/24

Problem 1

a) The characteristic equation $r^2 - 2r + 1 = (r-1)^2 = 0$ has a double root $r_1 = r_2 = 1$. Since $y_p(0) = y_p'(0) = 0$, we have $y_{p,\text{hom}} = 0$, and thus

$$y_p(x) = e^x \int_0^x \left(\int_0^t \tau^p \, d\tau \right) dt = e^x \int_0^x \frac{t^{p+1}}{p+1} \, dt = \frac{x^{p+2}}{(p+1)(p+2)} e^x,$$

for all $x \geq 0$. Hence,

$$z_p(x) = \frac{p^2}{(p+1)(p+2)} x^{p+2} e^x, \quad \text{for all } x \in [0,1],$$

so we conclude that

$$\boxed{z_\infty(x) = \lim_{p\to\infty} z_p(x) = \left\{ \begin{array}{ll} 0 & \text{for all } x \in [0,1) \\ e & \text{for } x = 1 \end{array} \right. .}$$

b) Since each z_p is continuous on $[0,1]$, but z_∞ fails to be continuous, we conclude that the convergence cannot be uniform.

Problem 2

We begin by analyzing the denominator. Recall that

$$\arctan(x) = x - x^3/3 + \mathcal{O}(x^5),$$

and thus $\arctan(x) - x = -x^3/3 + \mathcal{O}(x^5)$, so we need to choose a so that

$$\sin(x + \ln(1+x)) - 2x(1 + \sin(ax)) = Ax^3 + \mathcal{O}(x^4),$$

for some constant $A = A(a)$. If this is the case, then we clearly have

$$\lim_{x\to 0} \frac{\sin(x + \ln(1+x)) - 2x(1+\sin(ax))}{\arctan(x) - x} = -3A.$$

To do this, first note that

$$y = x + \ln(1+x) = x + (x - x^2/2 + x^3/3 + \mathcal{O}(x^4)) = 2x - x^2/2 + x^3/3 + \mathcal{O}(x^4),$$

and thus

$$\begin{aligned}\sin(y) &= y - y^3/3! + \mathcal{O}(y^5) = 2x - x^2/2 + x^3/3 + \mathcal{O}(x^4)\\ &- \frac{1}{6}(2x - x^2/2 + x^3/3 + \mathcal{O}(x^4))^3 + \mathcal{O}(x^5)\\ &= 2x - x^2/2 + x^3/3 - \frac{1}{6}(8x^3 + \mathcal{O}(x^4)) + \mathcal{O}(x^5)\\ &= 2x - x^2/2 - x^3 + \mathcal{O}(x^4).\end{aligned}$$

Furthermore,

$$2x(1+\sin(ax)) = 2x(1 + ax + \mathcal{O}(x^3)) = 2x + 2ax^2 + \mathcal{O}(x^4),$$

and thus

$$\begin{aligned}\sin(x+\ln(1+x)) - 2x(1+\sin(ax)) &= 2x - x^2/2 - x^3 + \mathcal{O}(x^4) - (2x + 2ax^2 + \mathcal{O}(x^4))\\ &= -(2a+1/2)x^2 - x^3 + \mathcal{O}(x^4).\end{aligned}$$

We thus see that we must choose $a = -1/4$ to kill the x^2-term, in which case $A = -1$. We conclude that

$$\boxed{a = -1/4 \quad \text{and} \quad \lim_{x\to 0} \frac{\sin(x+\ln(1+x)) - 2x(1-\sin(x/4))}{\arctan(x) - x} = 3}.$$

Problem 3

a) Since $\sin(\pi(x+k)) = \sin(\pi x)\cos(\pi k) = (-1)^k \sin(\pi x)$ for all k and x, we have

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(\pi(x+k))}{\ln(1+k)} x^k = \sin(\pi x) \sum_{k=1}^{\infty} \frac{(-x)^k}{\ln(1+k)} = \sin(\pi x) \sum_{k=0}^{\infty} a_k x^k,$$

where $a_0 = 0$ and

$$a_k = \frac{(-1)^k}{\ln(1+k)}, \quad k \geq 1.$$

Since $\limsup_{k\to\infty} |a_k|^{1/k} = 1$, we see that the radius of convergence equals 1, and thus the series $f(x)$ converges absolutely for all $|x| < 1$ and diverges for all $|x| > 1$ (as long as x is not an integer). Indeed, note for any integer m, we have

$$f(m) = \sum_{k=1}^{\infty} \underbrace{\frac{\sin(\pi(m+k))}{\ln(1+k)}}_{=0} m^k = 0,$$

which clearly converges absolutely (each term equals 0). We conclude that

$$\boxed{f(x) \begin{cases} \text{converges absolutely for all } |x| \leq 1 \text{ and for all } x \in \mathbb{Z} \\ \text{diverges for all non-integer } |x| > 1. \end{cases}}$$

b) Define
$$f(x,y) = \frac{\sin(x^2+2xy+y^2)}{x^2+y^2}, \quad (x,y) \neq (0,0).$$
We first consider limits along $y=0$. Then,
$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{\sin(x^2)}{x^2} = 1.$$
If we instead consider limits along $y=-x$, then
$$\lim_{x\to 0} f(x,-x) = \lim_{x\to 0} 0 = 0,$$
so the (two-variable) limit does not exist.

Problem 4

Let
$$f(x) = \frac{\ln(1+x^2)}{x}, \quad x \geq 1.$$
Note that
$$f'(x) = \frac{\frac{2x^2}{1+x^2} - \ln(1+x^2)}{x^2} = \frac{2x^2 - (1+x^2)\ln(1+x^2)}{(1+x^2)x^2}$$
and
$$\frac{2x^2}{1+x^2} < \ln(1+x^2), \quad \text{for all sufficiently large } x.$$
Indeed, the left-hand side tends to 2 as $x \to \infty$, while the right-hand side tends to infinity. We conclude that there exists some integer k_o such that
$$f'(x) < 0, \quad \text{for all } x \geq k_o,$$
and thus f is strictly decreasing on $[k_o, \infty)$. Let us now write
$$s_n := \sum_{k=1}^{n} f(k) = \underbrace{\sum_{k=1}^{k_o-1} f(k)}_{=:A} + \underbrace{\sum_{k=k_o}^{n} f(k)}_{=:B_n},$$
for $n \geq k_o$. We use that f is decreasing on $[k_o, \infty)$, and thus
$$f(k) \geq f(x) \geq f(k+1), \quad \text{for all } k \leq x \leq k+1,$$
whence
$$f(k) \geq \int_k^{k+1} f(x)\,dx \geq f(k+1), \quad \text{for all } k \geq k_o.$$
Hence,
$$B_n \geq \sum_{k=k_o}^{n} \int_k^{k+1} f(x)\,dx = \int_{k_o}^{n+1} f(x)\,dx \geq \sum_{k=k_o}^{n} f(k+1) = B_n - f(k_o) + f(n+1), \tag{0.1}$$

for all $n \geq k_o$. Note that

$$\begin{aligned}\int_{k_o}^{n+1} f(x)\,dx &= \int_{k_o}^{n+1} \frac{\ln(1+x^2)}{x}\,dx \\ &= \left\{u = \ln(1+x^2),\;\; x = (e^u-1)^{1/2},\;\; dx = \frac{e^u\,du}{2(e^u-1)^{1/2}}\right\} \\ &= \frac{1}{2}\int_{\ln(1+k_o^2)}^{\ln(1+(n+1)^2)} \frac{ue^u}{e^u-1}\,du \\ &= \frac{1}{2}\int_{\ln(1+k_o^2)}^{\ln(1+(n+1)^2)} u\,du + \frac{1}{2}\int_{\ln(1+k_o^2)}^{\ln(1+(n+1)^2)} \frac{u}{e^u-1}\,du \\ &= \frac{1}{4}(\ln^2(1+(n+1)^2) - \ln^2(1+k_o^2)) + c_n,\end{aligned}$$

where the sequence (c_n) is given by

$$c_n = \frac{1}{2}\int_{\ln(1+k_o^2)}^{\ln(1+(n+1)^2)} \frac{u}{e^u-1}\,du.$$

Since $u \mapsto \frac{u}{e^u-1}$ is integrable, we see that (c_n) is bounded. Hence,

$$\lim_{n\to\infty} \frac{1}{(\ln n)^2}\int_{k_o}^{n+1} f(x)\,dx = 1.$$

Since

$$\lim_{n\to\infty} \frac{f(n+1)}{(\ln n)^2} = 0,$$

we conclude from (4.1) that

$$\lim_{n\to\infty} \frac{B_n}{(\ln n)^2} = 1,$$

and thus

$$\boxed{\lim_{n\to\infty} \frac{s_n}{(\ln n)^2} = \lim_{n\to\infty} \frac{B_n}{(\ln n)^2} = 1}.$$

Problem 5

a) Fix $x, y \in [0,1]$. Since (f_n) converges pointwise to f, we have

$$|f(x) - f(y)| = \lim_{n\to\infty} |f_n(x) - f_n(y)|.$$

Since $|f_n(x) - f_n(y)| \leq L|x-y|$ for all n, we conclude that

$$|f(x) - f(y)| \leq L|x-y|.$$

Since x, y are arbitrary, we are done.

b) Since (f_n) converges pointwise to f, we can, for every $m = 0, 1, \dots, N$, find $n_\varepsilon^{(m)}$ such that

$$|f_n(m/N) - f(m/N)| < \varepsilon/3, \quad \text{for all } n \geq n_\varepsilon^{(m)}.$$

Define $n_\varepsilon = \max(n_\varepsilon^{(0)}, n_\varepsilon^{(1)}, \dots, n_\varepsilon^{(N)})$. Then,

$$|f_n(m/N) - f(m/N)| < \varepsilon/3, \quad \text{for all } m = 0, 1 \dots, N,$$

for all $n \geq n_\varepsilon$.

c) Since $x \in [m/N, (m+1)/N]$, we have $|x - m/N| \leq 1/N$, and thus

$$|f_n(x) - f_n(m/N)| \leq L|x - m/N| < L/N < \varepsilon/3,$$

and, by a),

$$|f(x) - f(m/N)| \leq L|x - m/N| < L/N < \varepsilon/3,$$

d) Note that

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \max_{m=0,1,\dots,N} \left(\sup_{x\in[m/N,(m+1)/N]} |f_n(x) - f(x)| \right),$$

and, by b) and c), for all $x \in [m/N, (m+1)/N]$ and $n \geq n_\varepsilon$,

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_n(m/N)| + |f_n(m/N) - f(m/N)| + |f(m/N) - f(x)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

This shows that

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| < \varepsilon, \quad \text{for all } n \geq n_\varepsilon.$$