Chalmers/GU
Mathematics sciences

EXAM

TMA947/MMG621
NONLINEAR OPTIMISATION

Date: 25–01–08
Time: 8^{30}–13^{30}
Aids: Chalmers approved calculator
Number of questions: 7; a passed question requires 2 points of 3.
Questions are *not* numbered by difficulty.
To pass requires 10 points and three passed questions.

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Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

(3p) **Question 1**

(Unconstrained optimization - Newton's method with Levenberg-Marquardt modification)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x) = \frac{1}{6}x_1^3 + \frac{1}{6}x_2^3 - 2x_1 - x_2 - x_1x_2$ and consider the unconstrained optimization problem

$$\text{minimize } f(x).$$

Starting in the point $x^0 = [0, 0]^T$, perform one iteration of Newton's method with the Levenberg-Marquardt modification. That is, compute the next point x^1.

When performing the iteration, you have to:

- use, as modification parameter, the smallest integer $\gamma \geq 0$ such that the conditions needed are fulfilled.
- use exact line search when computing the step length α.

Hint 1: $\frac{1}{6}\left(\frac{5}{3}\right)^3 + \frac{1}{6}\left(\frac{4}{3}\right)^3 = \frac{7}{6}$.

Hint 2: For $g : \mathbb{R} \to \mathbb{R}$, given by $g(z) = \frac{7}{6}z^3 - \frac{20}{9}z^2 - \frac{14}{3}z$, the optimal solution to $\min\ g(z)$ subject to $z \geq 0$ is $z = \frac{40+2\sqrt{1723}}{63}$. You can use this result directly without proving it.

Hint 3: You may find the following identity useful:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Question 2

(Linear programming)

(2p) a) Consider the linear programming problem

$$\begin{aligned} \text{minimize} \quad & 2x_1 + x_2 - x_3 + 3x_4 \\ \text{subject to} \quad & x_2 - 2x_3 + 4x_4 \leq 1 \\ & -x_1 + x_2 + 2x_3 \leq 1 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

Solve the problem using the Simplex method. Start with x_2 and x_3 as basic variables.

(1p) b) If you find that an optimal solution exists, then use your calculations to decide whether it is unique or not. If the problem is unbounded, then use your calculations to specify a ray of unboundedness of the objective value.

Question 3

(global convergence of exterior penalty method)

Consider the problem

$$(\mathrm{P}) \qquad \begin{cases} \text{minimize} & f(x), \\ \text{subject to} & g_i(x) \leq 0, \quad i = 1, \dots, m, \\ & h_j(x) = 0, \quad j = 1, \dots, \ell, \\ & x \in \mathbb{R}^n, \end{cases}$$

and let

$$S = \{ x \in \mathbb{R}^n \mid g_i(x) \leq 0, \text{ for } i = 1, \dots, m, \text{ and } h_j(x) = 0, \text{ for } j = 1, \dots, \ell \}.$$

Also consider the transformed problem

$$(\mathrm{P}_\nu) \qquad \begin{cases} \text{minimize} & f(x) + \nu \check{\chi}_S(x), \\ \text{subject to} & x \in \mathbb{R}^n, \end{cases}$$

where

$$\check{\chi}_S(x) = \sum_{i=1}^{m} \psi(\max\{0, g_i(x)\}) + \sum_{j=1}^{\ell} \psi(h_j(x)).$$

(1p) a) Define $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$. What condition must the function $\psi : \mathbb{R} \to \mathbb{R}_+$ satisfy for us to call (P_ν) an exterior penalty transformation of (P)?

(2p) b) A function $\psi : \mathbb{R} \to \mathbb{R}_+$ that fulfills the conditions asked for in part a) is called an exterior penalty function.

Prove the following theorem. Do so using basic results from the course. If you rely on other results when performing your proof of the theorem, then those results must be stated explicitly; they may however be utilized without proof.

THEOREM: *Let ψ be an exterior penalty function, and assume that* (P) *has at least one globally optimal solution. For each value of ν, let x^*_ν be a globally optimal solution to* (P_ν)*. Then every limit point of the sequence $\{x^*_\nu\}$, $\nu \to \infty$, is a globally optimal solution to* (P)*.*

Hint: The following result might be useful. You may use it without proving it.

LEMMA: *Let $x^*_{\nu_1}$ and $x^*_{\nu_2}$ be globally optimal to* (P_ν) *for penalty parameters ν_1 and ν_2, respectively. If $\nu_1 \leq \nu_2$, then $f(x^*_{\nu_1}) \leq f(x^*_{\nu_2})$.*

Question 4

(True or False)

The below three claims should be assessed. For each claim: Clearly state whether it is true or false. Provide an answer together with a short (but complete) motivation.

(1p) a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x) = x_1 x_2 - e^{x_2}.$$

Claim: The point $([3,\, 1]^T,\, 0)$ belongs to the epigraph of f, that is, $([3,\, 1]^T,\, 0) \in \operatorname{epi} f \subset \mathbb{R}^2 \times \mathbb{R}$.

(1p) b) Let $g_1 : \mathbb{R}^2 \to \mathbb{R}$ and $g_2 : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$g_1(x) = x_1^2 + x_2^2,$$
$$g_2(x) = -(x_1 - 2)^2 - (x_2 - 2)^2 + 1,$$

respectively, and let $S = \{x \in \mathbb{R}^2 \mid g_i(x) \leq 0 \text{ for } i = 1, 2\}$.

Claim: The set S is convex.

(1p) c) Consider the problem

$$\text{(P)} \qquad \begin{cases} \text{minimize} & f(x), \\ \text{subject to} & g_i(x) \leq 0, \quad i = 1, \ldots, m, \end{cases}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ are all convex functions. Assume that Slater's constraint qualifier holds for (P), and that x^* is a KKT point. That means that x^* is a globally optimal solution to (P).

Claim: If $g_k(x^*) = 0$ (that is, constraint k is active), but the corresponding multiplier in the KKT system is equal to zero (that is, $\mu_k = 0$), then x^* is also globally optimal to the problem

$$\text{(P')} \qquad \begin{cases} \text{minimize} & f(x), \\ \text{subject to} & g_i(x) \leq 0, \quad i = 1, \ldots, k-1, k+1, \ldots, m. \end{cases}$$

Question 5

(Lagrangian duality and global optimality conditions)

Consider the problem

$$\begin{aligned} \text{minimize } & x_1^3 + x_2, \\ \text{subject to } & x_1^2 - x_2 \leq 0, \\ & x_1 \geq 1, \\ & 0 \leq x_2 \leq 10. \end{aligned}$$

(2p) a) Perform a Lagrange relaxation of the first constraint, that is, of the constraint $x_1^2 - x_2 \leq 0$, with a multiplier $\mu \geq 0$, and derive the dual function $q(\mu)$.

Hint: For $X := \{x \in \mathbb{R}^2 \mid x_1 \geq 1,\ 0 \leq x_2 \leq 10\}$, an explicit solution to $\min_{x \in X} \mathscr{L}(x, \mu)$ can be found for each μ in this case.

(1p) b) Using the computations in a), with $X := \{x \in \mathbb{R}^2 \mid x_1 \geq 1,\ 0 \leq x_2 \leq 10\}$, verify that $\hat{x} = [1, 1]^T$ and $\hat{\mu} = 1$ satisfy the global optimality conditions.

(3p) Question 6

(Feasible directions and constraint qualifiers)

Let $g_i : \mathbb{R}^2 \to \mathbb{R}$, for $i = 1, 2$, be defined by

$$\begin{aligned} g_1(x) &= -x_1^2 + x_2, \\ g_2(x) &= -x_1^2 - x_2, \end{aligned}$$

respectively, and let $S = \{x \in \mathbb{R}^2 \mid g_i(x) \leq 0 \text{ for } i = 1, 2\}$. Moreover, let $\tilde{x} = [0, 0]^T$. In this point compute the cone of feasible directions (denoted $R_S(\tilde{x})$), the tangent cone ($T_S(\tilde{x})$), the inner gradient cone ($\mathring{G}(\tilde{x})$), and the gradient cone ($G(\tilde{x})$). Does Abadie's constraint qualifier hold in $\tilde{x}$?

Hint: The definitions of some of the cones are as follows.

$$\begin{aligned} R_S(\tilde{x}) :=& \{p \in \mathbb{R}^2 \setminus \{0\} \mid \exists\, \bar{\alpha} > 0 \text{ such that } x + \alpha p \in S\ \forall\, \alpha \in [0, \bar{\alpha}]\}, \\ T_S(\tilde{x}) :=& \{p \in \mathbb{R}^2 \mid \exists \text{ sequences } \{x^{(k)}\}_{k=1}^{\infty} \subset S \text{ and } \{\lambda^{(k)}\}_{k=1}^{\infty} \subset (0, \infty) \text{ such that} \\ & \lim_{k \to \infty} x^{(k)} = \tilde{x} \text{ and } \lim_{k \to \infty} \lambda^{(k)}(x^{(k)} - \tilde{x}) = p\} \\ \mathring{G}(\tilde{x}) :=& \{p \in \mathbb{R}^2 \mid \nabla g_i(\tilde{x})^T p < 0 \text{ for all } i \text{ such that } g_i(\tilde{x}) = 0\}. \end{aligned}$$

Moreover, it always holds that $\text{cl}(R_S(\tilde{x})) \subset T_S(\tilde{x})$ where cl denotes the closure of a set.

Question 7

(Choice of algorithm)

In each of the following questions, an optimization problem and a list of algorithms are given. For each question, state which of the algorithms (as they are taught in the course) **can *always* be used** to solve the type of optimization problem stated. If none of the algorithms can be used, state this clearly.

(1p) a)

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0, \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.
Algorithms:

- Exterior penalty method,
- Newton's method,
- Simplex method.

(1p) b)

$$\begin{aligned} \text{minimize} \quad & x^T Q x + c^T x \\ \text{subject to} \quad & Ax \leq b, \end{aligned}$$

where $Q \in \mathbb{R}^{n \times n}$, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, and where A and b are such that Slater's constraint qualifier holds for the problem.
Algorithms:

- Interior penalty method,
- Simplex method,
- Steepest descent method.

(1p) c)

$$\begin{aligned} \text{maximize} \quad & q(\mu) \\ \text{subject to} \quad & \mu \geq 0, \end{aligned}$$

where $q(\mu)$ is the Lagrangian dual problem of some nonlinear optimization problem minimize $f(x)$ subject to $g_i(x) \leq 0$, for $i = 1, \ldots, m$, and $x \in X$, obtained by relaxing the constraints $g_i(x) \leq 0$, for $i = 1, \ldots, m$.
Algorithms:

- Gradient projection algorithm,
- Frank-Wolfe method,
- Interior penalty method.

Hint: See Question 5a) on this exam.

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Mathematics

EXAM SOLUTION

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NONLINEAR OPTIMISATION

Date: 25–01–08
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Note: the solutions presented here are brief in relation to the requirements on your answers, in particular regarding your motivations.

(3p) Question 1

(Unconstrained optimization - Newton's method with Levenberg-Marquardt modification)

We have that

$$\nabla f(x) = \begin{bmatrix} \frac{1}{2}x_1^2 - 2 - x_2 \\ \frac{1}{2}x_2^2 - 1 - x_1 \end{bmatrix}, \qquad \nabla^2 f(x) = \begin{bmatrix} x_1 & -1 \\ -1 & x_2 \end{bmatrix}$$

and therefore

$$\nabla f(x^0) = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \qquad \nabla^2 f(x^0) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

The eigenvalues of $\nabla^2 f(x^0)$ are $\lambda = \pm 1$, and therefore $\gamma = 2$ is the sought modification parameter, i.e., the smallest integer such that $\nabla^2 f(x^0) + \gamma I \succ 0$. The search direction is found by solving $(\nabla^2 f(x^0) + 2I)p = -\nabla f(x^0)$, which gives that

$$p = -\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

To compute the step length α using exact line search, we have to solve the problem

$$\begin{aligned} \min_{\alpha \geq 0} \ f(x^0 + \alpha p) &= \min_{\alpha \geq 0} \ \frac{1}{6}\left(\frac{5}{3}\alpha\right)^3 + \frac{1}{6}\left(\frac{4}{3}\alpha\right)^3 - 2\left(\frac{5}{3}\alpha\right) - \left(\frac{4}{3}\alpha\right) - \left(\frac{5}{3}\alpha\right)\left(\frac{4}{3}\alpha\right) \\ &= \min_{\alpha \geq 0} \ \frac{7}{6}\alpha^3 - \frac{20}{9}\alpha^2 - \frac{14}{3}\alpha, \end{aligned}$$

where the second equality uses Hint 1 (can also be directly calculated). By Hint 2, we know that the optimal solution to this problem is $\alpha = \frac{40+2\sqrt{1723}}{63}$. This means that

$$x^1 = x^0 + \alpha p = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{40 + 2\sqrt{1723}}{63}\frac{1}{3}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \frac{40 + 2\sqrt{1723}}{189}\begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

Question 2

(Linear programming)

(2p) a) Transforming the problem to standard form gives

$$\begin{aligned} \text{minimize} \quad & 2x_1 + x_2 - x_3 + 3x_4 \\ \text{subject to} \quad & x_2 - 2x_3 + 4x_4 + s_1 = 1 \\ & -x_1 + x_2 + 2x_3 + s_2 = 1 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

As stated in the problem, we start with x_2 and x_3 as basic variables.

Iteration 1:
With $x_B = [x_2,\, x_3]^T$ and $x_N = [x_1,\, x_4,\, s_1,\, s_2]^T$,

$$B = \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 4 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \quad c_B^T = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad c_N^T = \begin{bmatrix} 2 & 3 & 0 & 0 \end{bmatrix}.$$

$x_B = B^{-1}b = [1,\, 0]^T$, and this is thus a BFS (as expected). The reduced costs are

$$\tilde{c}_N^T = \begin{bmatrix} 2 & 3 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & -1 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 4 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} = [9/4,\, 0,\, -3/4,\, -1/4],$$

and hence $(x_N)_3 = s_1$ enters the basis. $B^{-1}N_3 = 1/4[2,\, -1]^T$. The only positive element is the first one, and thus the minimum ratio test gives that $(x_B)_1 = x_2$ leaves the basis

Iteration 2:
With $x_B = [x_3,\, s_1]^T$ and $x_N = [x_1,\, x_2,\, x_4,\, s_2]^T$,

$$B = \begin{bmatrix} -2 & 1 \\ 2 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 1 & 4 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}, \quad c_B^T = \begin{bmatrix} -1 & 0 \end{bmatrix}, \quad c_N^T = \begin{bmatrix} 2 & 1 & 3 & 0 \end{bmatrix}.$$

$x_B = B^{-1}b = [1/2,\, 2]^T$. The reduced costs are

$$\tilde{c}_N^T = \begin{bmatrix} 2 & 1 & 3 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 \end{bmatrix} \frac{-1}{2} \begin{bmatrix} 0 & -1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} = [3/2,\, 3/2,\, 3,\, 1/2].$$

Since the reduced costs are larger than or equal to zero, this BFS is optimal. Thus $x^* = [x_1^*,\, x_2^*,\, x_3^*,\, x_4^*]^T = [0,\, 0,\, 1/2,\, 0]^T$.

(1p) b) Since the reduced costs are strictly positive, the solution is unique.

Question 3

(global convergence of exterior penalty method)

(1p) a) ψ must be continuous, and $\psi(s) = 0$ if and only if $s = 0$; see Section 13.1.1 in the book.

(2p) b) See Theorem 13.3 in the book.

Question 4

(True or False)

(1p) a) False. This is verified by $f([3, 1]^T) = 3 - e^1 > 0$.

(1p) b) True. $g_1(x) \leq 0$ only for $x = 0$. Moreover, $g_2(0) < 0$, so $S = \{0\}$ which is a convex set.

(1p) c) True. If x° is an inner point to the constraints in (P), then it is also an inner point to the constraints in (P'), and thus Slater's constraint qualifier holds also for (P'). Since the problem is convex, this means that KKT is necessary and sufficient for global optimality. Moreover, since $\mu_k = 0$, it is easily verified that since x^* is a KKT point for (P), it is also a KKT point for (P').

Question 5

(Lagrangian duality and global optimality conditions)

(2p) a) Let $X := \{x \in \mathbb{R}^2 \mid x_1 \geq 1,\ 0 \leq x_2 \leq 10\}$. The dual function $q(\mu) = \inf_{x\in X} \mathscr{L}(x,\mu)$ is given by

$$q(\mu) = \begin{array}{rl} \min & x_1^3 + \mu x_1^2 + (1-\mu)x_2 \\ \text{subject to} & x \in X \end{array} = \begin{cases} 1+\mu & \text{if } 0 \leq \mu \leq 1, \\ 11 - 9\mu & \text{if } \mu \geq 1, \end{cases}$$

where the minimizers of the Lagrangian are attained in $x = [1,0]^T$ for $0 \leq \mu < 1$, in points $x = [1,a]^T$ for $a \in [0,10]$ for $\mu = 1$, and in $x = [1,10]^T$ for $\mu \geq 1$.

(1p) b) Let $X := \{x \in \mathbb{R}^2 \mid x_1 \geq 1,\ 0 \leq x_2 \leq 10\}$ and $g(x) = x_1^2 - x_2$. A point $(x^*,\mu^*) \in \mathbb{R}^2 \times \mathbb{R}$ is said to satisfy the global optimality conditions if

$$\begin{aligned} x^* &\in \operatorname*{arg\,min}_X\ \mathcal{L}(x,\mu^*), \\ x^* \in X \text{ and } g(x^*) &\leq 0, \\ \mu^* &\geq 0, \\ \mu^* g(x^*) &= 0. \end{aligned}$$

We were told to consider the point $\hat{x} = [1,1]^T$ and $\hat{\mu} = 1$. From a), we know that $\arg\min_X\ \mathcal{L}(x,1) = \{x = [1,a]^T \mid a \in [0,10]\}$ and thus $\hat{x} \in \arg\min_X\ \mathcal{L}(x,\hat{\mu})$. Moreover, $\hat{x} \in X$ and $g(\hat{x}) = 1^2 - 1 = 0 \leq 0$. Next, $\hat{\mu} = 1 \geq 0$. Finally, $\hat{\mu} g(\hat{x}) = 1 \cdot 0 = 0$. This shows that $\hat{x} = [1,1]^T$ and $\hat{\mu} = 1$ satisfy the global optimality conditions.

(3p) Question 6

(Feasible directions and constraint qualifiers)

We first note that $g_1(\tilde{x}) = g_2(\tilde{x}) = 0$, and hence both constraints are active. We have the following definitions for three of the cones

$$\begin{aligned} R_S(\tilde{x}) :=& \{p \in \mathbb{R}^2 \setminus \{0\} \mid \exists\, \bar{\alpha} > 0 \text{ such that } x + \alpha p \in S\ \forall\, \alpha \in [0,\bar{\alpha}]\}, \\ \mathring{G}(\tilde{x}) :=& \{p \in \mathbb{R}^2 \mid \nabla g_1(\tilde{x})^T p < 0,\ g_2(\tilde{x})^T p < 0\}, \\ G(\tilde{x}) :=& \{p \in \mathbb{R}^2 \mid \nabla g_1(\tilde{x})^T p \leq 0,\ g_2(\tilde{x})^T p \leq 0\}. \end{aligned}$$

Moreover, we know that the following inclusions always hold:

$$\operatorname{cl}(R_S(\tilde{x})) \subseteq T_S(\tilde{x}) \subseteq G(\tilde{x}).$$

To compute the different cones, we compute the gradients of the constraints:

$$\nabla g_1(\tilde{x}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \nabla g_2(\tilde{x}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

To compute $R_S(\tilde{x})$, let $p = [p_1, p_2]^T$. For any $p_1 \neq 0$ and any $p_2 > 0$, we have that

$$g_1(\tilde{x} + \alpha p) = -\alpha^2 p_1^2 + \alpha p_2 > 0$$

for all $\alpha < p_1^2/p_2$, and where we have that $p_1^2/p_2 > 0$. This means that such directions are not in the cone of feasible directions. Similarly, for $p_1 = 0$ and any $p_2 > 0$

$$g_1(\tilde{x} + \alpha p) = \alpha p_2 > 0$$

for all $\alpha > 0$. Therefore, such directions are not in the cone of feasible directions. And analogous reasoning can be done for $p_2 < 0$ and g_2, which means that no directions with $p_2 \neq 0$ can be in $R_S(\tilde{x})$. However, for $p_2 = 0$

$$g_1(\tilde{x} + \alpha p) = g_2(\tilde{x} + \alpha p) = -\alpha^2 p_1^2 < 0$$

for all $\alpha > 0$. We therefore conclude that $R_S(\tilde{x}) = \{p \in \mathbb{R}^2 \setminus \{0\} \mid p_2 = 0\}$, or equivalently that $R_S(\tilde{x}) = \{p \in \mathbb{R}^2 \mid p_2 = 0, p_1 \neq 0\}$. This also means that $\text{cl}(R_S(\tilde{x})) = \{p \in \mathbb{R}^2 \mid p_2 = 0\}$.

To compute $G(\tilde{x})$, we note that it consists of all $p = [p_1, p_2]^T$ such that

$$0 \geq \nabla g_1(\tilde{x})^T p = \begin{bmatrix} 0 & 1 \end{bmatrix}^T \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = p_2$$

and

$$0 \geq \nabla g_2(\tilde{x})^T p = \begin{bmatrix} 0 & -1 \end{bmatrix}^T \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = -p_2.$$

Therefore, $G(\tilde{x}) = \{p \in \mathbb{R}^2 \mid p_2 = 0\}$.

To compute the tangent cone, we note that $\text{cl}(R_S(\tilde{x})) = G(\tilde{x}) = \{p \in \mathbb{R}^2 \mid p_2 = 0\}$, and therefore by the inclusions stated above we must have that $\text{cl}(R_S(\tilde{x})) = T_S(\tilde{x}) = G(\tilde{x}) = \{p \in \mathbb{R}^2 \mid p_2 = 0\}$. This shows that Abadie's constraint qualifier holds in $\tilde{x}$.

Finally, to compute $\mathring{G}(\tilde{x})$ we note that it consists of all $p = [p_1, p_2]^T$ such that

$$0 > \nabla g_1(\tilde{x})^T p \begin{bmatrix} 0 & 1 \end{bmatrix}^T \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = p_2$$

and

$$0 > \nabla g_2(\tilde{x})^T p \begin{bmatrix} 0 & -1 \end{bmatrix}^T \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = -p_2.$$

But both these inequalities cannot hold at the same time and thus $\mathring{G}(\tilde{x}) = \emptyset$.

Question 7

(Choice of algorithm)

(1p) a) Exterior penalty method and Simplex method.

Note: Newton's method cannot be used for several reasons: the problem is not unconstrained, and the Hessian of the cost function is the zero-matrix, that is, it is not invertible.

(1p) b) Interior penalty method.

Note 1: The Simplex method cannot be used since the problem is not (necessarily) linear. The steepest descent method cannot be used since the problem is not unconstrained.

Note 2: Without the assumption on that there is a point $\tilde{x}$ such that $A\tilde{x} < b$, which in the question was expressed as that Slater's constraint qualifier holds, then the interior penalty method might not be possible to use either.

(1p) c) None of the algorithms can be used.

Note: The reason is the same for all of the methods, namely that under the conditions given it is not sure that $q(\mu)$ is differentiable everywhere. (See the solution to Question 5a) on this exam for an example of a $q(\mu)$ that is not differentiable everywhere.)