Chalmers/GU
Mathematics sciences

EXAM

TMA947/MMG621 NONLINEAR OPTIMISATION

Date:	21–10–28
Time:	8^{30}–13^{30}
Aids:	Text memory-less calculator, English-Swedish dictionary
Number of questions:	7; a passed question requires 2 points of 3.
	Questions are *not* numbered by difficulty.
	To pass requires 10 points and three passed questions.
Examiner:	Ann-Brith Strömberg, Emil Gustavsson (070 290 83 00)

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

(3p) Question 1

(Simplex method)

Consider the following linear program:

$$\begin{aligned} \text{minimize} \quad & z = x_1 - x_2, \\ \text{subject to} \quad & \phantom{z = {}} x_1 + x_2 \geq 1, \\ & \phantom{z = {}} x_1 + 2x_2 \leq 4, \\ & \phantom{z = x_1 + 2{}} x_2 \geq 0. \end{aligned}$$

Solve this problem using phase I (so that you begin with a unit matrix as the first basis) and phase II of the simplex method. If the problem has an optimal solution, then present the optimal solution in both the original variables and in the variables used in the standard form. If the problem is unbounded, then use your calculations to find a direction of unboundedness in both the original variables and in the variables used in the standard form.

Question 2

(LP duality)

Consider the linear programming problem to

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u}, \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}, \boldsymbol{c} \in \mathbb{R}^n, \boldsymbol{b} \in \mathbb{R}^m, \boldsymbol{l} \in \mathbb{R}^n$, and $\boldsymbol{u} \in \mathbb{R}^n$

(2p) a) Construct the LP dual of this problem.

(1p) b) Show that the dual problem is always feasible.

(3p) Question 3

(characterization of convexity in C^1)

Let $f \in C^1$ on an open convex set S. Establish the following characterization of the convexity of f on S:

$$f \text{ is convex on } S \quad \iff \quad f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{x}), \text{ for all } \boldsymbol{x}, \boldsymbol{y} \in S$$

Question 4

(True or False)

The below three claims should be assessed. For each claim: state whether it is true or false. Provide an answer together with a short but complete motivation.

(1p) a) Assume that you have solved a linear program (LP) with the simplex method, and at the found optimal solution all reduced costs of the non-basic variables are strictly positive.

Claim: The solution found is the unique optimal solution to the problem (i.e., there do not exist multiple optimal solutions).

(1p) b) Suppose $f \in C^2$. Assume that at some iteration point $\boldsymbol{x} \in \mathbb{R}^n$, there exists a solution $\boldsymbol{p}$ to the search direction-finding problem of Newton's method.

Claim: The direction $\boldsymbol{p}$ defines a descent direction to f at $\boldsymbol{x}$.

(1p) c) Let $f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x}$ where $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$ is a symmetric and invertible matrix.

Claim: The function f is convex on $\mathbb{R}^n$.

Question 5

(exterior penalty method)

Consider the problem to

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := x_1^2 + x_2^2, \\ \text{subject to} \quad & h(\boldsymbol{x}) := x_1 + x_2 - 1 = 0. \end{aligned}$$

We consider solving the problem by using the exterior penalty method with the quadratic penalty function $\psi(s) = s^2$. The penalty problem is to

$$\underset{\boldsymbol{x} \in \mathbb{R}^n}{\text{minimize}} \, f(\boldsymbol{x}) + \nu \hat{\chi}_S(\boldsymbol{x}),$$

where $\hat{\chi}_S(\boldsymbol{x}) = \psi(h(\boldsymbol{x}))$, for positive, increasing values of the penalty parameter $\nu > 0$.

(1p) a) State the sequence of solutions to the penalty problem as a function of the penalty parameter ν.

(2p) b) Show that the sequence of solutions to the penalty problem converges to the unique optimal solution to the original problem when $\nu \to \infty$. Note: You need to show that the point the sequence converges to is optimal.

(3p) Question 6

(KKT)

Consider the problem to

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}(x_1 - 5)^2 + \frac{1}{2}(x_2 - 3)^2, \\ \text{subject to} \quad & x_1 + x_2 \leq 5, \\ & 0 \leq x_j \leq 3, \quad j = 1, 2. \end{aligned}$$

Solve the problem using the KKT conditions (i.e., find the optimal solution to the problem and verify that it is optimal).

Are the KKT conditions necessary for optimality? Are the KKT conditions sufficient for optimality? Motivate your answers.

Question 7

(Modelling)

As a hedge fund manager you are responsible for choosing which stocks to include in the hedge funds portfolio. You have a budget of M amount to invest and you have n stocks to choose from ($\{1, \dots, n\}$). Each stock $i \in \{1, \dots, n\}$ has an expected return of $r_i > 0$ (i.e., if you invest x amount of money in stock i, the expected return of the investment is $r_i x$).

You must respect the following requirements:

- You may only invest in a maximum of $k_{\max}$ stocks.
- You must invest in a minimum of $k_{\min}$ stocks.
- You can not invest in both stock 1 and stock 2.
- The maximum investment you can make in any stock is $m \leq M$.

(2p) a) Formulate a *linear integer program* that maximizes the return of the portfolio.

(1p) b) Now you want to maximize the minimum return for any stock you invest in. This means that you would like to maximize the following entity instead:

$$\min_{i \in \{1,\dots,n\} \,:\, x_i > 0} r_i x_i,$$

where x_i if the amount you invest in stock i. Formulate this new problem as a *linear integer program*. Hint: You may need to introduce a variable z for the minimum return of the stocks you choose.

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EXAM SOLUTION

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Note: the solutions presented here are brief in relation to the requirements on your answers, in particular regarding your motivations.

Question 1

(Simplex method)

We first rewrite the problem on standard form. We rewrite $x_1 = x_1^+ - x_1^-$ and introduce slack variables s_1 and s_2.

$$\begin{aligned} \text{minimize} \quad & x_1^+ - x_1^- - x_2, \\ \text{subject to} \quad & x_1^+ - x_1^- + x_2 - s_1 = 1, \\ & x_1^+ - x_1^- + 2x_2 + s_2 = 4, \\ & x_1^+, x_1^-, x_2, s_1, s_2 \geq 0. \end{aligned}$$

Phase I We introduce an artificial variable a and formulate our Phase I problem.

$$\begin{aligned} \text{minimize} \quad & a \\ \text{subject to} \quad & x_1^+ - x_1^- + x_2 - s_1 + a = 1, \\ & x_1^+ - x_1^- + 2x_2 + s_2 = 4, \\ & x_1^+, x_1^-, x_2, s_1, s_2 \geq 0. \end{aligned}$$

We now have a starting basis (a, s_2). Calculating the reduced costs we obtain $\tilde{c}_N = (-1, 1, -1, 1)^{\mathrm{T}}$, meaning that x_1^+ or x_2 should enter the basis. We choose x_2. From the minimum ratio rest, we get that a should leave the basis. This concludes phase I and we now have the basis (x_2, s_2).

Phase II Calculating the reduced costs, we obtain $\tilde{c}_N = (2, -2, 1)^{\mathrm{T}}\mathrm{T}$, meaning that x_1^- should enter the basis. From the minimum ratio test, we get that the outgoing variable is s_2. Updating the basis we now have (x_1^-, x_2) in the basis.

Calculating the reduced costs, we obtain $\tilde{c}_N = (0, 3, 2)^{\mathrm{T}} \geq 0$, meaning that the current basis is optimal. The optimal solution is thus $(x_1^+, x_1^-, x_2, s_1, s_2) = (0, 2, 3, 0, 0)$ which in the original variables means $(x_1, x_2) = (-2, 3)$, with optimal objective value $f^* = -5$.

Question 2

(Duality)

The dual problem is the problem to

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} + \boldsymbol{l}^{\mathrm{T}}\boldsymbol{z}_l + \boldsymbol{u}^{\mathrm{T}}\boldsymbol{z}_u, \\ \text{subject to} \quad & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} + \boldsymbol{z}_l + \boldsymbol{z}_u = \mathbf{c}, \\ & \boldsymbol{z}_l \geq \mathbf{0}, \\ & \boldsymbol{z}_u \leq \mathbf{0}. \end{aligned}$$

This problem is always feasible since for any $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}$, and $\boldsymbol{c}$ you can always set $\boldsymbol{y} = \mathbf{0}$ and solve the simple equation system $\boldsymbol{z}_l + \boldsymbol{z}_u = \boldsymbol{c}$ by letting $(\boldsymbol{z}_l)_i = c_i$ if $c_i > 0$, and $(\boldsymbol{z}_l)_i = 0$ otherwise. And letting $(\boldsymbol{z}_u)_i = c_i$ if $c_i < 0$, and $(\boldsymbol{z}_u)_i = 0$ otherwise.

Question 3

(characterization of convexity in C^1)

See Theorem 3.61 in textbook.

Question 4

(True or False)

a) True, since all reduced costs are positive no other variable can be included in the basis.

b) False, the direction may be an ascent direction.

c) False, the primal problem can be infeasible (if the dual is unbounded).

Question 5

(exterior penalty method)

(1p) a)

The penalty problem then becomes

$$\min_{\boldsymbol{x}\in\mathbb{R}^2} \left(f(\boldsymbol{x}) + \nu h(\boldsymbol{x})^2\right) = \min_{\boldsymbol{x}\in\mathbb{R}^2} \left(x_1^2 + x_2^2 + \nu(x_1 + x_2 - 1)^2\right),$$

where $\nu > 0$. Noting that this is a convex function for positive ν we can solve the problem by setting the gradient to zero. Then we obtain

$$\begin{aligned} 2x_1 + 2\nu x_1 + 2\nu x_2 - 2\nu &= 0, \\ 2x_2 + 2\nu x_1 + 2\nu x_2 - 2\nu &= 0, \end{aligned}$$

which has solution $\boldsymbol{x}_\nu = \frac{\nu}{1+2\nu}(1,1)^{\mathrm{T}}$.

(2p) b) Letting $\nu \to \infty$ we get that $\boldsymbol{x}_\nu \to (\frac{1}{2}, \frac{1}{2})^{\mathrm{T}}$. Analyzing the problem we see that this is a KKT point in the original problem. Since the original problem is convex, we have that the KKT conditions are sufficient for optimality, which means that the point $(\frac{1}{2}, \frac{1}{2})^{\mathrm{T}}$ is the optimal solution.

Question 6

(KKT)

Analyzing the problem we can see that at $\boldsymbol{x} = (3,2)^{\mathrm{T}}$ the constraints $g_1(\boldsymbol{x}) = x_1 + x_2 - 5 \leq 0$ and $g_2(\boldsymbol{x}) = x_1 - 3 \leq 0$ are active. We see that

$$\nabla f(\boldsymbol{x}) + \mu_1 \nabla g_1(\boldsymbol{x}) + \mu_2 \nabla g_2(\boldsymbol{x}) = \mathbf{0},$$

which at $\boldsymbol{x} = (3,2)^{\mathrm{T}}$ gives

$$\begin{bmatrix} -2 \\ -1 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \mu_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

with solution $\mu_1 = \mu_2 = 1$.

Since all constraints are affine, we have that the affine constraints CQ is fulfilled, meaning that the KKT conditions are necessary for optimality for this problem.

Since the problem is convex, the KKT conditions are also sufficient for optimality. This implies that $\boldsymbol{x} = (3,2)^{\mathrm{T}}$ is the optimal solution to the problem.

Question 7

(Modelling)

(2p) a) Let x_i be the amount of money invested in stock i, $i \in \{1, \dots, n\}$. Let the amount invested in each stock be a nonzero integer number (i.e., $x_i \in \{0, 1, 2, \dots\}$) and introduce the binary variables $y_i \in \{0, 1\}$ denote if we invest any money in stock i or not. Then the problem can be formulated as

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i \in \{1,\dots,n\}} r_i x_i, \\
\text{subject to} \quad & x_i \leq m y_i, \quad i = 1, \dots, n, \\
& x_i \geq y_i, \quad i = 1, \dots, n, \\
& k_{\min} \leq \sum_{i \in \{1,\dots,n\}} y_i \leq k_{\max}, \\
& y_1 + y_2 \leq 1, \\
& x_i \in \{0, 1, 2, \dots\}, \quad i = 1, \dots, n, \\
& y_i \in \{0, 1\}, \quad i = 1, \dots, n.
\end{aligned}
$$

(1p) b) Introduce a variable z for the minimum return among all chosen stocks. Then the model needs to be altered by replacing the objective function with z and adding the constraints

$$
z \leq r_i x_i + M(1 - y_i), \quad i = 1, \dots, n,
$$

in order to represent z as the minimum return of the chosen stocks.