Chalmers/GU
Mathematics sciences

EXAM

TMA947/MMG621
NONLINEAR OPTIMISATION

Date:	21–08-17
Time:	8^{30}–13^{30}
Aids:	All aids are allowed, but cooperation is not allowed
Number of questions:	7; passed on one question requires 2 points of 3.
	Questions are *not* numbered by difficulty.
	To pass requires 10 points and three passed questions.
Examiner:	Ann-Brith Strömberg (Course responsible: Emil Gustavsson)

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

Question 1

(Simplex method)

Consider the following linear program:

$$\begin{array}{lrcrcr} \text{minimize} & z = 2x_1 & + & 2x_2, & & \\ \text{subject to} & x_1 & - & x_2 & \leq & -2, \\ & 2x_1 & - & x_2 & \geq & 1, \\ & x_1, & & x_2 & \geq & 0. \end{array}$$

(1p) a) Convert the problem to standard form.

(2p) b) Solve the problem using phase I and phase II of the simplex method. Present an optimal solution, or a ray of unboudedness.

Question 2

(Duality)

Suppose we have an LP problem on the form

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & A\boldsymbol{x} \geq \boldsymbol{b}, \\ & B\boldsymbol{x} \leq \boldsymbol{d}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned}$$

where $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{d} \in \mathbb{R}^k$, $A \in \mathbb{R}^{m \times n}$, and $B \in \mathbb{R}^{k \times n}$

(1p) a) State the LP dual problem.

(2p) b) Suppose that this dual problem is feasible. Motivate whether or not the primal problem has an optimal solution.

(3p) ## Question 3

(Lagrangian relaxation)

Consider the quadratic optimization problem

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\boldsymbol{x}^{\mathrm{T}} Q \boldsymbol{x} + \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x}, \\ \text{subject to} \quad & A\boldsymbol{x} \geq \boldsymbol{b}, \end{aligned}$$

where $Q \in \mathbb{R}^{n \times n}$ is a positive definite and symmetric matrix, and $A \in \mathbb{R}^{m \times n}$.

Construct the Lagrangian dual problem by relaxing all the constraints and show that the dual problem itself is a quadratic optimization problem.

Hint: An explicit solution to the problem $\min_{x \in X} L(\boldsymbol{x}, \boldsymbol{\mu})$ can be found for each $\boldsymbol{\mu}$.

Question 4

(True or False)

The below three claims should be assessed. Are they true or false? Provide an answer together with a short but complete motivation.

(1p) a) Suppose that the function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is continuously differentiable on $\mathbb{R}^n$ and let G be a symmetric and positive definite matrix of dimension $n \times n$.

Claim: If $\nabla f(\boldsymbol{x}) \neq \boldsymbol{0}^n$ and the vector $\boldsymbol{d} \in \mathbb{R}^n$ fulfils $G\boldsymbol{d} = -\nabla f(\boldsymbol{x})$ it holds that $f(\boldsymbol{x} + t\boldsymbol{d}) < f(\boldsymbol{x})$ for small enough values of $t > 0$.

(1p) b) Consider a convex function $f : \mathbb{R}^n \mapsto \mathbb{R}$. Suppose that at some vector $\boldsymbol{x} \in \mathbb{R}^n$ the directional derivative of f in the direction of a given vector $\boldsymbol{p} \in \mathbb{R}^n$ is non-negative.

Claim: The vector $\boldsymbol{x}$ is a minimizer of f over $\mathbb{R}^n$.

(1p) c) *Claim:* If a function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is concave on $\mathbb{R}^n$ and $c \in \mathbb{R}$, then the set $\{\, \boldsymbol{x} \mid f(\boldsymbol{x}) \geq c \,\}$ is convex.

(3p) Question 5

(Frank-Wolfe)

Consider the problem to

$$\begin{aligned} \text{minimize} \quad & x_1^2 + x_2^2 + x_1x_2 - 3x_1 - 6x_2, \\ \text{subject to} \quad & x_1 \leq 4 - x_2, \\ & \frac{1}{2}x_2 \leq x_1, \\ & x_2 \leq 2, \\ & x_2 \geq 0 \end{aligned}$$

Start at the origo and perform two iterations of the Frank–Wolfe method. Give the upper and lower bounds on the optimal objective function value that the algorithm generates at each iteration, and give a theoretical motivation for them. State and motivate whether the resulting point (after two iterations) is optimal.

Question 6

(KKT conditions)

Consider the problem to:

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := x_2 - x_1^2, \\ \text{subject to} \quad & \frac{1}{2}\left(x_1^2 + x_2^2\right) \leq 2, \\ & x_2 + 1 \geq 0. \end{aligned}$$

(1p) a) Express the KKT conditions.

(2p) b) Find all KKT points. For each of the KKT points, state and motivate whether it is an optimal point.

(3p) Question 7

(Modelling)

There is a machine that can produce two different parts, pipes and plates. To produce one pipe takes T_{pipe} hours, while the corresponding time for one plate is T_{plate} hours. The profit for each pipe is c_{pipe} and for each plate c_{plate}. There are T hours of production time available. However, by paying a fixed cost of D another T_{extra} hours of production can be used. If more than 12 pipes are produced, then at least 4 plates must be produced. It is only possible to produce integer numbers of parts.

Formulate a linear integer program (a linear objective function, linear constraints, integrality restrictions on the variables) to determine the optimal production plan to maximize the profit.