Chalmers/GU
Mathematics sciences

EXAM

TMA947/MMG621
NONLINEAR OPTIMISATION

Date:	19–10–31
Time:	8^{30}–13^{30}
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3.
	Questions are *not* numbered by difficulty.
	To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Malin Nilsson, tel. 5325
Result announced:	19–11–11
	Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the problem (P) to:

$$
\begin{aligned}
\text{maximize} \quad z = \quad & x_1 + x_2, \\
\text{subject to} \quad & 2x_1 + x_2 \leq 2, \\
& x_2 \leq 1, \\
& -x_1 + x_2 \leq 1/2, \\
& x_1 - x_2 \leq 1/2, \\
& x_1, \quad x_2 \geq 0.
\end{aligned}
$$

(1p) a) Formulate the dual linear problem to (P) and convert the dual linear problem to standard form.

(1.5p) b) Solve the dual linear problem using phase I and phase II of the simplex method. Present an optimal solution to the dual linear problem or determine that no such exist.

(0.5p) c) Present an optimal solution to the original problem (P) or determine that no such exists. Utilize that the simplex algorithm computes the value of both primal- and dual variables.

(3p) Question 2

(unconstrained optimization)

Let $f(\boldsymbol{x}) := x_1^2 + 2x_1x_2 - 2x_2^2 + 4x_1$ and $\boldsymbol{x}^0 = (0,0)^{\mathrm{T}}$. Find the search directions at $\boldsymbol{x}^0$ for the following three unconstrained optimization methods:

a) The steepest descent method,

b) Newton's method,

c) Newton's method with the Levenberg–Marquardt modification using $\gamma = 8$ (where γ is the amount added to the diagonal of the Hessian).

In general, for which of the methods a)–c) are the directions found always descent directions? Motivate your answer.

(3p) Question 3

(Lagrangian relaxation)

Consider the problem to

$$
\begin{aligned}
\text{minimize} \quad & z = x_1 - 2x_2, \\
\text{subject to} \quad & x_1 - x_2 \geq 2, \\
& x_1 + x_2 \leq 5, \\
& x_1, \quad x_2 \geq 0.
\end{aligned}
$$

(2p) a) Lagrangian relax the first constraint. Use Lagrangian duality to obtain the optimal objective value z^*.

(1p) b) Use complementary slackness to obtain the optimal solution $\boldsymbol{x}^*$.

(3p) Question 4

(KKT conditions)

Consider the problem to

$$
\begin{aligned}
\text{minimize} \quad & x_1x_2 + x_1x_3 + x_2x_3, \\
\text{subject to} \quad & x_1 + x_2 + x_3 = 12.
\end{aligned}
$$

(2p) a) Write down the KKT conditions for the problem, and find all KKT points.

(1p) b) Does the problem have an optimal solution? Motivate!

(3p) Question 5

(modelling)

You are constructing a wooden product requiring N boards of lengths l_i, $i = 1, \dots, N$. Your local supplier currently has a stock of M boards with lengths L_j and at the prices p_j, $j = 1, \dots M$, where two boards of the same length and price are said to be of the same type. Let $\mathcal{S}_k \subset \{1, \dots, M\}$, denote all boards of type k and $d_k = p_j$, $j \in \mathcal{S}_k$, be the common price for board of type k, $k = 1, \dots, K$.

Moreover, each board bought can be cut, hence it can be enough for several boards of your wooden product. For example, if $L_1 = 3$, $l_1 = l_2 = 1$, then since $l_1 + l_2 \leq L_1$, the boards 1 and 2 of your wooden product can originate from board 1 in the stock.

The supplier also has an offer: every 4th board you buy of the same type, is for free.

Formulate an integer linear problem minimizing the cost of the boards purchased for your wooden product.

Question 6

(true or false)

Indicate for each of the following three statements whether it is true or false. Motivate your answers!

In each of the statements we consider $\boldsymbol{x}^* = (0, 0)^{\mathrm{T}}$ and the problem to:

$$\begin{aligned} \text{minimize} \quad & z = x_1, \\ \text{subject to} \quad & x_2^2 - x_1 x_2 + x_2 \leq 0, \\ & x_1, \quad x_2 \geq 0. \end{aligned}$$

(1p) a) *Claim:* $\boldsymbol{x}^*$ is a unique KKT Point to the problem.

(1p) b) *Claim:* Abadie's constraint qualification holds at $\boldsymbol{x}^*$.

(1p) c) *Claim:* The interior penalty method can converge to $\boldsymbol{x}^*$.

(3p) Question 7

(convergence of an exterior penalty method)

Let us consider a general optimization problem:

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}), \\ \text{subject to} \quad & \boldsymbol{x} \in S, \end{aligned} \tag{1}$$

where $S \in \mathbb{R}^n$ is a non-empty, closed set and $f : \mathbb{R}^n \to \mathbb{R}$ is a given differentiable function. We assume that the feasible set S of the optimization problem (1) is given by the system of inequality and equality constraints:

$$S = \{\boldsymbol{x} \in \mathbb{R}^n \quad | \quad \begin{aligned} g_i(\boldsymbol{x}) &\leq 0, \quad i = 1, \dots, m, \\ h_j(\boldsymbol{x}) &= 0, \quad j = 1, \dots, \ell.\} \end{aligned} \tag{2}$$

where $g_i \in C^0$, $i = 1, \dots, m$, and $h_j \in C^0$, $j = 1, \dots, \ell$.

We choose a function $\Psi : \mathbb{R} \to \mathbb{R}_+$ such that $\Psi(s) = 0$ if and only if $s = 0$ (typical examples of Ψ are $\Psi(s) = |s|$, or $\Psi(s) = s^2$), and introduce the function

$$\nu \check{X}_S(\boldsymbol{x}) = \nu \left(\sum_{i=1}^{m} \Psi(\max\{0, g_i(\boldsymbol{x})\}) + \sum_{j=1}^{\ell} \Psi(h_j(\boldsymbol{x})) \right) \tag{3}$$

where the real number ν is called a *penalty parameter*.

We assume that for every $\nu > 0$ the approximating optimization problem to

$$\text{minimize} \quad f(\boldsymbol{x}) + \nu \check{X}_S(\boldsymbol{x}) \tag{4}$$

has at least one optimal solution $\boldsymbol{x}^*_\nu$.

Prove the following theorem.

THEOREM 1. *Assume that the original constrained problem (1) possesses optimal solutions. Then, every limit point of the sequence $\{\boldsymbol{x}_\nu\}$, $\nu \to +\infty$, of globally optimal solutions to (4) is globally optimal in the problem (1).*

Chalmers/GU
Mathematics

EXAM SOLUTION

TMA947/MMG621
NONLINEAR OPTIMISATION

Date: 19–10–31
Examiner: Michael Patriksson

Note: the solutions presented here are brief in relation to the requirements on your answers, in particular regarding your motivations.

Question 1

(the simplex method)

(1p) a) The dual problem in standard form becomes:

$$\begin{array}{lrcccccccl} \text{minimize} & z = 2y_1 & + & y_2 & + & \frac{1}{2}y_3 & + & \frac{1}{2}y_4, & & \\ \text{subject to} & 2y_1 & & & - & y_3 & + & y_4 - s_1 & & = 1, \\ & y_1 & + & y_2 & + & y_3 & - & y_4 & - s_2 & = 1, \\ & y_1, & & y_2, & & y_3, & & y_4, \quad s_1, & s_2 & \geq 0. \end{array}$$

(1.5p) b) Introducing the artificial variable a_1, phase I gives the problem

$$\begin{array}{llcl} \text{minimize} & w = & & a_1, \\ \text{subject to} & 2y_1 \quad - y_3 + y_4 - s_1 & & + a_1 = 1, \\ & y_1 + y_2 + y_3 - y_4 \quad - s_2 & & = 1, \\ & y_1, \quad y_2, \quad y_3, \quad y_4, \quad s_1, \quad s_2, & & a_2 \geq 0. \end{array}$$

Using the starting basis $(a_1, y_2)^T$ gives

$$\boldsymbol{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{N} = \begin{pmatrix} 2 & -1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 & -1 \end{pmatrix}, \boldsymbol{x}_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \boldsymbol{c}_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \boldsymbol{c}_N = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The reduced costs, $\bar{\boldsymbol{c}}_N^T = \boldsymbol{c}_N^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{N}$, for this basis is $\bar{\boldsymbol{c}}_N^T = \begin{pmatrix} -2, & 1, & -1, & 1 & 0 \end{pmatrix}$, which means that y_1 enters the basis. $\boldsymbol{B}^{-1}\boldsymbol{N}_1 = \begin{pmatrix} 2 & 1 \end{pmatrix}^T$ thus the minimum ratio test implies that a_1 leaves.

Thus, we move on to phase II using the basis $\begin{pmatrix} y_1, & y_2 \end{pmatrix}^T$, and

$$\boldsymbol{B} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \boldsymbol{N} = \begin{pmatrix} -1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{pmatrix}, \boldsymbol{x}_B = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \boldsymbol{c}_B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \boldsymbol{c}_N = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}.$$

The new reduced costs are $\bar{\boldsymbol{c}}_N^T = \begin{pmatrix} 0, & 1, & \frac{1}{2}, & 1 \end{pmatrix}$. Since the reduced costs are all non-negative, the current BFS is optimal. The optimal solution to the dual problem is hence $\begin{pmatrix} y_1, & y_2, & y_3, & y_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}, & \frac{1}{2}, & 0, & 0 \end{pmatrix}$ with the objective value of $\frac{3}{2}$.

(.5p) c) Since the primal variables of our original problem are the dual variables of the dual problem, we get that $\boldsymbol{x}^T = \boldsymbol{c}_B^T \boldsymbol{B}^{-1} = \begin{pmatrix} \frac{1}{2}, & 1 \end{pmatrix}$.

Question 2

(unconstrained optimization)

a) For the steepest descent method:

$$\boldsymbol{p} = -\nabla f(\boldsymbol{x}^0) = (-4, 0)^T$$

b) For Netwon's method:

$$\boldsymbol{p} = -[\nabla^2 f(\boldsymbol{x})]^{-1}\nabla f(\boldsymbol{x}^0) = (-4/3, -2/3)^T$$

c) For Levemberg-Marquardt method:

$$\boldsymbol{p} = -[\nabla^2 f(\boldsymbol{x}) + \gamma I]^{-1}\nabla f(\boldsymbol{x}^0) = (-4/9, 2/9)^T$$

The methods a) and c) always finds descent directions (if γ is chosen large enough)

(3p) Question 3

(Lagrangian relaxation)

Lagrangian relax the first constraint, we can get:

$$L(\boldsymbol{x},\ \mu) = x_1 - 2x_2 + \mu(2 - x_1 + x_2) = (1-\mu)x_1 + (\mu - 2)x_2 + 2\mu.$$

$$q(\mu) = \max_{\boldsymbol{x}} L(\boldsymbol{x},\ \mu) = \begin{cases} 7\mu - 10, & \mu \in [0, 1.5) \quad x_1 = 0, x_2 = 5, \\ 0.5, & \mu = 1.5 \quad x_1 + x_2 = 5, \\ 5 - 3\mu & \mu \in (1.5, \infty) \quad x_1 = 5, x_2 = 0. \end{cases}$$

So $q^* = 0.5$, $\mu^* = 1.5$. Since the original problem is convex, and we have an interior point, by strong duality, we can get $z^* = q^* = 0.5$.
For complementary slackness, we need to fulfill $\mu_i^* g_i(\boldsymbol{x}^*) = 0$, since $\mu \neq 0$, so $g_i(\boldsymbol{x}^*) = 0$, which means $2 - x_1 + x_2 = 0$. Combine with $x_1 + x_2 = 5$, we can get $x^* = (3.5, 1.5)^T$. We can check that (x^*, μ^*) fulfilled all the conditions listed in Theorem 6.8, so x^* is the optimal solution for the original problem. The optimal value is 0.5.

(3p) Question 4

(KKT conditions)

(2p) a) The KKT conditions are

$$\nabla f(\boldsymbol{x}) + \lambda \nabla h(\boldsymbol{x}) = \begin{pmatrix} x_2 + x_3 \\ x_1 + x_3 \\ x_1 + x_2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

There is only one feasible point fulfilling the KKT conditions:

$$\bar{\boldsymbol{x}} = (4, 4, 4)^T$$

with $\gamma = -8$.

(1p) b) The problem is undounded. Take $x_1 = M$, $x_2 = M$ and $x_3 = 12 - 2M$ which is feasible. The objective value is $x_1x_2 + x_1x_3 + x_2x_3 = M^2 + M(12 - 2M) + M(12 - 2M) = 24M - 3M^2$. Let M tend to infinity and you get an undounded solution.

(3p) Question 5

(modelling)

Variables, let

- x_{ij} equal to one if the piece of length l_i is cut from the board of length L_j, and equal to zero otherwise, $i = 1, \ldots, N$, $j = 1, \ldots, M$.
- y_j equal to one if the board of length L_j is purchased, $j = 1, \ldots, M$.
- z_k be the number of times a discount has been retrieved for board of type k, $k = 1, \ldots, K$.

$$\begin{aligned}
\text{minimize} \quad & \sum_{j=1}^{M} p_j y_j - \sum_{k=1}^{K} d_k z_k, && && (1)\\
\text{s.t.} \quad & \sum_{i=1}^{N} l_i x_{ij} \leq L_j y_j, && j = 1, \ldots, M && (2)\\
& \sum_{j=1}^{M} x_{ij} = 1, && i = 1, \ldots, N, && (3)\\
& \sum_{j \in S_k} y_j \geq 4 z_k, && k = 1, \ldots, K, && (4)\\
& x_{ij} \in \{0, 1\}, && i = 1, \ldots, N,\ j = 1, \ldots, M && (5)\\
& y_j \in \{0, 1\}, && j = 1, \ldots, M. && (6)\\
& z_k \in \mathbb{Z}^+, && j = 1, \ldots, K. && (7)
\end{aligned}$$

Question 6

(true or false)

(1p) a) *True.* The KKT conditions becomes

$$\nabla f(\boldsymbol{x})+\sum_{i=1}^{3}\mu_i\nabla g_i(\boldsymbol{x}) = \begin{pmatrix}1\\0\end{pmatrix}+\mu_1\begin{pmatrix}-x_2\\2x_2-x_1+1\end{pmatrix}+\mu_2\begin{pmatrix}-1\\0\end{pmatrix}+\mu_3\begin{pmatrix}0\\-1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$g_i(\boldsymbol{x}) \le 0,\ \mu_i \ge 0,\ \mu_i g_i(\boldsymbol{x}) = 0,\ i = 1,2,3$$

Where $\mu_2 > 0 \Rightarrow \boldsymbol{x} = \begin{pmatrix}0\\0\end{pmatrix}$ and $\mu_2 = 0$ leads to an inconsistent system.

(1p) b) *True.* We check if the gradient cone and tangent cone are equal. The gradient cone is $G(\boldsymbol{x}^*) = \{\boldsymbol{p} \in \mathbb{R}^2 \,|\, x_2 \le 0, x_1 \ge 0, x_2 \ge 0\} = \{\boldsymbol{p} \in \mathrm{R}^2 \,|\, x_1 \ge 0, x_2 = 0\}$. For the tangent cone, let $\{\boldsymbol{x}^k\} \subset S$ be any sequence of points converging to $\boldsymbol{x}^*$, thus for any $\varepsilon > 0$ $\exists K$ such that $\boldsymbol{x}_1^k \le \varepsilon$, $\forall k \ge K$. Assuming that $\boldsymbol{x}_2^k > 0$ leads to a contradiction that $\boldsymbol{x}_1^k > 1$ thus $\boldsymbol{x}_2^k = 0$, $\forall k \ge K$. We thus get that $G(\boldsymbol{x}^*) = T_S(\boldsymbol{x}^*)$, i.e., Abadie's CQ holds.

(1p) c) *False.* Since any sequence of converging points must satisfy $\boldsymbol{x}_2^k = 0$, we have that there exist no sequence of strict interior points that converge to $\boldsymbol{x}^*$.

(3p)
Question 7

(convergence of an exterior penalty method)

See Theorem 13.3 in the course book.
