Chalmers/GU
Mathematics

EXAM

TMA947/MMG621 NONLINEAR OPTIMISATION

Date:	18–11–01
Time:	14^{00}–19^{00}
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3.
	Questions are *not* numbered by difficulty.
	To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Gustav Kettil, tel. 5325
Result announced:	18–11–21
	Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$
\begin{aligned}
\text{maximize} \quad & z = x_1 + 4x_2, \\
\text{subject to} \quad & x_1 + 3x_2 \leq 8, \\
& 2x_1 + x_2 \geq 4, \\
& x_2 \geq 0.
\end{aligned}
$$

(2p) a) Solve the problem using phase I and phase II of the simplex method.

Aid: You may utilize the identity

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.
$$

(1p) b) If an optimal solution exists, then use your calculations to decide whether it is unique or not. If the problem is unbounded, then use your calculations to specify a direction of unboundedness of the objective value.

(3p) **Question 2**

(necessary local and sufficient global optimality conditions)

Consider and optimization problem of the following general form:

$$\begin{aligned} &\text{minimize} \quad f(\boldsymbol{x}), && (1a)\\ &\text{subject to} \quad \boldsymbol{x} \in S, && (1b) \end{aligned}$$

where $S \subseteq \mathbb{R}^n$ is nonempty, closed and convex, and $f : \mathbb{R}^n \mapsto \mathbb{R} \cup \{+\infty\}$ is in C^1 on S.

Prove the following two propositions.

PROPOSITION 1. *If $\boldsymbol{x}^* \in S$ is a local minimum of f over S then it hold that*

$$\nabla f(\boldsymbol{x}^*)^T(\boldsymbol{x} - \boldsymbol{x}^*) \geq 0, \qquad \boldsymbol{x} \in S. \tag{2}$$

PROPOSITION 2. *Suppose that f is convex on S. Then*

$$\boldsymbol{x}^* \textit{is a global minimum of } f \textit{ over } S \qquad \Longleftrightarrow \qquad (2) \textit{ holds.}$$

Question 3

(Unconstrained optimization)

Consider the following optimization problem where the objective is to

$$\begin{aligned} &\text{minimize} \quad x_1^2 + 6x_1x_2 + x_2^2,\\ &\text{subject to} \quad \boldsymbol{x} \in \mathbb{R}^2. \end{aligned}$$

Let $\boldsymbol{x}_0 = (7, 7)^{\mathrm{T}}$

(2p) a) Start at $\boldsymbol{x}_0$ and perform one step of Levenberg–Marquardt method with an Armijo line search, using the multiplier $\gamma = 6$, the fraction requirement $\mu = 0.8$.

(1p) b) For this problem, will Levenberg–Marquardt method converges to a global optimal or not?

Question 4

(KKT conditions)

For the set described by the system (1) Abadie's CQ holds, but what about the following constraint qualifications?

$$S := \left\{ x_1, x_2 \in \mathbb{R} \;\middle|\; \begin{array}{ll} (x_1-1)^2 + x_2^2 & \leq 1 \\ (x_1+3)^2 + (x_2-4)^2 & \geq 25 \\ (x_1+3)^2 + (x_2+4)^2 & \geq 25 \\ x_2 & \leq 2 \end{array} \right\} \tag{1}$$

(1p) a) Show whether the affine CQ is satisfied or not.

(1p) b) Show whether the Slater CQ is satisfied or not.

(1p) c) Show whether the LICQ is satisfied or not at the point $\bar{\boldsymbol{x}}^{\mathrm{T}} = (0, 0)$.

(3p) Question 5

(modelling)

IT-support have grown tired of purchasing computer hardware with employee–specific performance requirements, and wishes to be aided in the decision. A computer consists of six essential components: motherboard (MB), power supply unit (PSU), CPU, GPU, RAM, and harddrive (HD). There exist several models (variants) of each component and the problem is to choose a variant for each component. For two of the components the choice of variant needs to be taken with extra care. First, each variant of the MB has a specific interface, making it incompatible with some variants of other components. Thus, a variant of MB cannot be used with some variants of other components. Second, the PSU distributes power to all other components of the computer. Hence, the power capacity of the PSU must exceed the total power usage of all other components. Both cost and power usage of each variant of a component, are known parameters. Only some variants of each component are considered to meet the performance requirements. Thus, only such acceptable variants are used in the optimization problem.

State an integer linear model that minimizes the hardware cost for a single computer by deciding the variant of each component.

Question 6

(true or false)

Indicate for each of the following three statements whether it is true or false. Motivate your answers!

(1p) a) In the interior penalty method we let the penalty tend to infinity.

(1p) b) The set $S := \{\, x_1, x_2 \in \mathbb{R} \mid x_2 \sin x_1 \geq 0, x_2^2 + x_1^3 \leq 27, x_2 \geq 0, x_1 \geq -3 \,\}$ is convex.

(1p) c) There exists a convex problem on the form $\min_{x \in \mathbb{R}^4} f(\boldsymbol{x})$ subject to $g_i(\boldsymbol{x}) \leq 0$, $i = 1, \dots, 3$, with an optimal primal solution $\boldsymbol{x}^*$, and Lagrangian multiplier vector $\boldsymbol{\mu}^*$, such that

$$\boldsymbol{x}^* = \begin{pmatrix} 3 \\ 4 \\ 7 \\ 5 \end{pmatrix}, \qquad \boldsymbol{\mu}^* = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{g}(\boldsymbol{x}^*) = \begin{pmatrix} -1 \\ -2 \\ -8 \end{pmatrix}.$$

(3p) Question 7

(linear programming duality)

Consider the following two polyhedral sets corresponding to the feasible sets of a pair of primal-dual linear programs:

$$X = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0} \,\},$$
$$Y = \{\, \boldsymbol{y} \in \mathbb{R}^m \mid \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}, \quad \boldsymbol{y} \geq \boldsymbol{0} \,\}.$$

Prove that if both sets X and Y are non-empty, then at least one of them must be unbounded.

Chalmers/GU
Mathematics

EXAM SOLUTION

TMA947/MMG621
NONLINEAR OPTIMISATION

Date: 18–11–01
Examiner: Michael Patriksson

Note: the solutions presented here are brief in relation to the requirements on your answers, in particular regarding your motivations.

Question 1

(the simplex method)

(2p) a) Rewrite the problem into standard form by letting $x_1 := x_1^+ - x_1^-$ and adding slack variables s_1 and s_2 to the left-hand side in the first and second constraint, respectively. Moreover, let $z := -z$ to get the problem on minimization form. Thus, we get the following linear program:

$$\begin{array}{lrcrcrcrcrl}
\text{minimize} & z = -x_1^+ & + & x_1^- & - & 4x_2, & & & & & \\
\text{subject to} & x_1^+ & - & x_1^- & + & 3x_2 & + & s_1 & & & = 8, \\
& 2x_1^+ & - & 2x_1^- & + & x_2 & & & - & s_2 & = 4, \\
& x_1^+, & & x_1^-, & & x_2, & & s_1, & & s_2 & \geq 0.
\end{array}$$

Introducing the artificial variable a, phase I gives the problem

$$\begin{array}{lrcrcrcrcrcrl}
\text{minimize} & w = a, & & & & & & & & & & & \\
\text{subject to} & x_1^+ & - & x_1^- & + & 3x_2 & + & s_1 & & & & & = 8, \\
& 2x_1^+ & - & 2x_1^- & + & x_2 & & & - & s_2 & + & a & = 4, \\
& x_1^+, & & x_1^-, & & x_2, & & s_1, & & s_2, & & a & \geq 0.
\end{array}$$

Using the starting basis $(s_1, a)^T$ gives

$$\boldsymbol{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{N} = \begin{pmatrix} 1 & -1 & 3 & 0 \\ 2 & -2 & 1 & -1 \end{pmatrix}, \boldsymbol{x}_B = \begin{pmatrix} 8 \\ 4 \end{pmatrix}, \boldsymbol{c}_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \boldsymbol{c}_N = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The reduced costs, $\bar{\boldsymbol{c}}_N^T = \boldsymbol{c}_N^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{N}$, for this basis is $\bar{\boldsymbol{c}}_N^T = \begin{pmatrix} -2, & 2, & -1, & 1 \end{pmatrix}$, which means that x_1^+ enters the basis. The minimum ratio test implies that a leaves.

Thus, we move on to phase II using the basis $(s_1, x_1^+)^T$, and

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \boldsymbol{N} = \begin{pmatrix} -1 & 3 & 0 \\ -2 & 1 & -1 \end{pmatrix}, \boldsymbol{x}_B = \begin{pmatrix} 6 \\ 2 \end{pmatrix}, \boldsymbol{c}_B = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \boldsymbol{c}_N = \begin{pmatrix} 1 \\ -4 \\ 0 \end{pmatrix}.$$

The new reduced costs are $\bar{\boldsymbol{c}}_N^T = \begin{pmatrix} 0, & -3.5, & -0.5 \end{pmatrix}$ which means that x_2 enters the basis. The minimum ratio test implies that s_1 leaves.

Updating the basis, now with $(x_1^+, x_2)^T$, gives

$$\boldsymbol{B} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}, \boldsymbol{N} = \begin{pmatrix} -1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix}, \boldsymbol{x}_B = \begin{pmatrix} 0.8 \\ 2.4 \end{pmatrix}, \boldsymbol{c}_B = \begin{pmatrix} -1 \\ -4 \end{pmatrix}, \boldsymbol{c}_N = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The new reduced costs are $\bar{\mathbf{c}}_N^T = \begin{pmatrix} 0, & 1.4, & 0.2 \end{pmatrix}$. Since the reduced costs are all non-negative, the current BFS is optimal. Returning to the original variables, we obtain $(x_1, x_2) = (0.8, 2.4)^T$ as the optimal solution and -10.4 as the optimal value.

(1p) b) In the optimal BFS, the reduced cost corresponding to x_1^- is zero. Therefore, we can let x_1^- enter the basis without changing the objective. We do not obtain any leaving variable as minimum ratio implies that the problem is unbounded in that direction. This is simply the increasing x_1^+ and increasing x_1^- by the same amount (which can be any positive number). So the problem in standard form does not have a unique optimal solution. But the problem formulated in the original variables does since all these solutions correspond to $(8, 0)^T$, that is due to the reduced cost for x_2 and s_1 is positive. Replacing one free variable with two positive variables always implies that each solution is non-unique in the sense described above.

(3p) Question 2

(necessary local and sufficient global optimality conditions)

See Propositions 4.22 and 4.23 in the book.

Question 3

(Unconstrained optimization)

(2p) a) Set $f(\mathbf{x}) = x_1^2 + 6x_1x_2 + x_2^2$, the Hessian is $\nabla^2 f(\mathbf{x}) = \begin{bmatrix} 2 & 6 \\ 6 & 2 \end{bmatrix}$. Since the multiplier $\gamma = 6$, we get the new matrix $\nabla^2 f(\mathbf{x}) + \gamma \mathbf{I}^2 = \begin{bmatrix} 8 & 6 \\ 6 & 8 \end{bmatrix}$. By $[\nabla^2 f(\mathbf{x}_k) + \gamma_k \mathbf{I}^2]\mathbf{p}_k = -\nabla f(\mathbf{x}_k)$, we can get the search direction $\mathbf{p}_0$ for $\mathbf{x}_0$ is $(-4, -4)^T$.To determine the step length, we perform Armijo line search. Start from $\alpha = 1$, $f(\mathbf{x}_0 + \alpha \mathbf{p}_0) - f(\mathbf{x}_0) \leq \mu\alpha\nabla f(\mathbf{x}_o)^T \mathbf{p}_k$ is not fulfilled. Then take $\alpha = 1/2$, the inequality is fulfilled. So $\mathbf{x}_1 = \mathbf{x}_0 + \alpha \mathbf{p}_0 = (5, 5)^T$.

(1p) b) Let $x_1 = -\lambda$ and $x_2 = \lambda$, when $\lambda \to \infty$, the objective function value will go to $-\infty$, which means the problem is unbounded, so there is no global optimal. So Newtons Levenberg-Marquardt method can't converge to a global optimal.

Question 4

(KKT conditions)

(1p) a) Affine CQ doesn't hold since by definition it only applies to affine constraints.

(1p) b) Slater CQ requires inequality constraints to be level sets of convex functions $g(x) \leq 0$. However, $g = 25 - (x_1 + 3)^2 - (x_2 - 4)^2$ is a strictly concave function, this since the hessian, $\nabla^2 g(x) = -2\mathbf{I}$, is negative definte.

(1p) c) Three constraints are active in the point $\bar{\mathbf{x}}^{\mathrm{T}} = (0, 0)$, hence the gradients of these constraints should be linearly independent for LICQ to be satisfied. But this is impossible since the dimension of the space is two and as the number of active constraints is three. Could also be verified by computing the gradients $\nabla g_1(\bar{\mathbf{x}})^{\mathrm{T}} = (-2, 0)$, $\nabla g_2(\bar{\mathbf{x}})^{\mathrm{T}} = (6, -8)$, and $\nabla g_3(\bar{\mathbf{x}})^{\mathrm{T}} = (6, 8)$. Then see that $6g_1(\bar{\mathbf{x}}) + g_2(\bar{\mathbf{x}}) + g_3(\bar{\mathbf{x}}) = \mathbf{0}$, i.e., they are linearly dependent and thus viloating LICQ.

(3p) Question 5

(modelling) Sets, let

- $\mathcal{K} = \{\texttt{mb}, CPU, GPU, PSU, \texttt{hd}\}$, be the set of components.
- $\mathcal{K}_- = \mathcal{K} \setminus \{\texttt{mb}\}$.
- $\mathcal{M}_k$ be the set of acceptable variants for component $k \in \mathcal{K}$.
- $\mathcal{P}_{mk} \subseteq \mathcal{M}_k$ be the set of variants of component $k \in \mathcal{K}_-$, incompatible with motherboard $m \in \mathcal{M}_{\texttt{mb}}$.

Parameters, let

- c_{km} be the cost of model $m \in \mathcal{M}_k$ of component $k \in \mathcal{K}$.
- p_{km} be the power of model $m \in \mathcal{M}_k$ of component $k \in \mathcal{K}$. (PSU has negative power)

Variables, let

- x_{km} be the binary choice of buying model $m \in \mathcal{M}_k$ of component $k \in \mathcal{K}$.

$$\begin{aligned}
\text{minimize} \quad & \sum_{k\in\mathcal{K}} \sum_{m\in\mathcal{M}_k} c_{km} x_{km}, && && (1)\\
\text{s.t.} \quad & \sum_{k\in\mathcal{K}} \sum_{m\in\mathcal{M}_k} p_{km} x_{km} \leq 0, && && (2)\\
& x_{\texttt{mb},m} + x_{kl} \leq 1, && l \in \mathcal{P}_{mk},\ m \in \mathcal{M}_{\texttt{mb}},\ k \in \mathcal{K}_-, && (3)\\
& \sum_{m\in\mathcal{M}_k} x_{km} = 1, && k \in \mathcal{K}, && (4)\\
& x_{km} \in \{0,1\}, && m \in \mathcal{M}_k, k \in \mathcal{K}. && (5)
\end{aligned}$$

(1) mininmize the costs, (2) must have excess power, (3) cannot choose a model incompatible with the MB, (4) each component needs to be installed, and (5) the choices are binary.

Question 6

(true or false)

(1p) a) False, the penalty goes to zero; allowing the optimal solution to be arbitrarily close to the boundary of the feasible set. Letting the penalty go to infinity would imply the original objective function redudant.

(1p) b) False, take two points $x_1 = (-2, 0)^T$ and $x_2 = (0, 5)^T$, then x_1 and x_2 are in S. But one convex combination of x_1 and x_2: $1/2 * (x_1 + x_2) = (-1, 5/2)$ is not in S, since $5/2 * sin(-1) < 0$.

(1p) c) False, complementary slackness conditions are not fulfilled ($\mu_2^* g_2(x^*) = -2 \neq 0$).

(3p) Question 7

linear programming duality Suppose, for example, that X is bounded. Then, there exists a bounded optimal solution for every value of the coefficient vector $\boldsymbol{c}$. Therefore, its dual must also have bounded optimal solutions for every value of $\boldsymbol{c}$. It then follows that the dual problem must have feasible solutions for every $\boldsymbol{c}$. Consider the cone

$$C := \{ \boldsymbol{y} \in \mathbb{R}^m \mid \boldsymbol{A}^{\mathrm{T}} \boldsymbol{y} \leq \boldsymbol{0}, \quad \boldsymbol{y} \geq \boldsymbol{0} \}.$$

By the Representation Theorem, the set Y is bounded if and only if C contains only the zero vector. Since the dual problem must have feasible solutions for every $\boldsymbol{c}$, choose $\boldsymbol{c} = -\boldsymbol{e}$, where $\boldsymbol{e}$ if the m-vector of ones. Then we have that the set

$$\{ \boldsymbol{y} \in \mathbb{R}^m \mid \boldsymbol{A}^{\mathrm{T}} \boldsymbol{y} \leq -\boldsymbol{e}, \quad \boldsymbol{y} \geq \boldsymbol{0} \},$$

is non-empty. Clearly, any of its members are non-zero, and moreover they belong to the cone C. Hence, C does not only contain the zero vector, ans so Y is unbounded.

The case when one assumes that Y is bounded is treated similarly.
