Chalmers/GU
Mathematics

**EXAM**

# TMA947/MMG621
# NONLINEAR OPTIMISATION

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| **Date:** | 18–08–21 |
| **Time:** | $8^{30}$–$13^{30}$ |
| **Aids:** | Text memory-less calculator, English–Swedish dictionary |
| **Number of questions:** | 7; passed on one question requires 2 points of 3. |
| | Questions are *not* numbered by difficulty. |
| | To pass requires 10 points and three passed questions. |
| **Examiner:** | Michael Patriksson |
| **Teacher on duty:** | Michael Patriksson, tel. 709-581812 |
| **Result announced:** | 18–09–09 |
| | Short answers are also given at the end of the exam on the notice board for optimization in the MV building. |

## Exam instructions

### When you answer the questions

*Use generally valid theory and methods.*
*State your methodology carefully.*

*Only write on one page of each sheet. Do not use a red pen.*
*Do not answer more than one question per page.*

### At the end of the exam

*Sort your solutions by the order of the questions.*
*Mark on the cover the questions you have answered.*
*Count the number of sheets you hand in and fill in the number on the cover.*

## Question 1

(the simplex method)

Consider the following linear program:

$$\begin{array}{lrl} \text{maximize} & z = & 3x_1 + 2x_2, \\ \text{subject to} & & 2x_1 + 3x_2 \leq 1, \\ & & x_1 - x_2 \geq 4, \\ & & x_1, \quad x_2 \geq 0. \end{array}$$

**(2p)** a) Solve the problem using phase I and phase II of the simplex method.

Aid: You may utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

**(1p)** b) If an optimal solution exists, then use your calculations to decide whether it is unique or not. If the problem is unbounded, then use your calculations to specify a direction of unboundedness of the objective value.

## Question 2

Suppose we have an LP problem (the primal one)

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & Ax \geq b, \\ & x \geq 0^n. \end{aligned}$$

**(1p)** State the LP dual problem.

**(2p)** Suppose that this dual problem is feasible. Prove, or disprove, whether the primal problem has an optimal solution, or not. Motivate clearly.

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## Question 3

(feasible direction methods)

Consider the problem to

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := (x_1 - 1/2)^2 + x_2^2, \\ \text{subject to} \quad & 0 \leq x_1 \leq 1, \\ & 0 \leq x_2 \leq 1. \end{aligned}$$

**(2p)** Draw the first two iterates obtained using the Frank-Wolfe algorithm starting in $\boldsymbol{x}_0 = (0, 1)^{\mathrm{T}}$.

**(1p)** Draw the first two iterates obtained using the simplicial decomposition algorithm starting in $\boldsymbol{x}_0 = (0, 1)^{\mathrm{T}}$.

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## (3p) Question 4

(on the SQP algorithm and the KKT conditions)

Consider the following nonlinear programming problem: find $\boldsymbol{x}^* \in \mathbb{R}^n$ that solves the problem to

$$\begin{aligned} \text{minimize} \;\; & f(\boldsymbol{x}), && && \text{(1a)} \\ \text{subject to} \;\; & g_i(\boldsymbol{x}) \leq 0, && i = 1, \dots, m, && \text{(1b)} \\ & h_j(\boldsymbol{x}) = 0, && j = 1, \dots, \ell, && \text{(1c)} \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g_i$, and $h_j : \mathbb{R}^n \to \mathbb{R}$ are given functions in $C^1$ on $\mathbb{R}^n$.

We are interested in establishing that the classic SQP subproblem tells us whether an iterate $\boldsymbol{x}_k \in \mathbb{R}^n$ satisfies the KKT conditions or not, thereby establishing a natural termination criterion for the SQP algorithm.

Given the iterate $\boldsymbol{x}_k$, the SQP subproblem is to

$$\begin{aligned} \underset{p}{\text{minimize}} \;\; & \frac{1}{2}\boldsymbol{p}^{\mathrm{T}}\boldsymbol{B}_k\boldsymbol{p} + \nabla f(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p}, && && \text{(2a)} \\ \text{subject to} \;\; & g_i(\boldsymbol{x}_k) + \nabla g_i(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p} \leq 0, && i = 1, \dots, m, && \text{(2b)} \\ & h_j(\boldsymbol{x}_k) + \nabla h_j(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p} = 0, && j = 1, \dots, \ell, && \text{(2c)} \end{aligned}$$

where the matrix $\boldsymbol{B}_k \in \mathbb{R}^{n \times n}$ is symmetric and positive semi-definite.

Establish the following statement: the vector $\boldsymbol{x}_k$ is a KKT point in the problem (1) if and only if $\boldsymbol{p} = \boldsymbol{0}^n$ is a globally optimal solution to the SQP subproblem (2). In other words, the SQP algorithm terminates if and only if $\boldsymbol{x}_k$ is a KKT point.

*Hint:* Compare the KKT conditions of (1) and (2).

## (3p) Question 5

(modelling)

A small municipality is forced to close one or several schools. Out of ten existing schools, at most three schools can be closed. The annual cost to keep school $i$ open is $c_i$ kr, where $i = 1, 2, \dots, 10$. School $i$ can educate a maximum of $k_i$ students.

The municipality is divided into $J$ home areas and there is a requirement that all students in an area belong to the same school. There are $b_j$ students in area $j$ and the average distance from area $j$ to school $i$ is $d_{ij}$ km. The estimated annual cost for student travels is set to $m$ kr per km and student.

Formulate a *linear integer program* to decide on which schools to keep and which ones to close, such that we minimize the total cost for schools and travels and fulfill the above listed requirement

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## Question 6

(true or false)

Indicate for each of the following three statements whether it is true or false. Motivate your answers!

**(1p)** a) For the phase I (when a BFS is *not* known a priori) problem of the simplex algorithm, the optimal value is always zero.

**(1p)** b) Suppose that the function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is continuously differentiable on $\mathbb{R}^n$ and let $\boldsymbol{G}$ be a symmetric and positive definite matrix of dimension $n \times n$. Then, if $\nabla f(\boldsymbol{x}) \neq \boldsymbol{0}^n$ and the vector $\boldsymbol{d}$ fulfils $\boldsymbol{G}\boldsymbol{d} = -\nabla f(\boldsymbol{x})$ it holds that $f(\boldsymbol{x} + t\boldsymbol{d}) < f(\boldsymbol{x})$ for small enough values of $t > 0$.

**(1p)** c) If the function $g : \mathbb{R}^n \mapsto \mathbb{R}$ is concave on $\mathbb{R}^n$ and $c \in \mathbb{R}$, then the set $\{\, \boldsymbol{x} \in \mathbb{R}^n \mid g(\boldsymbol{x}) \leq c \,\}$ is convex.

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## (3p) Question 7

(the gradient projection algorithm)

The gradient projection algorithm is a generalization of the steepest descent method to problems over convex sets. Given a feasible point $\boldsymbol{x}_k$, the next point is obtained according to $\boldsymbol{x}_{k+1} = \text{Proj}_X\left(\boldsymbol{x}_k - \alpha_k \nabla f(\boldsymbol{x}_k)\right)$, where $X$ is the convex set over which we minimize, $\alpha_k > 0$ is the step length, and $\text{Proj}_X(\boldsymbol{y}) = \arg\min_{\boldsymbol{x}\in X} ||\boldsymbol{x} - \boldsymbol{y}||$ denotes the closest point to $\boldsymbol{x}$ in $X$.

Consider the problem to

$$
\begin{aligned}
\text{minimize} \quad & f(\boldsymbol{x}) := x_1^2 + 2x_2^2 - 2x_1x_2 - 2x_1 - 3x_2, \\
\text{subject to} \quad & 0 \leq x_1 \leq 3, \\
& 0 \leq x_2 \leq 2.
\end{aligned}
$$

Start at the point $\boldsymbol{x}_0 = (0, 0)^{\mathrm{T}}$ and perform two iterations of the gradient projection algorithm using step length $\alpha_k = 1$ for all $k$. You may solve the projection problem in the algorithm graphically. Is the point obtained a global/local minimum? Motivate why/why not.

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**EXAM SOLUTION**

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# NONLINEAR OPTIMISATION

**Date:** 18–08–21
**Examiner:** Michael Patriksson

Note: the solutions presented here are brief in relation to the requirements on your answers, in particular regarding your motivations.

**(3p)** 

## Question 1

(the simplex method)

Rewrite the problem into standard form by and adding/subtracting slack variables $s_1$ and $s_2$ to the left-hand side in the first and second constraint, respectively. Moreover, let $z := -z$ to get the problem on minimization form. Thus, we get the following linear program:

$$\begin{array}{llrrrrl} \text{minimize} & z = & -3x_1 & - \ 2x_2, & & & \\ \text{subject to} & & 2x_1 & + \ 3x_2 & + s_1 & & = 1, \\ & & x_1 & - \ x_2 & & - s_2 & = 4, \\ & & x_1, & x_2, & s_1, & s_2 & \geq 0. \end{array}$$

Introducing the artificial variable $a$, phase I gives the problem

$$\begin{array}{llrrrrrl} \text{minimize} & w = & a, & & & & & \\ \text{subject to} & & 2x_1 & + \ 3x_2 & + s_1 & & & = 1, \\ & & x_1 & - \ x_2 & & - s_2 & + a & = 4, \\ & & x_1, & x_2, & s_1, & s_2, & a & \geq 0. \end{array}$$

Using the starting basis $(s_1, a)^T$ gives

$$\boldsymbol{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{N} = \begin{pmatrix} 2 & 3 & 0 \\ 1 & -1 & -1 \end{pmatrix}, \boldsymbol{x}_B = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \boldsymbol{c}_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \boldsymbol{c}_N = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The reduced costs, $\bar{\boldsymbol{c}}_N^T = \boldsymbol{c}_N^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{N}$, for this basis is $\bar{\boldsymbol{c}}_N^T = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix}$, which means that $x_1$ enters the basis. The minimum ratio test implies that $s_1$ leaves.

Updating the basis we now have $(x_1, a)^T$ in the basis. Calculating the reduced costs, we obtain $\bar{\boldsymbol{c}}_N^T = \begin{pmatrix} 5/2 & 1/2 & 1 \end{pmatrix}$, meaning that the current basis is optimal. The optimal solution is thus $a^* = 7/2$, since $a^* > 0$, so the original problem is infeasble.

Since the original problem is infeasible, so it is neither existing an optimal solution nor unbounded.

## Question 2

**(1p)** a) The LP dual problem is to:

$$\begin{aligned} \text{miximize} \quad & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}, \\ \text{subject to} \quad & A^{\mathrm{T}}y \leq c, \\ & y \quad \geq 0^m. \end{aligned}$$

**(2p)** b) If the dual problem has a finite optimal solution, then so does the primal problem. If the dual problem is unbounded, then the primal problem is infeasible. See Theorem 10.6 (Strong Duality Theorem).

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## Question 3

(feasible direction methods)

**(2p)** a) For the Frank-Wolfe algorithm, $y_1 = (1,0)^{\mathrm{T}}$, $x_1 = (0,1)^{\mathrm{T}}$, $y_2 = (0,0)^{\mathrm{T}}$, $x_2 = (9/20, 3/20)^{\mathrm{T}}$.

**(1p)** b) For the simplicial decomposition algorithm, $P_0 = \emptyset$, $y_1 = (1,0)^{\mathrm{T}}$, $P_1 = (1,0)^{\mathrm{T}}$, $x_1 = (3/4, 1/4)^{\mathrm{T}}$,$y_2 = (0,0)^{\mathrm{T}}$, $P_2 = (1,0)^{\mathrm{T}} \bigcup (0,0)^{\mathrm{T}}$, $x_2 = (1/2, 0)^{\mathrm{T}}$,

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**(3p)** **Question 4**

(on the SQP algorithm and the KKT conditions)

The result is based on a comparison between the KKT conditions of the original problem,

$$\begin{aligned} \text{minimize } & f(\boldsymbol{x}), && && \text{(1a)} \\ \text{subject to } & g_i(\boldsymbol{x}) \leq 0, && i = 1, \dots, m, && \text{(1b)} \\ & h_j(\boldsymbol{x}) = 0, && j = 1, \dots, \ell, && \text{(1c)} \end{aligned}$$

and those of the SQP subproblem,

$$\begin{aligned} \underset{p}{\text{minimize }} & \frac{1}{2}\boldsymbol{p}^{\mathrm{T}}\boldsymbol{B}_k\boldsymbol{p} + \nabla f(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p}, && && \text{(2a)} \\ \text{subject to } & g_i(\boldsymbol{x}_k) + \nabla g_i(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p} \leq 0, && i = 1, \dots, m, && \text{(2b)} \\ & h_j(\boldsymbol{x}_k) + \nabla h_j(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p} = 0, && j = 1, \dots, \ell. && \text{(2c)} \end{aligned}$$

We first note that the latter problem is a convex one (the matrix $\boldsymbol{B}_k$ was assumed to be positive semidefinite), and that the solution $\boldsymbol{p}_k$ is characterized by its KKT conditions, since the constraints are linear (so that Abadie's CQ is fulfilled). It remains to compare the two problems' KKT conditions. With $\boldsymbol{p}_k = \boldsymbol{0}^n$ they are in fact identical!

**(3p)**

## Question 5

(modelling)

Sets:

$I := \{1, ..., 10\}$, the set of schools.

The decision variables are:

$$x_i = \begin{cases} 1 & \text{keep school } i, \\ 0 & \text{otherwise,} \end{cases}$$

where $i \in I$.

$$y_{ij} = \begin{cases} 1 & \text{home area } j \text{ go to school } i, \\ 0 & \text{otherwise,} \end{cases}$$

where $i \in I$, $j \in J$.

Model:

$$\begin{array}{lrll}
\text{minimize} & x_i c_i + m b_j d_{ij}, & & \\
\text{subject to} & \sum_{i \in I} x_i \leq 9, & & \\
 & \sum_{i \in I} x_i \geq 7, & & \\
 & \sum_{j \in J} b_j y_{i,j} \leq k_i, & & i \in I, \\
 & y_{i,j} \leq x_i, & & i \in I, j \in J, \\
 & \sum_{i \in I} y_{i,j} = 1, & & j \in J, \\
 & x_i \in \{0,1\}, & & i \in I, \\
 & d_{ij} \in \{0,1\}, & & i \in I, j \in J.
\end{array}$$

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## Question 6

(true or false)

**(1p)** a) False. The original problem can be infeasible, which means the optimal value for phase I is higher than 0, like question 1 in this exam.

**(1p)** b) True. Since $\nabla f(\boldsymbol{x}) \neq \mathbf{0}^n$, and $\boldsymbol{G}$ is a symmetric and positive definite matrix of dimension $n \times n$, we have that $\nabla f(\boldsymbol{x})^{\mathrm{T}} \boldsymbol{d} = -\nabla f(\boldsymbol{x})^{\mathrm{T}} \boldsymbol{G}^{-1} \nabla f(\boldsymbol{x}) < 0$, so $\boldsymbol{d}$ is a decent direction. By defination of decent direction, the clam is true.

**(1p)** c) False. For example, $g(\boldsymbol{x}) = -x^2$ is concave, but $\{ \boldsymbol{x} \in \mathbb{R}^n \mid g(\boldsymbol{x}) \leq -1 \}$ is not convex.

**(3p)** **Question 7**

(the gradient projection algorithm)

$\nabla f(\boldsymbol{x}) = (2x_1 - 2x_2 - 2, 4x_2 - 2x_1 - 3)^{\mathrm{T}}$, $x_0 - \alpha_k \nabla f(\boldsymbol{x}_0) = (2, 3)^{\mathrm{T}}$, $x_1 = (2, 2)^{\mathrm{T}}$, $x_1 - \alpha_k \nabla f(\boldsymbol{x}_1) = (4, 1)^{\mathrm{T}}$, $x_2 = (3, 1)^{\mathrm{T}}$.

Since the feasible set is convex, there exists an interior point, so the Slater CQ holds. Since it is a convex problem, so the KKT conditions are both necessary and sufficient. $(3, 1)^{\mathrm{T}}$ is not a KKT point, so it is neither a global nor a local minimum.