Chalmers/GU
Mathematics

EXAM

TMA947/MMG621 NONLINEAR OPTIMISATION

Date:	17–01–10
Time:	8^{30}–13^{30}
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Michael Patriksson, tel. 709-581812
Result announced:	17–01–30 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$\begin{aligned}
\text{maximize} \quad & z = 5x_1 + 4x_2, \\
\text{subject to} \quad & x_1 \leq 7, \\
& x_1 - x_2 \leq 8. \\
& x_1, \quad x_2 \geq 0.
\end{aligned}$$

(2p) a) Solve the problem using phase I and phase II of the simplex method. If the problem has an optimal solution, then present the optimal solution in both the original variables and in the variables used in the standard form. If the problem is unbounded, then use your calculations to find a direction of unboundness in both the original variables and in the variables in the standard form.
Aid: Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(1p) b) Add a constraint to the linear program considered to obtain a uniquely solvable linear program. Present the optimal solution.

(3p)

Question 2

(finiteness of the simplex algorithm)

Establish the following statement: "If all of the basic feasible solutions are non-degenerate, then the simplex algorithm terminates after a finite number of iterations."

Further, if there exists an optimal solution to the problem, establish that the last iterate is an optimal one.

Question 3

(LP duality)

Consider the linear integer program

$$\begin{aligned} z^*_{IP} := \quad \min_{\boldsymbol{x}} \quad & \mathbf{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \\ & \boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{d}, \\ & \boldsymbol{x} \in \{0,1\}^n. \end{aligned} \tag{1}$$

Assume that problem (1) is feasible. Let $\{\boldsymbol{x} \mid \boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{d},\ \boldsymbol{x} \in \{0,1\}^n\} = \{\boldsymbol{x}^1, \dots, \boldsymbol{x}^N\}$. Consider the Lagrange dual problem

$$\begin{aligned} z^*_{LD} := \quad \max_{\boldsymbol{\mu}} \quad & q(\boldsymbol{\mu}), \\ \text{subject to} \quad & \boldsymbol{\mu} \geq \mathbf{0}, \end{aligned}$$

where

$$\begin{aligned} q(\boldsymbol{\mu}) := \quad \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{\mu}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}), \\ \text{subject to} \quad & \boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{d}, \\ & \boldsymbol{x} \in \{0,1\}^n. \end{aligned}$$

(1p) a) Show that z^*_{LD} is the optimal objective value of the following problem

$$\begin{aligned} \max_{y,\boldsymbol{\mu}} \quad & y \\ \text{subject to} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^i + \boldsymbol{\mu}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{x}^i - \boldsymbol{b}) \geq y, \quad \forall i = 1, \dots, N \\ & y \in \mathbb{R},\ \boldsymbol{\mu} \geq \mathbf{0} \end{aligned} \tag{2}$$

(1p) b) Show that problem (2) and the following problem have the same optimal objective value

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \in \operatorname{conv}\Big(\big\{\boldsymbol{x} \mid \boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{d},\ \boldsymbol{x} \in \{0,1\}^n\big\}\Big) \end{aligned}$$

(1p) c) Let z^*_{LP} denote the optimal objective value of the LP relaxation of (1), with the integrality constraint $\boldsymbol{x} \in \{0,1\}^n$ removed. Show that $z^*_{LP} \leq z^*_{LD} \leq z^*_{IP}$.

(3p) Question 4

(modelling)

We consider a stepped cantilever beam, which consists of five segments, as shown in Figure 1. Each segment has a rectangular cross-section with width b_i and height h_i to be designed. We assume that each section of the cantilever has the same length l. A vertical load P is applied at a fixed distance L from the support. This load causes the beam to deflect, and induces stress in each segment of the beam with Young's modulus E. Formulate an optimization model to minimize the volume of the beam, subject to constraints on bending stress in all five steps of the beam, to be less than an allowable stress $\sigma_{\max}$; the displacement constraint on the tip deflection to be less than the allowable deflection $\delta_{\max}$, and a specified aspect ration $a_{\max}$ to be maintained between the height and width of beam cross sections.

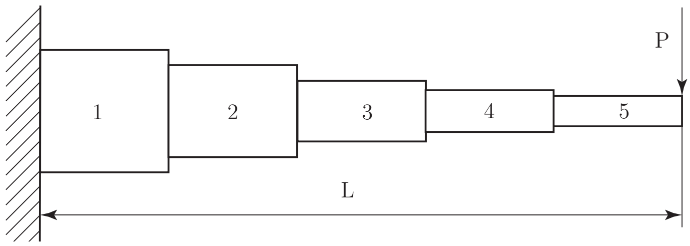

Figure 1: Stepped cantilever beam

Aid: The maximum bending stress at each segment of the beam is

$$\sigma_i = \frac{6PD_i}{b_i h_i^2},$$

where D_i is the maximum distance from the end load. The end deflection can be calculated using Castigliano's second theorem, which states that

$$\delta = \frac{\partial U}{\partial P},$$

where δ is the deflection of the beam, U is the energy stored in the beam due to the applied force P. The energy stored in a cantilever beam is given by

$$U = \int_0^L \frac{P^2 x^2}{2EI} \mathrm{d}x,$$

where I is the area moment of inertia. The moment of inertia of a beam segment with a rectangular cross-section is

$$I_i = \frac{b_i h_i^3}{12}.$$

Question 5

(true or false)

The below three individual claims should be assessed individually. Are they true or false, or is it impossible to say? For each of the three statements, provide an answer, together with a short—but complete— motivation.

(1p) a) Suppose we consider minimizing a function $f \in C^2$ over $\mathbb{R}^n$. Claim: all its stationary points have a positive semi-definite Hessian (i.e., matrix of second-order partial derivatives).

(1p) b) Consider the minimization of a continuous function $f : \mathbb{R}^n \to \Re$ over constraints of the form $g_i(\boldsymbol{x}) \leq 0$, $i = 1, 2, \ldots, m$, defined by the functions $g_i : \mathbb{R}^n \to \Re$. Derive the Lagrangian dual problem for this problem. Claim: the Lagrangian dual problem is a convex one.

(1p) c) Claim: In an optimization problem, a global optimum cannot be a local one.

(3p)
Question 6

(optimality conditions)

Prove that $\boldsymbol{x}^* = (1, 1/2, -1)^{\mathrm{T}}$ is optimal for the optimization problem

$$\begin{aligned} \text{minimize} \quad & z = (1/2)\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x} + \boldsymbol{q}^{\mathrm{T}}\boldsymbol{x} + r, \\ \text{subject to} \quad & -1 \leq x_i \leq 1, \quad i = 1, 2, 3, \end{aligned}$$

where

$$\boldsymbol{P} = \begin{pmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{pmatrix}, \quad \boldsymbol{q} = \begin{pmatrix} -22.0 \\ -14.5 \\ 13.0 \end{pmatrix}, \quad r = 1.$$

(3p) Question 7

(the basis of the SQP algorithm)

Consider the problem to minimize a function $f \in C^2$ over a set of equality constraints of the form $h_j(\boldsymbol{x}) = 0$, $j = 1, 2, \ldots, \ell$, where all functions h_j also are in C^2. Derive and motivate the subproblem of this algorithm.

Hint: utilize the standard optimality conditions for the original problem.

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EXAM SOLUTION

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NONLINEAR OPTIMISATION

Date: 17–01–10
Examiner: Michael Patriksson

Question 1

(the simplex method)

(2p) a) We first rewrite the problem on standard form by introducing slack variables s_1 and s_2. Consider the following linear program:

$$\begin{array}{lrrrrl} \text{minimize} & -5\,x_1 & -4x_2 & & & \\ \text{subject to} & x_1 & & +s_1 & & =7, \\ & x_1 - & x_2 & & +s_2 & =8, \\ & x_1, & x_2, & s_1, & s_2 & \geq 0. \end{array}$$

The starting basis is $(s_1, s_2)^{\mathrm{T}}$. The reduced costs for the non-basic variables x_1 and x_2 are $\tilde{\mathbf{c}}_N = (-5, -4)^{\mathrm{T}}$, meaning that x_1 enters the basis. From the minimum ratio test, we get that s_1 leaves the basis.

Updating the basis we now have $(x_1, s_2)^{\mathrm{T}}$ in the basis. Calculating the reduced costs, we obtain $\tilde{\mathbf{c}}_N = (5, -4)^{\mathrm{T}}$, meaning that x_2 enters basis. From the minimum ratio test we get that $\mathbf{B}^{-1}\mathbf{N}_{x_2} = (0, -2)^{\mathrm{T}} \leq \mathbf{0}$, meaning that the problem is unbounded. The direction of unboundness is $\mathbf{p} = (x_1, x_2, s_1, s_2) = (0, 1, 0, 2)^{\mathrm{T}}$ and $z \to \infty$ along the half-line $l(\mu) = (7, 0, 0, 8)^{\mathrm{T}} + \mu(0, 1, 0, 2)^{\mathrm{T}}$, $\mu \geq 0$.

(1p) b) For example $-x_1 + x_2 = 0$ can be added to get a uniquely solvable linear program. The optimal solution is then $\mathbf{x}^* = (7, 7, 0, 0)^T$ and $z^* = 63$.

(3p)
Question 2

(finiteness of the simplex algorithm)

Theorem 9.11 establishes the finite termination of the simplex method. The termination criterion is equivalent to the optimality conditions for the LP.

Question 3

(LP duality)

(1p) a) Since $q(\boldsymbol{\mu}) = \min\limits_{i=\{1,\dots,N\}} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^i + \boldsymbol{\mu}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{x}^i - \boldsymbol{b})$, the dual (maximization) problem can be written as

$$\begin{aligned} \max_{\boldsymbol{\mu}} \quad & \min_{i \in \{1,\dots,N\}} \ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^i + \boldsymbol{\mu}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{x}^i - \boldsymbol{b}) \\ \text{subject to} \quad & \boldsymbol{\mu} \geq \mathbf{0} \end{aligned}$$

This is equivalent to (2) in the problem statement.

(1p) b) The LP dual of (2) in the problem statement is

$$\begin{aligned} \min_{\boldsymbol{\nu}} \quad & \sum_{i=1}^{N} \nu_i(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^i) \\ \text{subject to} \quad & \sum_{i=1}^{N} \nu_i(\boldsymbol{A}\boldsymbol{x}^i - \boldsymbol{b}) \leq 0 \\ & \sum_{i=1}^{N} \nu_i = 1, \quad \boldsymbol{\nu} \geq \mathbf{0} \end{aligned}$$

This problem is equivalent to

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \in \operatorname{conv}\Big(\Big\{\boldsymbol{x} \mid \boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{d},\ \boldsymbol{x} \in \{0,1\}^n\Big\}\Big) \end{aligned} \tag{a}$$

because $\boldsymbol{x} \in \operatorname{conv}\Big(\Big\{\boldsymbol{x} \mid \boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{d},\ \boldsymbol{x} \in \{0,1\}^n\Big\}\Big)$ if and only if $\boldsymbol{x} = \sum_{i=1}^{N} \nu_i \boldsymbol{x}^i$ for some $\boldsymbol{\nu} \geq \mathbf{0}$, $\sum_{i=1}^{N} \nu_i = 1$. Problem (2) in the statement is feasible (e.g., $\boldsymbol{\mu} = \mathbf{0}$ and $y = \min_i \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^i$). In addition, the feasibility of (1) in the problem statement (i.e., the original integer program) implies that the dual of (2) in the problem statement is feasible. Hence, linear programming strong duality implies that the optimal objective values of (a) and (2) in the problem statement are the same.

(1p) c) If $\boldsymbol{x} \in \operatorname{conv}\Big(\Big\{\boldsymbol{x} \mid \boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{d},\ \boldsymbol{x} \in \{0,1\}^n\Big\}\Big)$ then $\boldsymbol{x}$ satisfies $\boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{d}$. Hence, the feasible set of (a) is included in the feasible set of the LP relaxation of (1) in the problem statement. Hence, $z^*_{LP} \leq z^*_{LD}$. Finally, the inequality $z^*_{LD} \leq z^*_{IP}$ is due to weak duality.

(3p) **Question 4**

(modelling)

The decision variables are:

b_i, $i = 1, \dots, 5$ width of segment i

h_i, $i = 1, \dots, 5$ height of segment i

Model

$$
\begin{aligned}
\text{minimize} \quad & l \sum_{i=1}^{5} b_i h_i, \\
\text{subject to} \quad & \frac{6Pl}{b_5 h_5^2} \leq \sigma_{\max}, \\
& \frac{6P(2l)}{b_4 h_4^2} \leq \sigma_{\max}, \\
& \frac{6P(3l)}{b_3 h_3^2} \leq \sigma_{\max}, \\
& \frac{6P(4l)}{b_2 h_2^2} \leq \sigma_{\max}, \\
& \frac{6P(5l)}{b_1 h_1^2} \leq \sigma_{\max}, \\
& \frac{Pl^3}{E}\left(\frac{244}{b_1 h_1^3} + \frac{148}{b_2 h_2^3} + \frac{76}{b_3 h_3^3} + \frac{28}{b_4 h_4^3} + \frac{4}{b_5 h_5^3}\right) \leq \delta_{\max}, \\
& \frac{h_i}{b_i} \leq a_{\max}, \; i = 1, \dots, 5, \\
& h_i \geq 0, b_i \geq 0, \; i = 1, \dots, 5.
\end{aligned}
$$

Question 5

(true or false)

(1p) a) False: at a stationary point the Hessian may have a negative eigenvalue, corresponding to an eigenvector $\boldsymbol{p}$, resulting in $\boldsymbol{p}^{\mathrm{T}} \nabla^2 f(\boldsymbol{x}) \boldsymbol{p} < 0$. This vector hence is a descent direction.

(1p) b) True: this is Theorem 6.4.

(1p) c) False: a global optimum is - by definition - also a local one.

(3p) Question 6

(optimality conditions)

The feasible set is nonempty and convex (three-dimensional box), z is C^1 on the feasible set and convex since its hessian $\mathbf{P}$ is positive semidefinite ($\mathbf{P}$ is symmetric and the upper left 1-by-1 corner of $\mathbf{P}$, 2-by-2 corner of $\mathbf{P}$ and $\mathbf{P}$ itself have positive determinants. Then Sylvester's criteria establishes the positive definiteness of $\mathbf{P}$. Eigenvalues of $\mathbf{P}$ can be found approximately, e.g. by bisection, instead to establish the positive definiteness of $\mathbf{P}$.).

Now we need to verify variational inequality to establish the global optimality of $\mathbf{x}^*$. The gradient of the objective function at $\boldsymbol{x}^*$ is

$$\nabla z(\boldsymbol{x}^*) = (-1, 0, 2)^{\mathrm{T}}.$$

Therefore the variational inequality is that

$$\nabla z(\boldsymbol{x}^*)^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{x}) = -1(y_1 - 1) + 2(y_3 + 1) \geq 0$$

for all $\boldsymbol{y}$ satisfying $-1 \geq y_i \geq 1$, which is clearly true. So $\mathbf{x}^*$ is a global optimum of the problem considered (Theorem 4.23).

(3p) Question 7

(the basis of the SQP algorithm)

See equation (13.25) in the course book: the subproblem is equivalent to a second-order approximation of the KKT conditions of the original problem.
