Chalmers/GU
Mathematics

EXAM

TMA947/MMG621
OPTIMIZATION, BASIC COURSE

Date:	16–08–25
Time:	House V, morning, 8^{30}–13^{30}
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Carl Lundholm (5325)
Result announced:	15–09–18 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$\begin{aligned}
\text{maximize} \quad & z = 3x_1 + 5\,x_2, \\
\text{subject to} \quad & \phantom{3x_1 +{}} 2\,x_2 \le 12, \\
& 3x_1 + 2\,x_2 \le 18. \\
& x_1 \ge 0, \\
& x_2 \ge 0.
\end{aligned}$$

(2p) a) Solve the problem using phase I and phase II of the simplex method.
Aid: Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(1p) b) Without solving the dual to the problem above, motivate clearly whether there are no optimal dual solutions, a unique optimal dual solution (if so, present it) or multiple optimal dual solutions (if so, present at least two of them).

Question 2

(the KKT conditions)

Consider the problem to find

$$\begin{aligned}
f^* := \operatorname*{infimum}_{x} \quad & f(\boldsymbol{x}), \\
\text{subject to} \quad & g_i(\boldsymbol{x}) \le 0, \qquad i = 1, \ldots, m,
\end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$, are given differentiable functions.

(1p) a) State the KKT conditions regarding locally optimal solutions to this problem.

(1p) b) Assume that there are two locally optimal solutions, $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$, to the problem at hand. Suppose that the feasible set at $\boldsymbol{x}^1$ satisfies the linear independence constraint qualification (LICQ). Does the vector $\boldsymbol{x}^1$ satisfy the KKT conditions? Does the vector $\boldsymbol{x}^2$ satisfy the KKT conditions?

(1p) c) Assume instead that there are two vectors, $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$, both satisfying the KKT conditions. Assume also that these are the only KKT points. Suppose that the feasible set, at $\boldsymbol{x}^1$, satisfies the linear independence constraint qualification (LICQ). Further, assume that there exists at least one locally optimal solution to the given problem. In terms of local or global optimality, what can be said about the vectors $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$?

(3p) Question 3

(Lagrangian duality)

Consider the optimization problem

$$\begin{aligned} \underset{x_1,x_2}{\text{minimize}} \quad & -x_1 - 2x_2 \\ \text{subject to} \quad & x_1^2 + x_2^2 \leq 1, \\ & x_1 + 0.5x_2 \leq 1. \end{aligned}$$

Write down the dual function and the corresponding Lagrangian dual problem. Can we say something about the differentiability of the dual function and the convexity of the dual problem? What is the duality gap (explain your answer)? Find the optimal primal and dual solutions, if they exist.

(3p) Question 4

(modelling)

You are responsible for the planning of a soccer tournament where all 14 teams in the Swedish national league will participate. The teams shall be put into two groups of 7 each, in which all teams will play each other once. The winners of the two groups will then play a final. The decision to make is which teams will play in which group. The objective is to minimize the total traveling distance for the matches in the two groups, not including the final match. The distances between the home towns of two teams i and j are given by the constants $d_{ij}(= d_{ji})$,

$i, j \in \{1, ..., 14\}$. The constants p_i, $i \in \{1, ..., 14\}$, represent the number of points team i took in the national league last year. Assume that the teams are sorted so that the team with the highest point is represented by $i = 1$, the team with second highest point by $i = 2$, and so on. You are not allowed to put the two teams with the highest p_i:s (team 1 and team 2) in the same group. Neither are you allowed to arrange the groups so that the difference between the sum of points of the teams in one group compared to the sum of points of the teams in the other group exceeds 20% of the total number of points. All games are played at the home ground of one of the two participating teams; which one is not important since $d_{ij} = d_{ji}$.

Your task is to model this problem as an integer program. All functions defined have to be differentiable and explicit!

Question 5

(true or false)

The below three claims should be assessed. Are they true or false? Provide an answer together with a short but complete motivation.

(1p) a) *Claim:* Suppose that a function $f : \mathbb{R}^n \to \mathbb{R}$ is minimized over a non-empty and bounded polyhedral set. Then there exists an optimal solution to the problem.

(1p) b) *Claim:* Suppose that you have solved an LP problem, and that you would like to easily find an optimal solution also to the integer version of the problem—where all variables are required to be integral. Then there is a simple procedure by which rounding each of the variable values individually, either up or down—you may identify such an optimal solution.

(1p) c) *Claim:* The Phase-I problem in the simplex method always has an optimal solution.

(3p) Question 6

(global convergence of a penalty method)

Consider a nonlinear optimization problem with the objective of minimizing a differentiable function f over a set S specified by constraints of the form $g_i(\boldsymbol{x}) \leq 0$,

$i = 1, \dots, m$, where each function g_i is in C^1. Define the classic exterior penalty method using a penalty function $\psi \in C^1$. Introduce the necessary assumptions on ψ, such that the penalty algorithm is well-defined, and describe the sufficient conditions on the sequence of vectors generated such that a limit point is stationary.

Question 7

(the KKT conditions)

Consider the problem to

$$\begin{aligned} \text{minimize} \quad & x_1^2 + x_2^2 + x_3^2 + x_4^2 \\ \text{subject to} \quad & x_1 + x_2 + x_3 + x_4 = 1 \\ & x_4 \leq A. \end{aligned}$$

(2p) a) Write down the KKT conditions and find the optimal solution of the problem above for all values of the parameter $A \in \mathbb{R}$.

(1p) b) Plot the graph of the objective function as a function of the parameter A.

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EXAM SOLUTION

TMA947/MMG621
NONLINEAR OPTIMISATION

Date: 16–08–25
Examiner: Michael Patriksson

Question 1

(the simplex method)

(2p) a) We first rewrite the problem on standard form. We introduce slack variables s_1 and s_2. Consider the following linear program:

$$\begin{array}{llrrrrl} \text{minimize} & z = & -3\,x_1 & -5\,x_2 & & & \\ \text{subject to} & & & 2\,x_2 & +s_1 & & = 12, \\ & & 3\,x_1 & +2x_2 & & +s_2 & = 18, \\ & & x_1, & x_2, & s_1, & s_2 & \geq 0. \end{array}$$

We start directly with phase II at the origin. The starting basis is $(s_1, s_2)^{\mathrm{T}}$. Calculating the reduced costs for the non-basic variables x_1, x_2 we obtain $\tilde{\boldsymbol{c}}_N = (-3, -5)^{\mathrm{T}}$, meaning that x_2 enters the basis. From the minimum ratio test, we get that s_1 leaves the basis.

Updating the basis we now have $(x_2, s_2)^{\mathrm{T}}$ in the basis. Calculating the reduced costs, we obtain $\tilde{\boldsymbol{c}}_N = (-12, 5/2)^{\mathrm{T}}$, meaning that x_1 enters the basis. From the minimum ratio test we get that s_2 leaves the basis.

Updating the basis we now have $(x_1, x_2)^{\mathrm{T}}$ in the basis. Calculating the reduced costs, we obtain $\tilde{\boldsymbol{c}}_N = (3/2, 2)^{\mathrm{T}}$, meaning that the current basis is optimal. The optimal solution is thus

$$\boldsymbol{x}^* = (x_1, x_2, s_1, s_2)^{\mathrm{T}} = (2, 6, 0, 0)^{\mathrm{T}},$$

with optimal objective value $f^* = 36$.

(1p) b) Since there is an optimal solution to the problem, Strong duality guarantees the existence of a dual optimal solution. The dual optimal solution is $\boldsymbol{y}^{*\mathrm{T}} = \boldsymbol{c}_B^{\mathrm{T}} \boldsymbol{B}^{-1} = (-3/2, -1)$. The optimal basis is not degenerate. The optimal solution is thus unique.

Question 2

(the KKT conditions)

(1p) a) See the Book, system (5.9).

(1p) b) The vector $\boldsymbol{x}^1$ satisfies the KKT conditions (5.9).

(1p) c) Nothing. (Under the conditions given, there may be optimal solutions that do not satisfy the KKT conditions.)

(3p) Question 3

(Lagrangian duality)

The Lagrange function is

$$\begin{aligned} L(x,\mu) &= -x_1 - 2x_2 + \mu_1(x_1^2 + x_2^2 - 1) + \mu_2(x_1 + 0.5x_2 - 1) \\ &= \underbrace{\mu_1 x_1^2 + (\mu_2 - 1)x_1}_{q_1(x_1)} + \underbrace{\mu_1 x_2^2 + (0.5\mu_2 - 2)x_2}_{q_2(x_2)} - \mu_1 - \mu_2. \end{aligned}$$

When $\mu_1 < 0$, $L(x,\mu)$ is strictly concave with respect to x which makes $\min_{x_1} q_1(x_1)$ and $\min_{x_2} q_2(x_2)$ unbounded from below. Similarly, when $\mu_1 = 0$, $L(x,\mu)$ is linear and at least one of $\min_{x_1} q_1(x_1)$ and $\min_{x_2} q_2(x_2)$ is unbounded from below. Only when $\mu_1 > 0$ is $L(x,\mu)$ strictly convex with respect to x, and $\min_x L(x,\mu)$ is finite. In this case, the minimizers of q_1 and q_2 are, respectively,

$$x_1(\mu) = \frac{1-\mu_2}{2\mu_1}, \quad x_2(\mu) = \frac{2-0.5\mu_2}{2\mu_1}. \tag{1}$$

Consequently, the dual function is

$$q(\mu) = \begin{cases} -\frac{1}{4\mu_1}\left((1-\mu_2)^2 + (2-0.5\mu_2)^2\right) - \mu_1 - \mu_2, & \text{when } \mu_1 > 0 \\ -\infty, & \text{when } \mu_1 \le 0 \end{cases}.$$

The dual problem is

$$\begin{aligned} \text{minimize} \quad & q(\mu) \\ \text{subject to} \quad & \mu_1 > 0 \end{aligned}.$$

The dual function q is differentiable as expressed. The dual problem is always convex.

Since the primal problem is convex and the Slater constraint qualifications hold, strong duality holds. Hence, the duality gap is zero and the optimal dual solution is attained (which is the same as the Lagrangian multiplier).

There are multiple ways to obtain the optimal primal and dual solutions. An approach is as follows: By graphically inspecting the primal problem, it can be seen that $(1,0)^T$ is the optimal primal solution. Then, by Theorem 6.9 in the text, if $x^\star = (1,0)^T$ and $\mu^\star = (\mu_1^\star, \mu_2^\star)^T$ are the optimal primal and dual pair, they must satisfy (1). This implies that $\mu_1^\star = -\frac{3}{2}$ and $\mu_2^\star = 4$.

(3p) Question 4

(modelling)

Introduce the binary variables

$$x_i = \begin{cases} 1 & \text{if team } i \text{ is in group 1} \\ 0 & \text{otherwise} \end{cases}, \qquad i = 1, \dots, 14.$$

The objective is to minimize the function

$$\sum_{i=1}^{13} \sum_{j=i+1}^{14} d_{ij}(x_i x_j + (1 - x_i)(1 - x_j)).$$

The constraints are

$$\begin{aligned} \sum_{i=1}^{14} x_i &= 7 \\ x_1 + x_2 &= 1 \\ \sum_{i=1}^{14} x_i p_i &= \sum_{i=1}^{14} (1 - x_i) p_i + 0.2 \sum_{i=1}^{14} p_i \\ \sum_{i=1}^{14} (1 - x_i) p_i &= \sum_{i=1}^{14} x_i p_i + 0.2 \sum_{i=1}^{14} p_i \\ x_i &\in \{0, 1\}, \qquad i = 1, \dots, 14 \end{aligned}$$

The first constraint makes sure that there are 7 teams in each group. The second constraint ensures that the two best teams are not in the same group. The third and the fourth constraints ensure that the groups are arranged so that the difference between the sum of points in the two groups are not bigger than 20% of the total points.

Question 5

(true or false)

(1p) a) False – f may be discontinuous, for example.

(1p) b) False – there may be *no* rounding that is even feasible.

(1p) c) True – the linear program describing the Phase I problem is a linear program with an objective function that is bounded from below by zero. Since the objective value is bounded the extreme point with the lowest objective value is optimal.

(3p) Question 6

(global convergence of a penalty method)

See Theorem 13.4.

Question 7

(the KKT conditions)

(2p) a) The KKT conditions are

$$\begin{aligned} \nabla f(\mathbf{x}) + \lambda \nabla h(\mathbf{x}) + \mu \nabla g(\mathbf{x}) = \begin{pmatrix} 2x_1 + \lambda \\ 2x_2 + \lambda \\ 2x_3 + \lambda \\ 2x_4 + \lambda + \mu \end{pmatrix} &= \mathbf{0}, & (1) \\ x_1 + x_2 + x_3 + x_4 &= 1, & (2) \\ x_4 &\leq A, & (3) \\ \mu &\geq 0, & (4) \\ \mu(x_4 - A) &= 0, & (5) \end{aligned}$$

giving that $x_1 = x_2 = x_3 = -\lambda/2$ and $x_4 = (-\lambda - \mu)/2$. From (1) we then get that $\lambda = (-2 - \mu)/4$ and thus $x_1 = x_2 = x_3 = 1/4 + \mu/8$ and $x_4 = 1/4 - 3\mu/8$.

From (2) we get that $3\mu/8 \geq 1/4 - A$; we treat the following three cases individually.

1. Assume that $A > 1/4$, implying that $\mu \geq 0$, $x_1 = x_2 = x_3 \geq 1/4$ and $x_4 = 1 - (x_1 + x_2 + x_3) \leq 1/4$. From (4) it follows that $\mu = 0$ and the optimal solution hence is $x_1 = x_2 = x_3 = x_4 = 1/4$.
2. $A = 1/4$ leads to the same optimal solution as the case above.
3. Assume that $A \leq 1/4$. Let $x_4 < A$; then $\mu = 0$ and $x_4 = 1/4 > A$. Therefore, $x_4 = A$ and $x_1 = x_2 = x_3 = 1/3(1 - A)$. Then, the

original problem reduces to the minimization of $1/3(1 - A^2) + A^2 = 1/3(1 - 2A + 4A^2)$ which is always $\geq 1/4$ and $1/3(1 - A^2) + A^2 = 1/4$ for $A = 1/4$. The optimal solution is thus $x_1 = x_2 = x_3 = x_4 = 1/4$.

(1p) b) The objective function of the problem considered can be written as a function of the parameter A as

$$f(A) = \begin{cases} \frac{1}{4} & \text{if } A \geq 1/4, \\ \frac{1}{3}(1 - 2A + 4A^2) & \text{otherwise.} \end{cases}$$