Chalmers/Gothenburg University
Mathematical Sciences

EXAM

TMA947/MMG621
OPTIMIZATION, BASIC COURSE

Date:	13–08–29
Time:	House V, morning, 8^{30}–13^{30}
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Cornelia Jareteg (0703-088304)
Result announced:	13–04–18 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program to

$$\begin{aligned} \text{minimize} \quad & x_1 - x_2, \\ \text{subject to} \quad & x_1 + x_2 \geq 1, \\ & x_1 + 2x_2 \leq 4, \\ & x_2 \geq 0. \end{aligned}$$

(2p) a) Solve this problem using phase I (so that you begin with a unit matrix as the first basis) and phase II of the simplex method. If the problem has an optimal solution, then present the optimal solution in both the original variables and in the variables used in the standard form. If the problem is unbounded, then use your calculations to find a direction of unboundedness in both the original variables and in the variables used in the standard form.

Aid: Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(1p) b) Is the solution obtained unique? Use your calculations from a) to motivate why/why not.

Question 2

(Lagrangian relaxation)

Consider the optimization problem to

$$\begin{aligned} \text{minimize} \quad & (x_1 - 4)^2 + (x_2 - 2)^2, && \text{(1a)} \\ \text{subject to} \quad & x_1 + x_2 \leq 4, && \text{(1b)} \\ & 0 \leq x_j \leq 4, \quad j = 1, 2. && \text{(1c)} \end{aligned}$$

(2p) a) Formulate and solve the dual problem obtained when Lagrangian relaxing the constraint (1b).

(1p) b) Construct an optimal solution to the primal problem (1) by using the information obtained in a).

(3p) Question 3

(algorithm choice)

For the following optimization problems, choose the most appropriate solution-method from the list below, in the sense that the method requiring more assumptions on a problem is deemed as more approriate (i.e., although a linear program can be solved with an exterior penalty method, this is deemed as less approriate than a pure linear programming solution method). Answers without motivation will be disregarded.

- The Simplex method
- The Frank–Wolfe method
- The subgradient method
- The exterior penalty method
- Newton's method with the Levenberg–Marquardt modification

(1p) a)

$$\begin{aligned} \text{minimize} \quad & (x_1^2 + 3x_2^2)e^{x_1+x_2}, \\ \text{subject to} \quad & x_1 - x_2 \leq 0, \\ & 0 \leq x_1, x_2 \leq 5 \end{aligned}$$

(1p) b)

$$\underset{\mu \geq 0}{\text{maximize}} \quad q(\mu),$$

where q is the Lagrange dual function to the problem in (a), formed by relaxing the first constraint (assume that $q(\mu)$ is easy to compute)

(1p) c)

$$\begin{aligned} \text{minimize} \quad & x_1^2 - 3x_2^2 + 2x_1x_2, \\ \text{subject to} \quad & (x_1 - 3)^2 + (x_2 - 4)^2 \leq 25, \\ & (x_1 + 1)^2 + (x_2 + 2)^2 \geq 16 \end{aligned}$$

Question 4

(cones and conditions)

Consider the problem to

$$\begin{aligned} \text{minimize} \quad & x_1 + x_2, \\ \text{subject to} \quad & \sin(\pi x_1) = 0, \\ & \sin(\pi x_2) = 0, \end{aligned}$$

the feasible set of which is denoted by S. Note that $S = \mathbf{Z}^2 = \{\boldsymbol{x} \mid x_1, x_2 \text{ integers}\}$.

(1p) a) Show that the tangent cone is $T_S(\boldsymbol{x}) = \{\mathbf{0}^m\}$ for all $\boldsymbol{x} \in S$. *Reminder*: the tangent cone is defined by

$$T_S(\boldsymbol{x}) := \left\{ \boldsymbol{p} \mid \boldsymbol{p} = \lim_{k\to\infty} \lambda_k(\boldsymbol{x}_k - \boldsymbol{x}),\ \lim_{k\to\infty} \boldsymbol{x}_k = \boldsymbol{x},\ \lambda_k \geq 0, \boldsymbol{x}_k \in S \text{ for all } k = 1, 2, \dots \right\}.$$

(2p) b) Find all KKT-points of the problem. Is any KKT-point a globally optimal solution? You may, if you so wish, assume that the Abadie CQ holds.

(3p)

Question 5

(modelling)

An online casino is running a promotion giving new players a gift item after a certain amount of money M has been used for betting. An optimization student who justs wants the gift item asks the question whether he/she should buy the item from a store or if he/she can find a betting strategy in which the worst case loss of money is less than the price of the item in a store.

Your task is therefore to formulate a linear programming model determining which bets to make in order to maximize the *guaranteed* payout (i.e., the worst case scenario) after exactly M SEK of bets have been made.

Assume that there is an available set of games $\mathcal{N} := \{1, \dots, N\}$ to bet on, each having a set $\mathcal{B}_i$ of K_i mutually exclusive possible bets $\mathcal{B}_i := \{1, \dots, K_i\}$ yielding a payout r_{ik} for $i \in \mathcal{N}$, $k \in \mathcal{B}_i$.

Example: There are two football matches to bet on: Inter Milan against AC Milan with payouts 10.0, 3.5, 1.1 and Juventus against Roma with payouts 2.0, 2.0, 2.0. A player betting all $M = 10000$ SEK on AC Milan winning versus in the (very unlikely) worst-case scenario loses all the money. A player who bets equal amounts 10000/6 on SEK on all six bets nets in the worst-case scenario (i.e., AC Milan winning and any result in Juventus against Roma) $1.1 \times 10000/6 + 2.0 \times 10000/6 = 5166.66\dots$ SEK.

(3p)

Question 6

(strong duality in linear programming)

Consider the following standard form of a linear program:

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{c}, \boldsymbol{x} \in \mathbb{R}^n$, and $\boldsymbol{b} \in \mathbb{R}^m$. State and prove the Strong Duality Theorem in linear programming.

Question 7

(true or false)

Indicate for each of the following three statements whether it is *true* or *false*. Motivate your answers!

(1p) a) For the phase I (when a BFS is *not* known a priori) problem of the simplex algorithm, the optimal value is always zero.

(1p) b) Suppose that the function $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable on $\mathbb{R}^n$ and let $\boldsymbol{G}$ be a symmetric and positive definite matrix of dimension $n \times n$. Then, if $\nabla f(\boldsymbol{x}) \neq \boldsymbol{0}^n$ and the vector $\boldsymbol{d}$ fulfils $\boldsymbol{G}\boldsymbol{d} = -\nabla f(\boldsymbol{x})$ it holds that $f(\boldsymbol{x} + t\boldsymbol{d}) < f(\boldsymbol{x})$ for small enough values of $t > 0$.

(1p) c) If the function $g : \mathbb{R}^n \to \mathbb{R}$ is *concave* on $\mathbb{R}^n$ and $c \in \mathbb{R}$, then the set $\{\, \boldsymbol{x} \in \mathbb{R}^n \mid g(\boldsymbol{x}) \leq c \,\}$ is convex.

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EXAM SOLUTION

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Date: 13–08–29
Examiner: Michael Patriksson

Question 1

(the simplex method)

(2p) a) We first rewrite the problem on standard form. We rewrite $x_1 = x_1^+ - x_1^-$ and introduce slack variables s_1 and s_2.

$$\begin{array}{lrl} \text{minimize} & x_1^+ - x_1^- - x_2 & \\ \text{subject to} & x_1^+ - x_1^- + x_2 - s_1 & = 1, \\ & x_1^+ - x_1^- + 2x_2 + s_2 & = 4, \\ & x_1^+, x_1^-, x_2, s_1, s_2 & \geq 0. \end{array}$$

Phase I

We introduce an artificial variable a and formulate our Phase I problem.

$$\begin{array}{lrl} \text{minimize} & a & \\ \text{subject to} & x_1^+ - x_1^- + x_2 - s_1 + a & = 1, \\ & x_1^+ - x_1^- + 2x_2 + s_2 & = 4, \\ & x_1^+, x_1^-, x_2, s_1, s_2 & \geq 0. \end{array}$$

We now have a starting basis (a, s_2). Calculating the reduced costs we obtain $\tilde{\mathbf{c}}_N = (-1, 1, -1, 1)^{\mathrm{T}}$, meaning that x_1^+ or x_2 should enter the basis. We choose x_2. From the minimum ratio rest, we get that a should leave the basis. This concludes phase I and we now have the basis (x_2, s_2).

Phase II

Calculating the reduced costs, we obtain $\tilde{\mathbf{c}}_N = (2, -2, 1)^{\mathrm{T}}$, meaning that x_1^- should enter the basis. From the minimum ratio test, we get that the outgoing variable is s_2. Updating the basis we now have (x_1^-, x_2) in the basis.

Calculating the reduced costs, we obtain $\tilde{\mathbf{c}}_N = (0, 3, 2)^{\mathrm{T}} \geq 0$, meaning that the current basis is optimal. The optimal solution is thus $(x_1^+, x_1^-, x_2, s_1, s_2)^{\mathrm{T}} = (0, 2, 3, 0, 0)^{\mathrm{T}}$, which in the original variables means $(x_1, x_2) = (-2, 3)^{\mathrm{T}}$, with optimal objective value $f^* = -5$.

(1p) b) The reduced costs are not all positive, so from the calculations we can not draw any conclusions regarding the uniqueness of the solution. However, the solution is unique in the original problem (draw the feasible set).

Question 2

(Lagrangian relaxation)

(2p) a) The dual problem is that to

$$\text{maximize}_{u \geq 0}\ q(u),$$

where q is the Lagrangian dual function defined as

$$\begin{aligned} q(u) &= \min_{0 \leq x_j \leq 4, j=1,2} \Big((x_1 - 4)^2 + (x_2 - 2)^2 + u(x_1 + x_2 - 4) \Big) &(1)\\ &= 20 - 4u + \min_{0 \leq x_1 \leq 4} \left(x_1^2 - 8x_1 + ux_1 \right) + \min_{0 \leq x_2 \leq 4} \left(x_2^2 - 4x_2 + ux_2 \right). &(2) \end{aligned}$$

The minimum of the two subproblems in (2) are attained at

$$x_1(u) = \frac{8-u}{2} \quad \text{and} \quad x_2(u) = \frac{4-u}{2}.$$

respectively. Inserting this into (1) we get that

$$q(u) = 2u - \frac{u^2}{2},$$

which attains is maximum when $q'(u) = 2 - u = 0$. So the optimal dual solution is $u^* = 2$ with dual objective value $q^* = q(u^*) = 2$.

(1p) b) At $u = 2$, we have that $x_1(u) = 3$ and $x_2(u) = 1$. This is a feasible solution to the primal problem with objective value 2, which is the same as the dual optimal value, implying that $\boldsymbol{x}^* = (3, 1)^{\mathrm{T}}$ is an optimal solution to the primal problem.

Question 3

(algorithm choice)

(1p) a) The Frank–Wolfe method is most appropriate; exterior penalty is also applicable but makes less assumptions on problem structure, the others are not applicable. (Differentiable objective function, the feasible set is a bounded polyhedron).

(1p) b) The subgradient method is most appropriate. The only other candidate for an applicable methods is the exterior penalty method, but it does not use the convexity of the problem. Further, without checking it is unclear whether the Lagrangian function is differentiable or not. (Lagrangian dual problems are convex, and the subgradients can easily be computed).

(1p) c) Exterior penalty is the only applicable method.

Question 4

cones and conditions

(1p) a) $T_S(\boldsymbol{x}) = \{\boldsymbol{0}\}$ for all $\boldsymbol{x} \in S$, since for any sequence $\{\boldsymbol{x}_k\} \subset S$, $\boldsymbol{x}_k \to \boldsymbol{x}$ we must have $\boldsymbol{x}_k = \boldsymbol{x}$ for all $k \geq K$ for some K.

(2p) b) For any $\boldsymbol{x} \in S$, since $T_S(\boldsymbol{x}) = \{\boldsymbol{0}\}$ and (by assumption) Abadie's CQ holds we have $G(\boldsymbol{x}) = T_S(\boldsymbol{x}) = \{\boldsymbol{0}\}$. Thus $G(\boldsymbol{x}) \cap F_0(\boldsymbol{x}) = \emptyset$, so all points $\boldsymbol{x} \in S$ are KKT-points (this can also be verified directly from solving the KKT-system). Although all feasible points are KKT-points, none is optimal, as the objective function is unbounded from below as $x_1 \to -\infty$.

(3p) Question 5

modelling

We declare variables x_{ik} for $i \in \mathcal{N}$, $k \in \mathcal{B}_i$, to be understood as the amount of cash bet k of game i. Further declare the variables y_i as the worst case payout from game i for $i \in \mathcal{N}$. A model can then be written as

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i \in \mathcal{N}} y_i & , & \qquad (1)\\
\text{subject to} \quad & y_i \leq r_{ik} x_{ik}, & i \in \mathcal{N}, k \in \mathcal{B}_i, & \qquad (2)\\
& \sum_{i \in \mathcal{N}} \sum_{k \in \mathcal{B}_i} x_{ik} = M, & & \qquad (3)\\
& x_{ik} \geq 0, & i \in \mathcal{N}, k \in \mathcal{B}_i. & \qquad (4)
\end{aligned}
$$

The objective function (1) maximizes the worst case scenario payout. The inequalities (2) models that the worst case scenario payout is less than the payout for any outcome. The equality (3) states that the total amount of bets to be made is M SEK. The final inequalities (4) are definitional and state that we cannot bet negative money.

(3p) Question 6

(strong duality in linear programming)

See Theorem 10.6 in The Book.

Question 7

(true or false)

(1p) a) False. If no feasible solution exists, the optimal value is > 0. If feasible solutions exist, the optimal value is $= 0$. (Section 9.1.2.)

(1p) b) True. We have that $\nabla f(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{d} = -\boldsymbol{d}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{d} < 0$, since $\boldsymbol{G}$ is a positive definite matrix (Section 2.2, page 37). Then, by Proposition 4.16, $\boldsymbol{d}$ is a descent direction for f at $\boldsymbol{x}$ since $\nabla f(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{d} < 0$. Hence (Definition 4.15) $\exists \delta > 0$ such that $f(\boldsymbol{x} + t\boldsymbol{d}) < f(\boldsymbol{x})$ for all $t \in (0, \delta]$.

(1p) c) False. Consider the function $g(x) = 4 - x^2$ and the two points $x^1 = -2$ and $x^2 = 3$ which belong to the set $S = \{ x \in \mathbb{R} \mid g(x) \leq 0 \}$. By Definitions 3.31 and 3.32, g is concave. The point $\frac{1}{2} \cdot x^1 + \frac{1}{2} \cdot x^2 = \frac{1}{2} \notin S$. Hence, by Definition 3.1, the set S is not convex.
