Chalmers/Gothenburg University
Mathematical Sciences

EXAM

TMA947/MMG620
OPTIMIZATION, BASIC COURSE

Date:	09–12–14
Time:	House V, morning
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Adam Wojciechowski (0703-088304)
Result announced:	10–01–08 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$\begin{array}{lrl} \text{minimize} & z = & x_1 + 2x_2 \\ \text{subject to} & & 2x_1 - 2x_2 \leq -2, \\ & & 2x_1 + \ \ x_2 \leq \ \ 2, \\ & & x_1 \qquad \ \ \in \ \mathbb{R}, \\ & & \qquad \ \ x_2 \geq \ \ 0. \end{array}$$

(2p) a) Solve this problem by using phase I and phase II of the simplex method.
[*Aid:* Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

for producing basis inverses.]

(1p) b) Consider the application of the simplex method to a general LP and suppose that you, unlike in the standard procedure taught in this course, at some iteration *a*) choose the entering variable to be a non-basic variable with a negative reduced cost but not having the most negative reduced cost, or *b*) choose the outgoing variable as a basic variable with the $B^{-1}N_{j^*}$ component > 0 but not fulfilling the minimum ratio test. Which of these choices is a critical mistake? Motivate *clearly* why that is the case (guessing without a clear motivation will not give you any points).

Question 2

(convexity)

Prove or disprove the following three claims.

(1p) a) $f(x_1, x_2, x_3) = x_1^4 + x_2^2 + 4x_2x_3 + 5x_3^2$ is convex for all $\boldsymbol{x} \in \mathbb{R}^3$.

(1p) b) $f(x_1, x_2) = \max\{2x_1 - x_2, x_2^2\}$ is convex for all $\boldsymbol{x} \in \mathbb{R}^2$.

(1p) c) $f(x_1, x_2) = 2(x_1^3 + x_2^3) + x_1^2 x_2^2 + 4x_2^2$ is convex near $\boldsymbol{x} = (0,0)^{\mathrm{T}}$ (i.e., there is a small ball around the origin in which the function is convex).

(3p) Question 3

(modeling)

Suppose you want to put up some new wallpaper in your house. You have done some measurements and came to the conclusion that you will need the following pieces of wallpaper: 10 pieces of length 2.7 m, 1 piece of length 2.3 m, 2 pieces of length 2.0 m, 5 pieces of length 1.5 m and 1 piece of length 1.2 m. You can buy the wallpaper in rolls of 10 m length each. Since you have little money left before Christmas, you want to minimize the number of rolls that you have to buy.

Introduce the necessary constants and variables and formulate an integer linear program (i.e., a model that after the relaxation of the integrality constraints is an LP) that will minimize the number of rolls that you need to buy and such that the optimal solution will tell you how the wallpaper rolls should be used (which piece to cut from which roll).

[*Remarks:* Do not solve the model. Make sure that you specify the meaning of each variable and constraint, so that we understand the model.]

(3p) Question 4

(gradient projection)

The gradient projection algorithm is a generalization of the steepest descent method to problems over convex sets. Given a point $\boldsymbol{x}^k$, the next point is obtained according to $\boldsymbol{x}^{k+1} = \mathrm{Proj}_X(\boldsymbol{x}^k - \alpha\nabla f(\boldsymbol{x}^k))$, where X is the convex set over which we minimize, $\alpha > 0$ is the step length, and $\mathrm{Proj}_X(\boldsymbol{y}) = \arg\min_{\boldsymbol{x}\in X} \| \boldsymbol{x} - \boldsymbol{y} \|$ denotes the closest point to $\boldsymbol{y}$ in X.

[*Remark:* if $X = \mathbb{R}^n$, then the method reduces to that of steepest descent.]

Consider the optimization problem to

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := (x_1 + x_2)^2 + 3(x_1 - x_2)^2 \ , \\ \text{subject to} \quad & 0 \leq x_1 \leq 2, \\ & 1 \leq x_2 \leq 3. \end{aligned}$$

Start at the point $\boldsymbol{x}^0 = (1\ 2)^{\mathrm{T}}$ and perform two iterations of the gradient projection algorithm using step length $\alpha = 1$. Note that the special form of the feasible region X makes projections very easy! Is the point obtained a global/local optimum? Motivate why/why not!

(3p) Question 5

(strong duality in linear programming)

Consider the following standard form of a linear program:

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{c}, \boldsymbol{x} \in \mathbb{R}^n$, and $\boldsymbol{b} \in \mathbb{R}^m$. State and prove the Strong Duality Theorem in linear programming.

(3p) Question 6

(the Fritz John conditions)

In the problem to minimize the C^1 function f over the set S, we assume that S is described by differentiable inequality constraints defined by the functions $g_i \in C^1(\mathbb{R}^n)$, $i = 1, \dots, m$, such that

$$S := \{\, \boldsymbol{x} \in \mathbb{R}^n \mid g_i(\boldsymbol{x}) \leq 0, \quad i = 1, \dots, m \,\}.$$

The Fritz John conditions state the following: *If $\boldsymbol{x}^* \in S$ is a local minimum of f over S then there exist multipliers $\mu_0 \in \mathbb{R}$, $\boldsymbol{\mu} \in \mathbb{R}^m$, such that*

$$\mu_0 \nabla f(\boldsymbol{x}^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(\boldsymbol{x}^*) = \boldsymbol{0}^n, \tag{1a}$$

$$\mu_i g_i(\boldsymbol{x}^*) = 0, \qquad i = 1, \dots, m, \tag{1b}$$

$$\mu_0, \mu_i \geq 0, \qquad i = 1, \dots, m, \tag{1c}$$

$$(\mu_0, \boldsymbol{\mu}^{\mathrm{T}})^{\mathrm{T}} \neq \boldsymbol{0}^{m+1}. \tag{1d}$$

Suppose that we, to the given problem, add the redundant constraint that

$$g_{m+1}(\boldsymbol{x}) := -\frac{1}{2}\|\boldsymbol{x} - \boldsymbol{x}_0\|^2 \leq 0,$$

where the vector $\boldsymbol{x}_0$ is an arbitrary feasible solution.

Establish that if this constraint is added to the problem at hand, then the Fritz John conditions are satisfied at $\boldsymbol{x}_0$.

Draw as many conclusions as can be made regarding the practical usefulness of the Fritz John conditions given the above revelation.

Question 7

(topics in linear programming)

(1p) a) Suppose that a linear program includes a free variable x_j. When transforming this problem into standard form, x_j is replaced by

$$\begin{aligned} x_j &= x_j^+ - x_j^-, \\ x_j^+, x_j^- &\geq 0. \end{aligned}$$

Show that no basic feasible solution can include both x_j^+ and x_j^- as non-zero basic variables.

(1p) b) Consider the linear program

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}. \end{aligned} \tag{1}$$

Assume that the objective function vector $\boldsymbol{c}$ cannot be written as a linear combination of the rows of $\boldsymbol{A}$. Show that (**??**) cannot have an optimal solution.

(1p) c) Consider the linear program

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned} \tag{P}$$

and the perturbed problem to

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \tilde{\boldsymbol{b}}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n. \end{aligned} \tag{P'}$$

Show that if (P) has an optimal solution, then the perturbed problem (P') cannot be unbounded (independently of $\tilde{\boldsymbol{b}}$).

Good luck!

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EXAM SOLUTION

TMA947/MAN280
OPTIMIZATION, BASIC COURSE

Date: 09–12–14

Examiner: Michael Patriksson

Question 1

(the simplex method)

(2p) a) First we need to transform the problem to standard form. The free variable x_1 is replaced with the difference of the two non-negative variables x_1^+ and x_1^-, $x_1 := x_1^+ - x_1^-$. The sign of the first constraint is changed, and a non-negative slack (surplus) variable s_1 is subtracted. In the second constraint, a non-negative slack variable s_2 is added. We get

$$\begin{array}{llrrrrrl}
\text{minimize} & z = & x_1^+ & - \ x_1^- & +2x_2, & & & \\
\text{subject to} & & -2x_1^+ & +2x_1^- & +2x_2 & -s_1 & & = 2, \\
& & 2x_1^+ & -2x_1^- & + \ x_2 & & +s_2 & = 2, \\
& & x_1^+, & x_1^-, & x_2, & s_1, & s_2 & \geq 0.
\end{array}$$

Now start phase 1 using an artificial variable $a \geq 0$ added in the first constraint. Use s_2 as the second basic variable.

$$\begin{array}{llrrrrrrl}
\text{minimize} & w = & & & & & & a, & \\
\text{subject to} & & -2x_1^+ & +2x_1^- & +2x_2 & -s_1 & & +a & = 2, \\
& & 2x_1^+ & -2x_1^- & + \ x_2 & & +s_2 & & = 2, \\
& & x_1^+, & x_1^-, & x_2, & s_1, & s_2, & a & \geq 0.
\end{array}$$

We start with the BFS given by $(a, s_2)^\mathrm{T}$. The vector of reduced costs for the non-basic variables x_1^+, x_1^-, x_2 and s_1 is $(2, -2, -2, 1)^\mathrm{T}$. We choose x_1^- as the entering variable. In the minimum ratio test, a is the only basic varible for which the corresponing component of B^-1N_2 is positive, and is therefore selected as the outgoing variable. No artificial variables are left in the basis, thus the reduced costs will be non-negative and we are optimal with $w^* = 0$. We proceed to phase 2.

The BFS is given by $\boldsymbol{x}_B = (x_1^-, s_2)^\mathrm{T}$, $\boldsymbol{x}_N = (x_1^+, x_2, s_1)^\mathrm{T}$ and the reduced costs with the phase 2 cos t vector are $\tilde{c}^\mathrm{T} = (0, 3, -\frac{1}{2})$. The reduced cost is negative for s_1 which is the only eligable incoming variable. $B^{-1}\boldsymbol{b} = (1, 4)^\mathrm{T}$ and $B^{-1}N_3 = (-\frac{1}{2}, -1)^\mathrm{T}$. Thus, the unboundedness criterion is fulfilled and we have that $z \to -\infty$ for

$$\begin{pmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} + \mu \begin{pmatrix} \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{pmatrix}, \quad \mu \to \infty,$$

or, in the original variables, along the line

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} - \mu \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \quad \mu \to \infty.$$

(1p) b) *a*) is not critical since with the reduced cost < 0 we still have a descent direction (in fact, selection of the most negative reduced cost is not the most efficient one in reality, and in all commercial softwares a more sophisticated selection method is used). However, *b*) is a major mistake. The minimum ratio test is used to decide how far we can move along the coordinate axis of the incoming variable and still stay feasible. Not selecting the minimum ratio implies that the incoming variable is given a value so high that one of the basic variables will turn negative (since we have $\boldsymbol{x}_B = B^{-1}\boldsymbol{b} - B^{-1}N\boldsymbol{x}_N$, and we wish to increase the value of the incoming variable in $\boldsymbol{x}_N$). Since the idea of the simplex method is to move from extreme point to extreme point (from BFS to BFS), this is a critical mistake, since our new point will not be a BFS.

Question 2

(convexity)

(1p) a) The claim is true. Clearly, x_1^4 is a convex function, and since we know that a sum of convex functions remains convex, what is left to check is if $x_2^2 + 4x_2x_3 + 5x_3^2 := g(x_2, x_3)$ is convex. A computation of the eigenvalues to the hessian of g shows that they are $\lambda = 6 \pm \sqrt{32} > 0$. Therefore, the hessian is positive semidefinite for all $\boldsymbol{x} \in \mathbb{R}^3$ and thus, g is convex. We conclude that f is convex.

(1p) b) The claim is true. Let $h(\boldsymbol{x}) := 2x_1 - x_2$ and $g(\boldsymbol{x}) := x_2^2$ and observe that they are both convex. f is not differentiable so we cannot use the same procedure as in *a*), instead we use the definition. Let $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ be two arbitrary points and let $\lambda \in (0, 1)$. Since h and g are convex, we have that

$$h(\lambda\boldsymbol{x}^1 + (1-\lambda)\boldsymbol{x}^2) \leq \lambda h(\boldsymbol{x}^1) + (1-\lambda)h(\boldsymbol{x}^2) \text{ and}$$
$$g(\lambda\boldsymbol{x}^1 + (1-\lambda)\boldsymbol{x}^2) \leq \lambda g(\boldsymbol{x}^1) + (1-\lambda)g(\boldsymbol{x}^2).$$

Therefore,

$$f(\lambda \boldsymbol{x}^1 + (1-\lambda)\boldsymbol{x}^2) = \max\left\{h(\lambda \boldsymbol{x}^1 + (1-\lambda)\boldsymbol{x}^2), g(\lambda \boldsymbol{x}^1 + (1-\lambda)\boldsymbol{x}^2)\right\} \leq$$
$$\max\left\{\lambda h(\boldsymbol{x}^1) + (1-\lambda)h(\boldsymbol{x}^2), \lambda g(\boldsymbol{x}^1) + (1-\lambda)g(\boldsymbol{x}^2)\right\} \leq$$
$$\max\left\{\lambda h(\boldsymbol{x}^1), \lambda g(\boldsymbol{x}^1)\right\} + \max\left\{(1-\lambda)h(\boldsymbol{x}^2), (1-\lambda)g(\boldsymbol{x}^2)\right\} =$$
$$\lambda f(\boldsymbol{x}^1) + (1-\lambda)f(\boldsymbol{x}^2),$$

where the last inequality comes from the obvious fact that

$$\max\{a+b, c+d\} \leq \max\{a, c\} + \max\{b, d\}.$$

(1p) c) The claim is false. The hessian to f is given by

$$\nabla^2 f(\boldsymbol{x}) = \begin{pmatrix} 12x_1 + 2x_2^2 & 4x_1x_2 \\ 4x_1x_2 & 12x_2^2 + 8 \end{pmatrix}$$

and we conclude that its eigenvalues at $\boldsymbol{x} = (0,0)^{\mathrm{T}}$ are $\lambda_1 = 8$ and $\lambda_2 = 0$, i.e., the matrix is positive semidefinite but not positive definite. Therefore we cannot conclude anything about the local convexity from this fact. But now look at the line given by

$$\begin{cases} x_1 = t \\ x_2 = 0 \end{cases} \text{ and let } \boldsymbol{x}^1 = \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix}, \ \boldsymbol{x}^2 = \begin{pmatrix} -\varepsilon \\ 0 \end{pmatrix}.$$

We have $f(\boldsymbol{x}^1) = 2\varepsilon^3$, $f(\boldsymbol{x}^2) = -2\varepsilon^3$ and $f(\lambda\boldsymbol{x}^1 + (1-\lambda)\boldsymbol{x}^2) = 2\varepsilon^3(2\lambda-1)^3$. Therefore, we get that $f(\lambda\boldsymbol{x}^1 + (1-\lambda)\boldsymbol{x}^2) > \lambda f(\boldsymbol{x}^1) + (1-\lambda)f(\boldsymbol{x}^2)$ when $(2\lambda-1)^3 > 2\lambda-1$ which is true for all $\lambda < 1/2$. This counterexample shows that f is not locally convex around the origin.

(3p) Question 3

(modeling)

Introduce the constants: the number of pieces needed, $n = 19$, the length of each roll $L = 10$ m and the length of piece i, b_i for $i = 1, \ldots, n$. Introduce the variables

$$x_{ij} = \begin{cases} 1 & \text{piece } i \text{ is in roll } j \\ 0 & \text{otherwise} \end{cases}, \quad i = 1, \ldots, n, \ j = 1, \ldots, n,$$

$$y_j = \begin{cases} 1 & \text{roll } j \text{ is used} \\ 0 & \text{otherwise} \end{cases}, \quad j = 1, \ldots, n.$$

The objective is

$$\text{minimize} \sum_{j=1}^{n} y_j.$$

The first constraint is that we may not cut more than L meters from each roll (and 0 if the roll is not used):

$$\sum_{i=1}^{n} b_i x_{ij} \leq L y_j, \quad j = 1, \dots, n.$$

The second contraint is that each piece i must be cut from exactly one roll:

$$\sum_{j=1}^{n} x_{ij} = 1, \quad j = 1, \dots, n.$$

Finally, x and y are integral variables:

$$x_{ij}, y_j \in \{0, 1\}, \quad i = 1, \dots, n, j = 1, \dots, n.$$

(3p) Question 4

(gradient projection)

Note first that the feasible region X is a square.

Iteration 1: $\boldsymbol{x}^0 = (1\ 2)^{\mathrm{T}}$, $\nabla f(\boldsymbol{x}^0) = (0\ 12)^{\mathrm{T}}$. $\boldsymbol{x}^0 - \alpha \nabla f(\boldsymbol{x}^0) = (1\ 2)^{\mathrm{T}} - (0\ 12)^{\mathrm{T}} = (1\ -10)^{\mathrm{T}}$. $\mathrm{Proj}_X(1\ -10)^{\mathrm{T}} = (1\ 1) = \boldsymbol{x}^1$.

Iteration 2: $\boldsymbol{x}^1 = (1\ 1)^{\mathrm{T}}$, $\nabla f(\boldsymbol{x}^1) = (4\ 4)^{\mathrm{T}}$. $\boldsymbol{x}^1 - \alpha \nabla f(\boldsymbol{x}^1) = (1\ 1)^{\mathrm{T}} - (4\ 4)^{\mathrm{T}} = (-3\ -3)^{\mathrm{T}}$. $\mathrm{Proj}_X(-3\ -3)^{\mathrm{T}} = (0\ 1)^{\mathrm{T}} = \boldsymbol{x}^2$.

We have convex constraints with an interior point, hence Slaters CQ imply that KKT is necessary for local optimality (We can use LICQ or the fact that the constraints are linear as well). The constraint $g_1 = -x_1$ and $g_2 = 1 - x_2$ are active. $\nabla f(\boldsymbol{x}^2) = (-4\ 8)^{\mathrm{T}}$, $\nabla g_1(\boldsymbol{x}^2) = (-1\ 0)^{\mathrm{T}}$, $\nabla g_2(\boldsymbol{x}^2) = (0\ -1)^{\mathrm{T}}$. The KKT conditions do not hold. Hence $\boldsymbol{x}^2$ is not a KKT point, and therefore it is not a local (nor a global) minimum.

(3p) Question 5

(strong duality in linear programming)

See Theorem 10.6 in The Book.

(3p) Question 6

(the Fritz John conditions)

Introducing the redundant constraint with multiplier μ_{m+1} results in the new Fritz John conditions:

$$\mu_0 \nabla f(\boldsymbol{x}^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(\boldsymbol{x}^*) - \mu_{m+1}(\boldsymbol{x}^* - \boldsymbol{x}_0) = \mathbf{0}^n, \tag{1a}$$

$$\mu_i g_i(\boldsymbol{x}^*) = 0, \quad i = 1, \dots, m+1, \tag{1b}$$

$$\mu_0, \mu_i \geq 0, \quad i = 1, \dots, m+1, \tag{1c}$$

$$(\mu_0, \boldsymbol{\mu}^{\mathrm{T}})^{\mathrm{T}} \neq \mathbf{0}^{m+2}. \tag{1d}$$

These conditions are satisfied by setting $\boldsymbol{x}^* = \boldsymbol{x}_0$, $\mu_0 = 0$, $\mu_i = 0$ for $i = 1, \dots, m$, and $\mu_{m+1} > 0$ arbitrarily.

The main conclusion is that since an arbitrary solution can be made to satisfy the Fritz John condition, it is not a very useful measure of optimality at all.

Question 7

(topics in linear programming)

(1p) a) The two new variables x_j^+ and x_j^- will have columns of the system matrix $\boldsymbol{A}$ that have the same absolute values, but have opposite signs, i.e., $\boldsymbol{a}_j^+ = -\boldsymbol{a}_j^-$. Since these two vectors are linearly dependent, no basis can include them both.

(1p) b) The dual linear program is that to

$$\begin{aligned} \text{maximize} \quad & w = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}, && (1)\\ \text{subject to} \quad & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{c}, && (2)\\ & \boldsymbol{y} \leq \boldsymbol{0}^n. && (3) \end{aligned}$$

If $\boldsymbol{c}$ cannot be written as a linear combination of the rows of $\boldsymbol{A}$, then the constraint (2) cannot be satisfied. Hence this dual problem cannot have an optimal solution.

(1p) c) The result follows from a simple argument based on weak duality. By assumption, the problem

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, && \text{(P)}\\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n \end{aligned}$$

has an optimal solution. Then, its dual problem, that to

$$\begin{aligned} \text{maximize} \quad & w = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}, && \text{(D)}\\ \text{subject to} \quad & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}, \end{aligned}$$

also has an optimal solution. Now, for any perturbation $\tilde{\boldsymbol{b}}$ of $\boldsymbol{b}$ the perturbed dual problem

$$\begin{aligned} \text{maximize} \quad & w = \tilde{\boldsymbol{b}}^{\mathrm{T}}\boldsymbol{y}, && \text{(D')}\\ \text{subject to} \quad & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}, \end{aligned}$$

at least has a nonempty feasible set. By the Weak Duality Theorem, then, its dual, to

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, && \text{(P')}\\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \tilde{\boldsymbol{b}}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n \end{aligned}$$

has feasible solutions with objective values not better than any objective values of the problem (D'). Hence, the perturbed problem (P') cannot have an unbounded solution. As the perturbation $\tilde{\boldsymbol{b}}$ was arbitrary, the result follows.

Good luck!