Chalmers/Gothenburg University
Mathematical Sciences

EXAM

TMA947/MMG620
OPTIMIZATION, BASIC COURSE

Date:	09–08–27
Time:	House V, morning
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Jacob Sznajdman (0762-721860)
Result announced:	09–09–17 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$\begin{array}{lrrrrrl}
\text{minimize} & z = & 2x_1 & - & x_2 & + & x_3, \\
\text{subject to} & & x_1 & + & 2x_2 & - & x_3 \leq 7, \\
 & & -2x_1 & + & x_2 & - & 3x_3 \leq -3, \\
 & & x_1, & & x_2, & & x_3 \geq 0.
\end{array}$$

(2p) a) Solve this problem by using phase I and phase II of the simplex method.
[*Aid:* Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

for producing basis inverses.]

(1p) b) If the problem in a) is infeasible or unbounded, perform a small modification of the cost vector and/or the right-hand side vector such that the modified problem will have at least one optimal solution. If the problem has an optimal solution, state for which values of the first component (the one which is 7 now) of the right-hand side vector the optimal basis remains being the optimal one.

(3p) Question 2

(modeling)

A wind power company has n wind turbines located in an area. In order to perform maintenance, m maintenance units are located in different depots. Each day, the management obtains a list of maintenance activities to be performed at the different wind turbines. The data can be transformed into work hours that a crew has to spend at the site. Let d_i be the number of maintenance hours that must be spent on repairing turbine i. If the repairs will not be completed, the turbine can not run and will generate a loss of e_i SEK. Each crew can work a maximum number of 8 hours, but in order to perform repairs at a site, the crew has to travel to the site and return to the depot afterwards. Let c_{ij} be the time that crew j has to travel in order to reach turbine i. Since there only is a limited space in each turbine, a maximum of 2 crews may work on the same turbine during the day.

Create a linear mixed integer programming model (that is, a model which becomes a linear programming (LP) problem if any integrality requirements were to be removed) that schedules the maintenance work of the maintenance units at the turbines during one day so that the cost of the production losses are minimized.

(3p) Question 3

(optimality conditions)

Farkas' Lemma can be stated as follows:

Let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{b}$ an $m \times 1$ vector. Then exactly one of the systems

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b}, \\ \boldsymbol{x} &\geq \boldsymbol{0}^n, \end{aligned} \tag{I}$$

and

$$\begin{aligned} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} &\leq \boldsymbol{0}^n, \\ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} &> 0, \end{aligned} \tag{II}$$

has a feasible solution, and the other system is inconsistent.

Prove Farkas' Lemma.

Question 4

(exterior penalty method)

Consider the following problem:

$$\begin{aligned} &\text{minimize} \quad f(x) := \frac{1}{2}(x_1)^2 - x_1x_2 + (x_2)^2, \\ &\text{subject to} \quad x_1 + x_2 - 1 = 0. \end{aligned}$$

(1p) a) By applying the KKT conditions to this problem, establish its (unique) exact primal–dual solution.

(1p) b) Apply the standard exterior quadratic penalty method for this problem, and show that the sequence of (explicitly stated) subproblem solutions converges to the unique primal solution.

(1p) c) From the theory of exterior penalty methods provide the corresponding sequence of estimates of the Lagrange multiplier, and show that it converges to the dual solution provided in a).

Question 5

(topics in convexity)

Let there be given a function $f : \mathbb{R}^n \to \mathbb{R}$.

(2p) a) Let f be once continuously differentiable (that is, $f \in C^1$) on $\mathbb{R}^n$. Establish the following equivalence relation:

$$f \text{ is convex on } \mathbb{R}^n \iff f(y) \geq f(x) + \nabla f(x)^{\mathrm{T}}(y - x), \quad \text{for all } x, y \in \mathbb{R}^n.$$

(1p) b) Let f be twice continuously differentiable (that is, $f \in C^2$) on $\mathbb{R}^n$. Establish the following equivalence relation:

$$f \text{ is convex on } \mathbb{R}^n \iff \nabla^2 f(x) \text{ is positive semi-definite on } \mathbb{R}^n .$$

(3p)
Question 6

(Lagrangian duality)

Consider the problem to

$$\begin{aligned} \text{minimize} \quad & -x_1 - 0.5x_2, \\ \text{subject to} \quad & (x_1 - 1)^2 + (x_2 - 1)^2 \geq 1, \\ & x_1^2 + x_2^2 \leq 1. \end{aligned}$$

Formulate the Lagrangian dual problem. Can we say something about convexity and differentiability of the dual problem? Let q be the Lagrangian dual function. Evaluate $q(1, 1/2)$ and $f(0, 1)$; what does this say about the optimal value of the primal problem f^*? Solve the primal problem graphically. Does this problem have a dual gap (i.e. is $f^* = q^*$) ? Motivate you answer!.

Question 7

(true or false claims in optimization)

For each of the following three claims, decide whether it is true or not. Motivate your answers! (Unless there is a clear motivation, no credits will be given.)

(1p) a) Consider the linear program

$$\begin{array}{lrl} \text{minimize} & z = & c_1x_1 + c_2x_2 + c_3x_3, \\ \text{subject to} & & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \le b_1, \\ & & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \le b_2, \\ & & a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \le b_3, \\ & & x_1, \quad x_2, \quad x_3 \ge 0, \end{array}$$

where $a_{ij} \in \mathbb{R},\ i = 1,2,3;\ j = 1,2,3$ and $b_j \in \mathbb{R},\ j = 1,2,3$ are such that there is at least one feasible point. Suppose that a fourth variable $x_4 \ge 0$ is added to the problem with cost coefficient c_4 and constraint coefficients $a_{j4},\ j = 1,2,3$.

<u>Claim:</u> No matter the values of c_4 and $a_{j4},\ j = 1,2,3$, the dual to the extended problem will never be unbounded.

(1p) b) <u>Claim:</u> The polyhedron in $\mathbb{R}^3$ defined by the following system

$$\begin{aligned} x_1 \qquad\qquad\quad &\le 1, \\ 2x_2 \qquad &\le 2, \\ 2x_1 + 2x_2 + 2x_3 &\le 7, \\ x_3 &\le 1, \\ x_1 + x_2 + x_3 &\le 3, \end{aligned}$$

has an extreme point at $(1,1,1)^{\mathrm{T}}$.

(1p) c) Consider the non-linear program

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}), \\ \text{subject to} & \boldsymbol{g}(\boldsymbol{x}) \le \boldsymbol{0}^m. \end{array}$$

Suppose that there is a feasible point $\boldsymbol{x}^*$ fulfilling the KKT conditions, that a CQ is fulfilled at $\boldsymbol{x}^*$ and that there is a feasible point $\boldsymbol{y}$ arbitrarily close to $\boldsymbol{x}^*$ with $f(\boldsymbol{y}) > f(\boldsymbol{x}^*)$.

<u>Claim:</u> $\boldsymbol{x}^*$ is a local minimum to the problem.

Good luck!

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EXAM SOLUTION

TMA947/MAN280
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Date: 09–08–27

Examiner: Michael Patriksson

Question 1

(the simplex method)

(2p) a) To transform the problem to standard form, first change the sign on the second contraint and then add a non-negative slack variable to the first constraint and subtract a non-negative slack (surplus) variable from the second. We get

$$\begin{array}{llrrrrrl}
\text{minimize} & z = 2x_1 - & x_2 + & x_3, & & & & \\
\text{subject to} & x_1 + 2x_2 - & x_3 + s_1 & & & = & 7, \\
& 2x_1 - & x_2 + 3x_3 & & -s_2 & = & 3, \\
& x_1, & x_2 & x_3, & s_1, & s_2 \geq & 0.
\end{array}$$

Now start phase 1 using an artificial variable $a \geq 0$ added in the second constraint. s_1 can be used as a second basic variable.

$$\begin{array}{llrrrrrrl}
\text{minimize} & w = & & & & & a, & & \\
\text{subject to} & x_1 + 2x_2 - & x_3 + s_1 & & & & & = & 7, \\
& 2x_1 - & x_2 + 3x_3 & & -s_2 + a & & & = & 3, \\
& x_1, & x_2 & x_3, & s_1, & s_2, & a \geq & & 0.
\end{array}$$

We start with the BFS given by $(s_1, a)^{\mathrm{T}}$. In the first iteration of the simplex algorithm, x_3 has the least reduced cost (-3) and is chosen as the incoming variable. The minimum ratio test then shows that a should leave the basis. By updating the basis and computing the reduced costs we see that we are now optimal with $w^* = 0$ and we proceed to phase 2.

The BFS is given by $\boldsymbol{x}_B = (s_1, x_3)^{\mathrm{T}}$, $\boldsymbol{x}_N = (x_1, x_2, s_2)^{\mathrm{T}}$ and the reduced costs with the phase 2 cost vector are $\tilde{\boldsymbol{c}}^{\mathrm{T}}_{(x_1,x_2,s_2)} = (\frac{4}{3}, -\frac{2}{3}, \frac{1}{3})$. The reduced cost is negative for x_2 which is the only eligable incoming variable. $\boldsymbol{B}^{-1}\boldsymbol{b} = (8, 1)^{\mathrm{T}}$ and $\boldsymbol{B}^{-1}\boldsymbol{N}_{x_2} = (\frac{5}{3}, -\frac{1}{3})^{\mathrm{T}}$, so the minimum ratio test shows that s_1 should leave the basis. Updating the basis and computing the new reduced costs gives $\tilde{\boldsymbol{c}}^{\mathrm{T}}_{(x_1,s_1,s_2)} = (2, \frac{2}{5}, \frac{1}{5}) \geq \boldsymbol{0}$ and thus the optimality condition is fulfilled for the current basis. We have $\boldsymbol{x}^*_B = (\frac{24}{5}, \frac{13}{5})^{\mathrm{T}}$, or in the original variables, $\boldsymbol{x}^* = (x_1, x_2, x_3)^* = (0, \frac{24}{5}, \frac{13}{5})^{\mathrm{T}}$, with the optimal value $z^* = -\frac{11}{5}$.

(1p) b) The reduced costs are not affected by the right-hand-side vector, so the only thing that has to be checked is when the current basis stays feasible.

$$\boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0} \Leftrightarrow \frac{1}{3}\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b_1 \\ 3 \end{pmatrix} \geq \boldsymbol{0} \Leftrightarrow \begin{cases} b_1 + 1 \geq 0 \\ 3 \qquad \geq 0 \end{cases} \Leftrightarrow b_1 \geq -1$$

Thus, the current basis stays optimal for all $b_1 \geq -1$.

(3p) Question 2

(modeling)

Introduce the variables x_{ij} = number of workhours that the crew j spends in turbine i,

$$y_{ij} = \begin{cases} 1, & \text{crew } j \text{ performs maintenance at turbine } i, \\ 0, & \text{otherwise;} \end{cases}$$

$$z_i = \begin{cases} 1, & \text{the turbine } i \text{ is not operational,} \\ 0, & \text{otherwise.} \end{cases}$$

The model is

$$\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{n} z_i e_i \\
\text{subject to} \quad & d_i - \sum_{j=1}^{m} x_{ij} \leq d_i z_i, \quad i \in \{1, \dots, n\}, \\
& x_{ij} \leq d_j y_{ij}, \quad i \in \{1, \dots, n\}, j \in \{1, \dots, m\}, \\
& \sum_{i=1}^{n} y_{ij} \leq 2, \quad i \in \{1, \dots, n\}, j \in \{1, \dots, m\}, \\
& \sum_{i=1}^{n} x_{ij} + \sum_{i=1}^{n} 2c_{ij} y_{ij} \leq 8, \quad j \in \{1, \dots, m\}, \\
& x_{ij} \geq 0, y_{ij}, z_i \in \{0, 1\} \quad i \in \{1, \dots, n\}, j \in \{1, \dots, m\}.
\end{aligned}$$

Question 3

(optimality conditions)

See The Book, Theorem 10.10.

Question 4

(exterior penalty method)

(1p) a) Direct application of the KKT conditions yield that $\boldsymbol{x}^* = (\frac{3}{5}, \frac{2}{5})^{\mathrm{T}}$ and $\lambda^* = -1/5$ uniquely.

(1p) b) Letting the penalty parameter be $\nu > 0$, it follows that $\boldsymbol{x}_\nu = \frac{\nu}{1+5\nu}(3,2)^{\mathrm{T}}$. Clearly, as $\nu \to \infty$ convergence to the optimal primal–dual solution follows.

(1p) c) From the stationarity conditions of the penalty function $\boldsymbol{x} \mapsto f(\boldsymbol{x}) + \lambda h(\boldsymbol{x}) + \nu|h(\boldsymbol{x})|^2$ follow that $\boldsymbol{x}_\nu$ fulfills $\nabla f(\boldsymbol{x}_\nu) + [2\nu h(\boldsymbol{x}_\nu)]\nabla h(\boldsymbol{x}_\nu) = 0^2$, and hence a proper Lagrange multiplier estimate comes out as $\lambda_\nu := 2\nu h(\boldsymbol{x}_\nu)$. Insertion from b) yields $\lambda_\nu = \frac{-\nu}{1+5\nu}$, which tends to $\lambda^* = -\frac{1}{5}$ as $\nu \to \infty$.

Question 5

(topics in convexity)

(2p) a) See Theorem 3.40.

(1p) b) See Theorem 3.42.

(3p) Question 6

(Lagrangian dual)

$L(x,\mu) = -x_1 - 1/2x_2 + \mu_1(x_1^2 + x_2^2 - 1) + \mu_2(1 - (x_1 - 1)^2 - (x_2 - 1)^2)$. The dual function is $q(\mu) = \min_x(L(x,\mu)) = \min_{x_1} \underbrace{(-x_1 + \mu_1 x_1^2 - \mu_2(x_1 - 1)^2)}_{q_1(x_1)}$

$+ \min_{x_2} \underbrace{(-1/2x_1 + \mu_1 x_1^2 - \mu_2(x_1 - 1)^2) + \mu_2}_{q_2(x_2)}.$

$\frac{dq_1}{dx_1} = -1+2\mu_1 x_1 - 2\mu_2(x_1-1)$ and $\frac{d^2q_1}{dx_1^2} = 2(\mu_1-\mu_2)$. We notice that q_1 is strictly convex for $\mu_1 > \mu_2$ and strictly concave for $\mu_1 < \mu_2$ and linear for $\mu_1 = \mu_2$. For $\mu_1 > \mu_2$ the minimum is attained attained at $x_1 = \frac{1-2\mu_2}{2(\mu_1-\mu_2)}$ and is $-\infty$ for $\mu_1 < \mu_2$. Similarly for q_2 we obtain $x_2 = \frac{1/2-2\mu_2}{2(\mu_1-\mu_2)}$. Simplifying and inserting into L yields $q(\mu) = \frac{8(3-2\mu_2)\mu_2-16\mu_1^2-5}{16(\mu_1-\mu_2)}$ if $\mu_1 > \mu_2$. If $\mu_1 = \mu_2$ the derivatives of q_1 and q_2 can not be zero simultaneosly. We therefore have $q_1(\mu) = -\infty$ or $q_2(\mu) = -\infty$. We therefore have $q(\mu) = -\infty$ if $\mu_1 \leq \mu_2$.

The dual problem can be formulated as $\max_{\mu \geq 0} q(\mu)$. The dual problem is always convex; in the pressent case it is also differentiable.

$q(1, 1/2) = -13/8$ and $f(0, 1) = -1/2$; we can therefore conclude (by weak duality) that $-13/8 \leq f^* \leq -1/2$.

Drawing the feasible region together with the linear objective gives the optimal solution $x^* = (1, 0)$, $f^* = -1$.

The problem is non-convex, hence a dual gap can exist. Assume there is no duality gap, then according to Theorem 6.7 $L(x^*, \mu^*) = \min_x L(x, \mu^*)$. If μ^* is optimal then $\mu_1^* > \mu_2^*$. Since the function $L(\cdot, \mu)$ is strictly convex, the minimum is obtained at $\nabla_x L(\cdot, \mu) = 0$. Therefore $1 = \frac{1-2\mu_2}{2(\mu_1-\mu_2)}$ and $0 = \frac{1/2-2\mu_2}{2(\mu_1-\mu_2)}$. This yields $\mu_2 = 1/4$ and $\mu_1 = 1/2$. Since $q(1/2, 1/4) = -1$ no duality gap exists.

Question 7

(true or false claims in optimization)

(1p) a) True. The important implication is that if a problem is unbounded, then its dual must be infeasible. The adding of an extra variable relaxes the original problem. Since there is a feasible point to the original problem, the extended problem will also have a feasible solution (e.g., by setting $x_4 = 0$). If the dual to the extended problem is unbounded the primal problem (dual to the dual) must be infeasible. This is not the case and the claim is proved.

(1p) b) True. The equality subsystem at $(1, 1, 1)^{\mathrm{T}}$ consists of all rows but the third, so

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

The rank of $\tilde{A}$ is 3 since the first three rows are linearly independent. So, $rank(\tilde{A}) = n$ which implies that the proposed point is an extreme point (in this case corresponding to a degenerate basis).

(1p) c) False. A counterexample in $\mathbb{R}^2$ is given by the problem defined by $f(\boldsymbol{x}) = x_2$, $g(\boldsymbol{x}) = -x_1^2 - x_2$ at the point $\boldsymbol{x}^* = (0, 0)^{\mathrm{T}}$. The conditions are fulfilled, but all balls around $\boldsymbol{x}^*$ contain points with smaller objective values.