Chalmers/Gothenburg University
Mathematical Sciences

EXAM

TMA947/MAN280
OPTIMIZATION, BASIC COURSE

Date:	08–03–25
Time:	House V, morning
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Adam Wojciechowski (0762-721860)
Result announced:	08–04–02 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$\begin{array}{lrcl}
\text{minimize} & z = -x_1 - x_2, & & \\
\text{subject to} & -x_1 - 2x_2 - x_3 & = & 2, \\
& 3x_1 + x_2 & \leq & -1, \\
& x_2, \; x_3 & \geq & 0, \\
& x_1 & \in & \mathbb{R} \text{ (free)}.
\end{array}$$

(2p) a) Solve this problem by using phase I and phase II of the simplex method.
[*Aid:* Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

for producing basis inverses.]

(1p) b) Motivate using the solution from a) and the relationships between primal and dual problems why there cannot exist a vector $\boldsymbol{u} = (u_1, u_2, u_3)^{\mathrm{T}}$ fulfilling the following system of constraints:

$$\begin{pmatrix} -1 & 3 & 0 \\ -2 & 1 & 1 \\ 2 & -1 & 0 \end{pmatrix} \boldsymbol{u} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad u_1 \geq 0, \; u_2 \leq 0, \; u_3 \geq 0.$$

Question 2

(modelling)

Consider the mixed-integer problem (MIP) of minimizing the linear function $f(\boldsymbol{x}, \boldsymbol{y})$ over the set $X \times Y$, where $X = \{\, \boldsymbol{x} \in \{0,1\}^n \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \,\}$ and $Y = \{\, \boldsymbol{y} \in \mathbb{R}^m \mid y_i \geq 0, \; i = 1, \ldots, m \,\}$.

(1p) a) Formulate the mixed-integer problem as *one* non-linear program using *only continuous* variables and continuous constraints.

(2p) b) Assume that $n = 1$. Explain how to solve the mixed-integer problem by solving *a number of* linear programs. Formulate these programs.

Question 3

(topics in Lagrangian duality)

Consider the problem to find

$$\begin{aligned} f^* := \underset{x}{\text{infimum}} \;\; & f(\boldsymbol{x}), \\ \text{subject to} \;\; & g_i(\boldsymbol{x}) \le 0, \qquad i = 1, \dots, m, \\ & \boldsymbol{x} \in X, \end{aligned} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \dots, m$, are given functions, and $X \subseteq \mathbb{R}^n$.

Consider also the Lagrangian dual problem to find

$$q^* := \underset{\boldsymbol{\mu} \ge \boldsymbol{0}^m}{\text{supremum}} \;\; q(\boldsymbol{\mu}), \tag{2}$$

where

$$q(\boldsymbol{\mu}) = \underset{x \in X}{\text{infimum}} \;\; L(\boldsymbol{x}, \boldsymbol{\mu}),$$

and the function $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is defined by

$$L(\boldsymbol{x}, \boldsymbol{\mu}) = f(\boldsymbol{x}) + \sum_{i=1}^{m} \mu_i g_i(\boldsymbol{x}).$$

(1p) a) Establish that the optimization problem (2) is a convex problem.

(1p) b) Suppose that all the functions f and g_i, $i = 1, 2, \dots, m$, are continuous and that X is nonempty, closed and bounded. Establish that the function q is finite on $\mathbb{R}^m$.

(1p) c) Take as an example $f(x) := x$, $m = 1$ and $g_1(x) = \frac{1}{2}x^2$, and $X := \mathbb{R}$. What is the optimal primal solution (if any)? What is the optimal dual solution (if any)? Letting $\Gamma := f^* - q^*$ denote the “duality gap” of the problem, what is the value of Γ in this instance?

(3p) **Question 4**

(complementarity slackness theorem)

Consider the primal–dual pair of linear programs given by

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned} \tag{1}$$

and

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \\ \text{subject to} \quad & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \geq \boldsymbol{c}, \\ & \boldsymbol{y} \geq \boldsymbol{0}^m. \end{aligned} \tag{2}$$

THEOREM 1 (Complementary Slackness Theorem) *Let $\boldsymbol{x}$ be a feasible solution to (1) and $\boldsymbol{y}$ a feasible solution to (2). Then $\boldsymbol{x}$ is optimal to (1) and $\boldsymbol{y}$ optimal to (2) if and only if*

$$x_j(c_j - \boldsymbol{y}^{\mathrm{T}}\boldsymbol{A}_{\cdot j}) = 0, \qquad j = 1, \dots, n, \tag{3a}$$

$$y_i(\boldsymbol{A}_{i\cdot}\boldsymbol{x} - b_i) = 0, \qquad i = 1 \dots, m, \tag{3b}$$

where $\boldsymbol{A}_{\cdot j}$ is the j^{th} column of $\boldsymbol{A}$ and $\boldsymbol{A}_{i\cdot}$ the i^{th} row of $\boldsymbol{A}$. ∎

Prove this theorem. If you wish to refer to other theorems from The Book in your proof, then state (but do not prove) those theorems, as they apply to the problem given.

Question 5

(quadratic programming)

(1p) a) Consider the quadratic problem:

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{x}^{\mathrm{T}}\boldsymbol{H}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \end{aligned} \tag{QP}$$

where $\boldsymbol{H} \in \mathbb{R}^{n\times n}$ is symmetric and positive definite, $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ has full row rank, $\boldsymbol{x} \in \mathbb{R}^n$, and $\boldsymbol{b} \in \mathbb{R}^m$. Set up the KKT-conditions and find the optimal Lagrange multipliers.

(1p) b) The Lagrange dual problem to (QP) is also a quadratic problem. State the (quadratic) dual problem and show that the dual solution is identical to the Lagrange multipliers in problem a).

(1p) c) Let $\boldsymbol{Z} \in \mathbb{R}^{n\times(n-m)}$ be the null-space matrix to $\boldsymbol{A}$ in (QP), i.e., $\boldsymbol{AZ} = \boldsymbol{0}^{m\times(n-m)}$. Assume $\boldsymbol{H}$ is neither positive definite nor positive semidefinite, but that $\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{HZ}$ is positive semidefinite. Is a local optimal solution in (QP) a global optimal solution? Answer true or false, and motivate your answer!

(3p) Question 6

(the Frank-Wolfe algorithm)

Consider the problem

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2}x_1^2 - \tfrac{1}{2}(x_2-1)^2, \\ \text{subject to} \quad & \qquad\qquad x_1 \leq 2, \\ & \qquad 0 \leq x_2 \leq 2, \\ & 1-4x_1 \leq x_2 \leq 1+4x_1. \end{aligned}$$

Start at $\boldsymbol{x}^0 = (1,1)^{\mathrm{T}}$ and perform *one*(!) complete iteration with the Frank-Wolfe algorithm. Is the resulting vector $\boldsymbol{x}^1$ a KKT-point? Is it a local minimum? Is it a global minimum? Motivate your answers!

Question 7

(nonlinear optimization solves interesting problems)

(1p) a) Fermat's Last Theorem states that there are no solutions in the positive integers of the equation

$$x^n + y^n = z^n,$$

for $n \geq 3$. Re-state this problem as a continuous nonlinear program, whose optimal solution reveals the answer to the above question.

(1p) b) Show that for any symmetric and positive definite matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ there exists a positive number c such that

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Ax} \geq c\|\boldsymbol{x}\|^2, \qquad \boldsymbol{x} \in \mathbb{R}^n.$$

(1p) c) An $n \times n$ matrix $\boldsymbol{A}$ is said to be *invertible* if there exists for each vector $\boldsymbol{y} \in \mathbb{R}^n$ a unique vector $\boldsymbol{x} \in \mathbb{R}^n$ such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$. (Then, there is a unique $n \times n$ matrix $\boldsymbol{A}^{-1}$ such that $\boldsymbol{A}^{-1}\boldsymbol{y} = \boldsymbol{x}$ precisely if $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$. The matrix $\boldsymbol{A}^{-1}$ is then denoted the *inverse* of $\boldsymbol{A}$.)

Show that if $\boldsymbol{A}$ is a positive definite and symmetric $n \times n$ matrix then $\boldsymbol{A}$ is invertible.

Good luck!

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EXAM SOLUTION

TMA947/MAN280
APPLIED OPTIMIZATION

Date: 08–03–25
Examiner: Michael Patriksson

Question 1

(the simplex method)

(2p) a) To transform the problem to standard form, first replace the free variable x_1 with the non-negative variables x_1^+ and x_1^- such that $x_1 = x_1^+ - x_1^-$. Then change sign on the inequality constraint and substract a slack variable $s_1 \geq 0$. A BFS cannot be found directly, hence begin with a phase 1 problem using artificial variables $a_1, a_2 \geq 0$ in both constraints. The objective is to minimize $a_1 + a_2$. Start with the BFS given by (a_1, a_2) in the basis. In the first iteration of the simplex algorithm, x_1^- is the only variable with a negative reduced cost (-4), and is therefore the only eligable incoming variable. The minimum ratio test shows that a_2 should leave the basis. In the next iteration, s_1 is the only variable with a negative reduced cost $(-\frac{1}{3})$ and is chosen as the incoming variable. The minimum ratio test shows that x_3 should leave. No artificial variables are left in the basis, and we can proceed to phase 2.

The reduced costs in the first iteration of the phase 2 problem are

$$\tilde{c}^{\mathrm{T}}_{(x_1^+, x_2, x_3)} = (0, 1, 1) \geq \mathbf{0},$$

and thus the optimality condition is fulfilled for the current basis. We have $\boldsymbol{x}_B^* = (5, 2)^{\mathrm{T}}$, or, in the original variables, $\boldsymbol{x}^* = (x_1, x_2, x_3)^* = (-2, 0, 0)^{\mathrm{T}}$, with the optimal value $z^* = 2$.

(1p) b) The dual to the LP is given by

$$\begin{array}{lrrrl}
\text{maximize} & w = & 2\,x_1 - & y_2, & \\
\text{subject to} & & -\;\; y_1 + & 3y_2 & = -1, \\
& & -\,2y_1 + & y_2 & \leq -1, \\
& & -\;\; y_1 & & \leq \;\; 0, \\
& & y_1 & & \in \mathbb{R} \text{ (free)}, \\
& & & y_2 & \leq \;\; 0.
\end{array}$$

The primal problem has an optimal solution. Then, from strong duality, so does the dual problem. Add a slack variable $y_3 \geq 0$ in the second constraint and let the dual optimal solution be $\boldsymbol{y}^*$. There cannot exist a solution $\boldsymbol{u}$ to the given system, since if that would be the case, then $\boldsymbol{y}^* + \boldsymbol{u}$ would be feasible in the dual with a larger objective value (from the third row in the system). This is a contradiction to $\boldsymbol{y}^*$ being optimal.

Question 2

(modelling)

(1p) a) One possibility is the following formulation:

$$
\begin{aligned}
\min \quad & f(\boldsymbol{x}, \boldsymbol{y}), \\
\text{s.t.} \quad & x_j(1 - x_j) = 0, \qquad j = 1, \dots, n, \\
& \boldsymbol{Ax} \leq \boldsymbol{b}, \\
& y_i \geq 0, \qquad i = 1, \dots, m.
\end{aligned}
\tag{NLP}
$$

(2p) b) Two problems must be solved. The optimal solution to MIP is given by the solution to the problem with the least optimal value. That is, the optimal value of MIP is

$$z^* = \min\{z_0^*, z_1^*\},$$

where

$$
\begin{aligned}
z_0^* = \min \quad & f(x, \boldsymbol{y}), \\
\text{s.t.} \quad & x = 0, \\
& \boldsymbol{Ax} \leq \boldsymbol{b}, \\
& y_i \geq 0, \qquad i = 1, \dots, m,
\end{aligned}
\tag{P^0}
$$

and

$$
\begin{aligned}
z_1^* = \min \quad & f(x, \boldsymbol{y}), \\
\text{s.t.} \quad & x = 1, \\
& \boldsymbol{Ax} \leq \boldsymbol{b}, \\
& y_i \geq 0, \qquad i = 1, \dots, m.
\end{aligned}
\tag{P^1}
$$

Question 3

(topics in Lagrangian duality)

(1p) a) See The Book, Theorem 6.4.

(1p) b) Under the assumptions on X, for any vector $\boldsymbol{\mu} \in \mathbb{R}^m$ the function $L(\cdot, \boldsymbol{\mu})$ is weakly coercive with respect to X (see The Book, Definition 4.5). By the continuity assumptions on f and g_i, $i = 1, \dots, m$, $L(\cdot, \boldsymbol{\mu})$ is also continuous. Hence, Weierstrass' Theorem 4.7 applies.

(1p) c) $x^* = 0$; the dual problem has no optimal solution; however, $f^* = q^* = 0$, whence the duality gap $\Gamma = 0$.

(3p) Question 4

(complementarity slackness theorem)

We first establish that if the system (3) is satisfied at $(\boldsymbol{x}, \boldsymbol{y})$ then the pair $(\boldsymbol{x}, \boldsymbol{y})$ is primal–dual optimal in (1), (2). By assumption, $\boldsymbol{x}$ (respectively, $\boldsymbol{y}$) is a feasible solution to the primal (respectively, dual) problem. By the Weak Duality Theorem 10.5, then, $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \leq \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$. The system (3) implies that in fact equality holds. This immediately, by the Corollary 10.6 to the Weak Duality Theorem, implies that the pair $(\boldsymbol{x}, \boldsymbol{y})$ must be optimal.

Suppose then that the pair $(\boldsymbol{x}, \boldsymbol{y})$ is primal–dual optimal in (1), (2). Then, $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$ holds, by the Strong Duality Theorem. In the string of inequalities

$$\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \leq \boldsymbol{y}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{x} \leq \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$$

provided by the Weak Duality Theorem 10.5, equality then must hold throughout. From the resulting two equalities then follow (3).

Question 5

(quadratic programming)

(1p) a) The KKT-conditions are:

$$2\boldsymbol{H}\boldsymbol{x} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda} = \boldsymbol{0}^n$$

The problem is a convex problem with linear constraints, so a feasible solution which fulfills the KKT-conditions is a global optimal solution. Since $\boldsymbol{H}$ is postive definite and hence invertible, we have:

$$\boldsymbol{x}^* = -\tfrac{1}{2}\boldsymbol{H}^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda},$$

and

$$\boldsymbol{A}\boldsymbol{x}^* = \boldsymbol{b} \Rightarrow -\tfrac{1}{2}\boldsymbol{A}\boldsymbol{H}^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda} = \boldsymbol{b}.$$

Since $\boldsymbol{A}$ has full row rank, $\boldsymbol{A}\boldsymbol{H}^{-1}\boldsymbol{A}^{\mathrm{T}}$ is invertible, and so

$$\boldsymbol{\lambda}^* = -2(\boldsymbol{A}\boldsymbol{H}^{-1}\boldsymbol{A}^{\mathrm{T}})^{-1}\boldsymbol{b}.$$

(1p) b) The dual function is found by minimizing the Lagrangian for each $\boldsymbol{\lambda}$. So

$$\boldsymbol{x}^*(\boldsymbol{\lambda}) = -\tfrac{1}{2}\boldsymbol{H}^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda},$$

which gives

$$q(\boldsymbol{\lambda}) = L(\boldsymbol{x}^*(\boldsymbol{\lambda}), \boldsymbol{\lambda}) = -\tfrac{1}{4}\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{H}^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda} - \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{b}.$$

In the dual problem, we want to maximize the dual function. Since we have equality constraints in the primal, we have no bounds on the dual variables:

$$\text{maximize} \quad q(\boldsymbol{\lambda}) := -\tfrac{1}{4}\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{H}^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda} - \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{b}. \qquad \text{(Dual QP)}$$

The dual problem is a convex unconstrained problem, and a dual optimal solution is therefore found by setting the gradient of q to zero, which yields $\boldsymbol{\lambda}^* = -2(\boldsymbol{A}\boldsymbol{H}^{-1}\boldsymbol{A}^{\mathrm{T}})^{-1}\boldsymbol{b}$.

(1p) c) Let $\hat{\boldsymbol{x}}$ be a feasible solution, i.e., $\boldsymbol{A}\hat{\boldsymbol{x}} = \boldsymbol{b}$. Then $\boldsymbol{x} = \hat{\boldsymbol{x}} + \boldsymbol{Z}\boldsymbol{p}$, $\boldsymbol{p} \in \mathbb{R}^{n-m}$ is also a feasible solution, and (QP) is equivalent to:

$$\underset{\boldsymbol{p}\in\mathbb{R}^{n-m}}{\text{minimize}} \; \boldsymbol{p}^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{H}\boldsymbol{Z}\boldsymbol{p} + 2\boldsymbol{p}^{\mathrm{T}}\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{H}\hat{\boldsymbol{x}} + \text{const.}$$

This is an unconstrained problem with a pos. semidef. Hessian, and hence it is a convex problem. A local optimal solution is a global optimal solution.

(3p) Question 6

(The Frank-Wolfe algorithm)

Since the objective function is nonconvex, we cannot provide any lower bounds from the subproblem solutions. An upper bound is $f(\boldsymbol{x}^0) = 0.5$. At $\boldsymbol{x}^0 = (1,1)^{\mathrm{T}}$, $\nabla f(\boldsymbol{x}^0) = (1,0)^{\mathrm{T}}$; $\boldsymbol{y}^0 = (0,1)^{\mathrm{T}}$; $\operatorname{argmin}_{\alpha\in[0,1]}\varphi(\alpha) = 1$, where $\varphi(\alpha) = f(\boldsymbol{x}^0 + \alpha(\boldsymbol{y}^0 - \boldsymbol{x}^0))$; $\boldsymbol{x}^1 = (0,1)^{\mathrm{T}}$; $\nabla f(\boldsymbol{x}^1) = (0,0)^{\mathrm{T}}$. A new upper bound is $f(\boldsymbol{x}^1) = 0$. The vector $\boldsymbol{x}^1$ is a KKT-point (set all Lagrange multipliers to zero). It is not a local minimum, however, since for example $\boldsymbol{x}(t) = \boldsymbol{x}^1 + (t, 4t)^{\mathrm{T}}$ is feasible for $0 \le t \le 0.25$, and for $t > 0$, $f(\boldsymbol{x}(t)) = -\frac{15}{2}t^2 < f(\boldsymbol{x}^1)$. [There are two global minima: $\boldsymbol{x}^* = (0.25, 0)^{\mathrm{T}}$ and $\boldsymbol{x}^* = (0.25, 2)^{\mathrm{T}}$.]

Question 7

(nonlinear optimization solves interesting problems)

(1p) a) Let $\varepsilon > 0$ be any small enough number. The optimization problem is to find

$$\begin{aligned} f^* = \underset{(x,y,z,n)\in\mathbb{R}^4_+}{\text{minimum}} \ & f(x,y,z,n) := (x^n + y^n - z^n)^2, \\ \text{subject to} \ & \sin \pi x = \sin \pi y = \sin \pi z = \sin \pi n = 0, \\ & xyz \geq \varepsilon, \\ & n \geq 3. \end{aligned}$$

If $f^* > 0$ then Fermat's Last Theorem has been proved. (Which it already has by other means.)

(1p) b) Consider the problem to

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \ & f(\boldsymbol{x}) := \boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}, \\ \text{subject to} \ & \|\boldsymbol{x}\| = 1. \end{aligned}$$

An optimal solution, say $\boldsymbol{x}^*$, exists due to Weierstrass' Theorem, as the sphere is non-empty, closed and bounded. For each non-zero vector $\boldsymbol{x} \in \mathbb{R}^n$, the vector $\|\boldsymbol{x}\|^{-1}\boldsymbol{x}$ is a feasible solution; hence, $\|\boldsymbol{x}\|^{-2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} \geq (\boldsymbol{x}^*)^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}^* =: c$.

(1p) c) Choose $\boldsymbol{y} \in \mathbb{R}^n$ aribtrarily. To prove existence and uniqueness of a solution to the equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$, consider the minimization of $f(\boldsymbol{x}) := \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}^{\mathrm{T}}\boldsymbol{x}$ over $\boldsymbol{x} \in \mathbb{R}^n$.

The function f is coercive on $\mathbb{R}^n$, whence Weierstrass' Theorem applies; the problem has an optimal solution. As it is unconstrained, we know that stationarity is a necessary condition, so we set the gradient of f to zero: $\nabla f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{y} = \boldsymbol{0}^n$, and so we know that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$ holds. To establish uniqueness, we may observe that $\boldsymbol{A}\boldsymbol{x}^1 = \boldsymbol{A}\boldsymbol{x}^2 = \boldsymbol{y}$ implies that $\boldsymbol{A}(\boldsymbol{x}^1 - \boldsymbol{x}^2) = \boldsymbol{0}^n$ and hence that $(\boldsymbol{x}^1 - \boldsymbol{x}^2)^{\mathrm{T}}\boldsymbol{A}(\boldsymbol{x}^1 - \boldsymbol{x}^2) = 0$. By positive definiteness this implies that $\boldsymbol{x}^1 = \boldsymbol{x}^2$. We are done.