Chalmers/GU
Mathematics

EXAM

TMA947/MAN280
OPTIMIZATION, BASIC COURSE

Date:	07–08–30
Time:	House V, morning
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Peter Lindroth (0762-721860)
Result announced:	07–09–13 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the Simplex method)

Consider the following linear program:

$$\begin{array}{llrrrl}
\text{minimize} & z = & x_1 & -x_2 & +x_3, & \\
\text{subject to} & & x_1 & +2x_2 & -2x_3 & \leq 0, \\
& & -x_1 & & +x_3 & \leq -1, \\
& & x_1, & x_2, & x_3 & \geq 0.
\end{array}$$

(2p) a) Solve this problem by using phase I and phase II of the simplex method. [*Aid:* Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

for producing basis inverses.]

(1p) b) Is the solution unique? Motivate!

(3p) Question 2

(strong duality in linear programming)

Consider the following standard form of a linear program:

$$\begin{aligned}
\text{minimize} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\
\text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\
& \boldsymbol{x} \geq \boldsymbol{0}^n,
\end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{c}, \boldsymbol{x} \in \mathbb{R}^n$, and $\boldsymbol{b} \in \mathbb{R}^m$. State and prove the Strong Duality Theorem in linear programming.

Question 3

(exterior penalty method)

Consider the following problem:

$$\begin{aligned} &\text{minimize} \quad f(x) := \frac{1}{2}(x_1)^2 - x_1x_2 + (x_2)^2, \\ &\text{subject to} \quad x_1 + x_2 - 1 = 0. \end{aligned}$$

(1p) a) By applying the KKT conditions to this problem, establish its (unique) exact primal–dual solution.

(1p) b) Apply the standard exterior quadratic penalty method for this problem, and show that the sequence of (explicitly stated) subproblem solutions converges to the unique primal solution.

(1p) c) From the theory of exterior penalty methods provide the corresponding sequence of estimates of the Lagrange multiplier, and show that it converges to the solution provided in a).

Question 4

(true or false claims in optimization)

For each of the following three claims, your task is to decide whether it is true or false. Motivate your answers!

(1p) a) The vector $\boldsymbol{p} := (-1, 1)^{\mathrm{T}}$ is a descent direction for $f(\boldsymbol{x}) := (x_1 + x_2^2)^2$ at $\boldsymbol{x} = (1, 0)^{\mathrm{T}}$.

(1p) b) Suppose $f \in C^2$. If, at some iteration point $\boldsymbol{x} \in \mathbb{R}^n$ there exists a solution $\boldsymbol{p}$ to the search direction-finding problem of Newton's method then it defines a descent direction for f at $\boldsymbol{x}$.

(1p) c) Suppose $f \in C^1$. If $\boldsymbol{p} \in \mathbb{R}^n$ is a descent direction for f at $\boldsymbol{x}$ then the Armijo step length rule will in a finite number of steps provide a vector $\bar{\boldsymbol{x}}$ with $f(\bar{\boldsymbol{x}}) < f(\boldsymbol{x})$.

(3p) **Question 5**

(least-squares minimization)

Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^{m\times 1}$. If $m > n$, the system $\boldsymbol{Ax} = \boldsymbol{b}$ is over-determined and has in general no exact solution. Systems of this type appear e.g. in applications when a linear function is to be fitted to experimental data. The linear least-squares solution to the system is the vector

$$\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}} ||\boldsymbol{Ax} - \boldsymbol{b}||_2$$

If the rank of $\boldsymbol{A}$ is n, motivate using optimality conditions and derive the closed form of the least-squares solution

$$\boldsymbol{x}^* = (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}.$$

(3p) **Question 6**

(modelling)

You are assigned to model the planning of personnel for a construction project. The project will take 24 months to complete. To meet your requirements, you must have d_t construction workers working during month t, $t = 1, \cdots, 24$. If you recruit any workers in month t, you need to send them to a short introduction course. The course costs a fixed amount of k SEK regardless of the number of workers participating in the course, and w SEK for each participant. The salary for a construction worker is r SEK per month. Also, you cannot hire anyone for a period shorter than 3 months. Before the project starts ($t = 0$), you have no working personnel.

Your task is to minimize the total cost, subject to the above mentioned requirements. Formulate this as an optimization problem. [*Hint:* Let x_{ij} be the number of workers recruited at the beginning of month i to the end of month j.]

Question 7

(duality in linear and nonlinear optimization)

(1p) a) Consider the LP problem to

$$\text{minimize} \quad z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \quad + \boldsymbol{d}^{\mathrm{T}}\boldsymbol{v}$$

$$
\begin{aligned}
\text{subject to} \quad & \boldsymbol{A}_1\boldsymbol{x} + \boldsymbol{B}\boldsymbol{v} \geq \boldsymbol{b}_1, \\
& \boldsymbol{A}_2\boldsymbol{x} = \boldsymbol{b}_2, \\
& \sum_{k=1}^{\ell} v_k = a, \\
& \boldsymbol{x} \geq \boldsymbol{0}^n, \\
& \boldsymbol{v} \geq \boldsymbol{0}^\ell,
\end{aligned}
$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{v} \in \mathbb{R}^\ell$, $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{d} \in \mathbb{R}^\ell$, $\boldsymbol{A}_1 \in \mathbb{R}^{m_1 \times n}$, $\boldsymbol{A}_2 \in \mathbb{R}^{m_2 \times n}$, $\boldsymbol{B} \in \mathbb{R}^{m_1 \times \ell}$, $\boldsymbol{b}_1 \in \mathbb{R}^{m_1}$, $\boldsymbol{b}_2 \in \mathbb{R}^{m_2}$, and $a \in \mathbb{R}$.

State its LP dual problem.

(2p) b) Consider the strictly convex quadratic optimization problem to

$$
\begin{aligned}
\text{minimize} \quad & f(\boldsymbol{x}) := 2x_1^2 + x_2^2 - 4x_1 - 6x_2, && \text{(1a)} \\
\text{subject to} \quad & -x_1 + 2x_2 \leq 4. && \text{(1b)}
\end{aligned}
$$

For this problem, do the following:

[1] explicitly state its Lagrangian dual function q and its Lagrangian dual problem, associated with the Lagrangian relaxation of the constraint (1b);

[2] solve this Lagrangian dual problem and provide the optimal Lagrange multiplier μ^*;

[3] provide the globally optimal solution $\boldsymbol{x}^*$ to the problem (1); and

[4] prove that strong duality holds, that is, prove that $q(\mu^*) = f(\boldsymbol{x}^*)$ holds.

Good luck!

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EXAM SOLUTION

TMA947/MAN280
APPLIED OPTIMIZATION

Date: 07–08–30
Examiner: Michael Patriksson

Question 1

(the Simplex method)

(2p) a) After changing sign of the second inequality and adding two slack variables s_1 and s_2, a BFS cannot be found directly. We create the phase I problem through an added artificial variable a_1 in the second linear constraint; the value of a_1 is to be minimized.

We use the BFS based on the variable pair (s_1, a_1) as the starting BFS for the phase I problem. In the first iteration of the Simplex method x_1 is the only variable with a negative reduced cost; hence x_1 is picked as the incoming variable. The minimum ratio test shows that s_1 should leave the basis. In the next iteration the reduced cost for varialbe x_3 is negative, and x_3 is picked as the incoming variable. The minimum ratio test shows that a_1 should leave the basis. We have found an optimal basis, $x_B = (x_1, x_3)^{\mathrm{T}}$, to the phase I problem. We proceed to phase II, since the basis is feasible in the original problem.

Starting phase II with this BFS, we see that all reduced costs are positive, $\tilde{c}_N = (3, 2, 3)^{\mathrm{T}} > 0$, and thus the BFS is optimal. $x_B = B^{-1}b = (2, 1)^{\mathrm{T}}$ so $x^* = (2, 0, 3)^{\mathrm{T}}$ and $z^* = c_B^{\mathrm{T}} x_B = 3$.

(1p) b) Yes. The reduced costs are positive.

(3p) Question 2

(strong duality in linear programming)

See Theorem 10.6 in The Book.

Question 3

(exterior penalty method)

(1p) a) Direct application of the KKT conditions yield that $\boldsymbol{x}^* = (\frac{3}{5}, \frac{2}{5})^{\mathrm{T}}$ and $\lambda^* = -1/5$ uniquely.

(1p) b) Letting the penalty parameter be $\nu > 0$, it follows that $\boldsymbol{x}_\nu = \frac{\nu}{1+5\nu}(3, 2)^{\mathrm{T}}$. Clearly, as $\nu \to \infty$ convergence to the optimal primal–dual solution follows.

(1p) c) From the stationarity conditions of the penalty function $\boldsymbol{x} \mapsto f(\boldsymbol{x}) + \lambda h(\boldsymbol{x}) + \nu |h(\boldsymbol{x})|^2$ follow that $\boldsymbol{x}_\nu$ fulfills $\nabla f(\boldsymbol{x}_\nu) + [2\nu h(\boldsymbol{x}_\nu)] \nabla h(\boldsymbol{x}_\nu) = 0^2$, and hence a proper Lagrange multiplier estimate comes out as $\lambda_\nu := 2\nu h(\boldsymbol{x}_\nu)$. Insertion from b) yields $\lambda_\nu = \frac{-\nu}{1+5\nu}$, which tends to $\lambda^* = -\frac{1}{5}$ as $\nu \to \infty$.

Question 4

(true or false claims in optimization)

(1p) a) True. $\nabla f(\boldsymbol{x})^{\mathrm{T}} \boldsymbol{p} = -2$.

(1p) b) False. Suppose, for example, that the Hessian of f at $\boldsymbol{x}$ is negative definite, and that $\boldsymbol{x}$ is not a stationary point. Then the Newton direction is well-defined but it is an ascent direction.

(1p) c) True. The result follows rather immediately from the definition of descent direction.

(3p) Question 5

(least-squares minimization)

We wish to minimize $||\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}||_2$ or equivalently $f(\boldsymbol{x}) = ||\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}||_2^2$ over $\boldsymbol{x} \in \mathbb{R}^n$, i.e. we have a unconstrained optimization problem. We rewrite

$$f(\boldsymbol{x}) = (\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b})^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} - \boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b} - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b}$$

The hessian of $f(\boldsymbol{x})$ is $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ and is always positive semi-definite since $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{v} = ||\boldsymbol{A}\boldsymbol{v}||^2 \geq 0 \quad \forall \boldsymbol{v}$. Thus, the minimization problem is convex and from the optimality conditions we know that stationarity is sufficient for a point to be optimal.

We have $\bigtriangledown f(\boldsymbol{x}^*) = \boldsymbol{0} \Longleftrightarrow \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}^* = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}$. If the rank of $\boldsymbol{A}$ is n then $||\boldsymbol{A}\boldsymbol{v}||^2 > 0 \quad \forall \boldsymbol{v} \neq \boldsymbol{0}$, the hessian is positive definite and therefore invertible and we get the wanted result.

(3p) Question 6

(modelling)

Introduce the variables:

x_{ij} Number of persons recruited in the beginning of month i to the end of month j, $i = 1, \dots, 24$, $j = i, \dots, 24$

y_t Value one if anyone is recruited month t, zero otherwise.

The objective is

$$\min \quad \sum_i \sum_j w x_{ij} + \sum_i y_i k + \sum_{t=1}^{24} \sum_{i \le t, j \ge t} x_{ij} r$$

and the constraints are

$$\begin{aligned}
\sum_{i \le t, j \ge t} x_{ij} &\ge d_t, && t = 1, \dots, 24, \\
x_{ij} &= 0, && \forall (i,j) : i = j, i = j+1, && (1) \\
M y_i &\ge \sum_j x_{ij}, && i = 1, \dots, 24 && (2) \\
y_t &\in \mathbb{B} \\
x_{ij} &\in \mathbb{Z}_+
\end{aligned}$$

where M is a big number. Constraint (1) sets the required work force. Constraint (1) sets the recruitments to more than 3 months. Constraint (2) is present for setting the auxiliary variable y.

Question 7

(duality in linear and nonlinear optimization)

(1p) a) The LP dual is to

$$\begin{array}{lllll}
\text{maximize} & w = \boldsymbol{b}_1^{\mathrm{T}} \boldsymbol{y}_1 & + \boldsymbol{b}_2^{\mathrm{T}} \boldsymbol{y}_2 & + a y_3 & \\
\text{subject to} & \boldsymbol{A}_1^{\mathrm{T}} \boldsymbol{y}_1 & + \boldsymbol{A}_2^{\mathrm{T}} \boldsymbol{y}_2 & & \le \boldsymbol{c}, \\
& \boldsymbol{B}^{\mathrm{T}} \boldsymbol{y}_1 & & + \boldsymbol{1}^{\ell} y_3 & \le \boldsymbol{d}, \\
& \boldsymbol{y}_1 \ge \boldsymbol{0}^{m_1}, & \boldsymbol{y}_2 \in \mathbb{R}^{m_2}, y_3 \in \mathbb{R}, & &
\end{array}$$

where $\boldsymbol{1}^{m_1}$ is the m_1-vector of ones.

(2p) b) With $g(\boldsymbol{x}) := -x_1 + 2x_2 - 4$, the Lagrange function becomes

$$\begin{aligned} L(\boldsymbol{x}, \mu) &= f(\boldsymbol{x}) + \mu g(\boldsymbol{x}) \\ &= 2x_1^2 + x_2^2 - 4x_1 - 6x_2 + \mu(-x_1 + 2x_2 - 4). \end{aligned}$$

Minimizing this function over $\boldsymbol{x} \in \mathbb{R}^2$ yields [since $L(\cdot, \mu)$ is a strictly convex quadratic function for every value of μ, it has a unique minimum for every value of μ] that its minimum is attained where its gradient is zero. This gives us that

$$\begin{aligned} x_1(\mu) &= (4 + \mu)/4; \\ x_2(\mu) &= 3 - \mu. \end{aligned}$$

Inserting this into the Lagrangian function, we define the dual objective function as

$$q(\mu) = L(\boldsymbol{x}(\mu), \mu) = \cdots = -2\left(\frac{4+\mu}{4}\right)^2 - (3 - \mu)^2 - 4\mu.$$

This function is to be maximized over $\mu \geq 0$. We are done with task [1].

We attempt to optimize the one-dimensional function q by setting the derivative of q to zero. If the resulting value of μ is non-negative, then it must be a global optimum; otherwise, the optimum is $\mu^* = 0$.

We have that $q'(\mu) = \cdots = 1 - \frac{9\mu}{4}$, so the stationary point of q is $\mu = 4/9$. Since its value is positive, we know that the global maximum of q over $\mu \geq 0$ is $\mu^* = 4/9$. We are done with task [2].

Our candidate for the global optimum in the primal problem is $\boldsymbol{x}(\mu^*) = \frac{1}{9}(10, 23)^{\mathrm{T}}$. Checking feasibility, we see that $g(\boldsymbol{x}(\mu^*)) = 0$. Hence, without even evaluating the values of $q(\mu^*)$ and $f(\boldsymbol{x}(\mu^*))$ we know they must be equal, since $q(\mu^*) = f(\boldsymbol{x}(\mu^*)) + \mu^* g(\boldsymbol{x}(\mu^*)) = f(\boldsymbol{x}(\mu^*))$, due to the fact that we satisfy complementarity. We have proved that strong duality holds, and therefore task [4] is done.

By the Weak Duality Theorem 7.4 follows that if a vector $\boldsymbol{x}$ is primal feasible and $f(\boldsymbol{x}) = q(\mu)$ holds for some feasible dual vector μ, then $\boldsymbol{x}$ must be the optimal solution to the primal problem. (And μ must be optimal in the dual problem.) Task [4] is completed by the remark that this is exactly the case for the pair $(\boldsymbol{x}(\mu^*), \mu^*)$.