Chalmers/GU
Mathematics

EXAM

TMA947/MAN280
APPLIED OPTIMIZATION

Date:	05–08–25
Time:	House V, morning
Aids:	Text memory-less calculator
Number of questions:	7; passed on one question requires 2 points of 3.
	Questions are *not* numbered by difficulty.
	To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Niclas Andréasson (0762-721860)
Result announced:	05–09–09
	Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid methods and theory.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the Simplex method)

Consider the following linear program:

$$\begin{aligned}
\text{minimize} \quad z = \quad & x_2, \\
\text{subject to} \quad & x_1 \leq \frac{3}{2}, \\
& 2x_1 + 3x_2 \geq 6, \\
& x_1, \quad x_2 \geq 0.
\end{aligned}$$

(2p) a) Solve this problem by using phase I and phase II of the simplex method.

[*Aid:* Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

for producing basis inverses.]

(1p) b) Suppose that $c_1 = 0$ (the cost coefficient of x_1) changes to $c_1 = 3$. Establish whether the optimal basis in the problem solved in a) is optimal in this new problem, or provide an optimal basis to this new problem if it is not.

Question 2

(the Karush–Kuhn–Tucker conditions)

Consider the nonlinear program to

$$\begin{aligned}
\text{minimize} \quad & f(\boldsymbol{x}) := x_1, \\
\text{subject to} \quad & x_1^2 + x_2^2 \leq 2, \\
& (x_1 - 2)^2 + (x_2 - 2)^2 \leq 2.
\end{aligned}$$

(1p) a) Establish theoretically or graphically that $\boldsymbol{x}^* = (1, 1)^{\mathrm{T}}$ is the unique globally optimal solution.

(2p) b) Are the KKT conditions satisfied at $\boldsymbol{x}^*$? Verify!

If they are not, explain why, and relate your explanation to the known results on necessary and sufficient optimality conditions.

Question 3

(The Frank–Wolfe method)

Consider the nonlinear program to

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - 2x_2 + \frac{1}{4}x_1x_2, \\ \text{subject to} \quad & \boldsymbol{x} \in X := \{\, \boldsymbol{x} \in \mathbb{R}^2 \mid 0 \le x_j \le 1, \quad j = 1, 2 \,\}. \end{aligned}$$

(2p) a) Starting at the origin, solve this problem using the Frank–Wolfe method, using exact line searches. Apply at most two iterations. For each iteration, provide the smallest known interval wherein the optimal value (that is, f^*) of the problem lies.

(1p) b) Suppose the problem is to be solved using the simplicial decomposition method. Explain this method briefly, in particular the main difference(s) to the Frank–Wolfe method.

What is the maximum number of iterations that this method may need to converge to an optimal solution? Explain your answer in theory, and then apply it to the given problem. Do *not* compute the answer by applying the simplicial decomposition method to the problem.

Question 4

(convexity of functions)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function.

(1p) a) Prove that f is convex on $\mathbb{R}^n$ if and only if

$$f(\boldsymbol{y}) \ge f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{x}), \qquad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n.$$

(2p) b) Suppose that f is in C^2 on $\mathbb{R}^n$. Prove that f is convex on $\mathbb{R}^n$ if and only if its Hessian $\nabla^2 f(\boldsymbol{x})$ is positive semidefinite for every $\boldsymbol{x} \in \mathbb{R}^n$.

Question 5

(linear programming duality and optimality)

Consider the linear program

$$\begin{array}{lrrl} \text{minimize} & z = -x_1 & -2x_2, & \\ \text{subject to} & x_1 & & \leq 1, \\ & & x_2 & \leq 2, \\ & x_1 & +x_2 & \leq 3, \\ & x_1, & x_2 & \geq 0. \end{array}$$

(1p) a) Solve the problem geometrically, and utilize the primal–dual optimality conditions in linear programming to provide *all* optimal solutions to its dual.

(2p) b) Looking at the primal problem geometrically reveals that $\boldsymbol{x}^*$ can be represented by three primal bases. However, it may be that some of these three bases may *not* be optimal; such a basis can of course not be a terminal basis when applying the simplex method to the problem. Provide the three basic feasible solutions that correspond to $\boldsymbol{x}^*$, and investigate which one (ones) is (are) primal optimal (that is, dual feasible).

Question 6

(fixed points)

Consider the quadratic equation

$$f(x) := x^2 + ax + b = 0, \tag{1}$$

where $a, b \in \mathbb{R}$. We are interested in finding a real root of the equation (that is, a zero of f) if one exists, using a convergent algorithm that is based on elementary operations only. The present question illustrates how some pocket calculators work (or, used to work).

Consider first the fixed point iteration

$$x_0 \in S; \qquad x_{k+1} := \frac{x_k^2 + b}{-a}, \quad k = 0, 1, \dots, \tag{2}$$

where S is a closed and bounded interval containing a solution to (1). Observe that $x_{k+1} = x_k$ holds if and only if x_k is a root, in which case the iterations would

Calculator	“Root” of f
SHARP-EL 506S	0
TI-36X SOLAR	0
TI Programmable 58	10^{-6}
Java, double precision	$7.292255\ldots \cdot 10^{-7}$

Table 1: “Roots” of f in (1) from some calculators.

be terminated. The convergence of the iteration (2) is of course not guaranteed for every value of a and b, as is evident from the fact that real solutions to (1) do not always exist.

(1p) a) Let $a = -12345678$ and $b = 9$. Using MS Calculator V. 5, an estimate of a root of f is $7.290000597780479\ldots \cdot 10^{-7}$. Interestingly, using some other calculators, answers can vary substantially; cf. Table 1.

Starting at $x_0 = 0$, utilize five iterations of the formula (2) and present the result. Does the algorithm seem to converge to a fixed point, that is, to a solution to the quadratic equation?

Compare the rate of convergence of the algorithm with that of the following alternative fixed point algorithm:

$$x_0 \in S; \qquad x_{k+1} := \sqrt{-(ax_k + b)}, \quad k = 0, 1, \ldots .$$

(Do not start at $x_0 = 0$, but at some larger value.) Which method is to prefer?

(2p) b) Provide a sufficient condition on a and b such that the convergence of the algorithm (2) is guaranteed, supposing that a root exists within S.

[*Hint:* Provide sufficient conditions such that the iteration defines a contraction.]

Question 7

(linear programming duality and matrix games)

Let $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^m$, and $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, and consider the canonical LP problem

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned}$$

and its associated dual LP problem. In the following, we denote the respective problem by (P) and (D).

(1p) a) If $m = n$ and $\boldsymbol{A}^{\mathrm{T}} = -\boldsymbol{A}$, we then say that the matrix $\boldsymbol{A}$ is *skew-symmetric*.

Suppose that in the problem (P), the matrix $\boldsymbol{A}$ is skew-symmetric and that $\boldsymbol{b} = -\boldsymbol{c}$ also holds. Establish that if an optimal solution to (P) exists, then $z^* = 0$ holds.

(2p) b) The problem studied in a) is known as a *self-dual* LP problem.

Consider again the canonical primal–dual pair (P), (D) of LP problems. Construct a self-dual LP problem in $n + m$ variables and $n + m$ linear constraints which is equivalent to (P), (D). By "equivalent" we refer to the property that any primal–dual optimal solutions $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ to the pair (P), (D) are obtained immediately as an optimal solution to the problem constructed. (In other words, we can solve any primal–dual pair of canonical LP problems as a self-dual LP problem in a higher dimension.)

[*Remark:* Self-dual LP problems arose perhaps first in applications of linear two-player matrix games, where the variable vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are associated with the two players' respective strategies, and the matrix $\boldsymbol{A}$ with the resulting pay-off, that is, the redistribution of wealth between the players. A self-dual LP corresponds to a linear two-player matrix game called "fair", because the fact that the optimal value is zero means that on average, the two players' optimal strategies will lead to no redistribution of wealth between the two players.]

Good luck!

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EXAM SOLUTION

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Date: 05–08–25
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Question 1

(the Simplex method)

(2p) a) After adding two slack variables, a BFS cannot be found directly. We create the phase I problem through an added artificial variable a_1 in the second linear constraint; the value of a_1 is to be minimized. We use the BFS based on the variable pair (s_1, a_1) as the starting BFS for the phase I problem, terminating the simplex method with the optimal BFS given by $(s_1, x_2) = (3/2, 2)$, which is a BFS for the original problem.

Starting phase II with this BFS, the optimal basis for the problem is given by $(x_1, x_2) = (3/2, 1)$.

(1p) b) In the new problem the reduced cost vector for the non-basic variables is given by $\bar{\boldsymbol{c}}_N^{\mathrm{T}} = (-7/3, 1/3)$, indicating that the BFS is not optimal in the new problem. After one iteration of the simplex method, the optimal BFS reached is given by $(s_1, x_2) = (3/2, 2)$; hence $\boldsymbol{x}^* = (0, 2)^{\mathrm{T}}$.

Question 2

(the Karush–Kuhn–Tucker conditions)

(1p) a) $\boldsymbol{x}^* = (1, 1)^{\mathrm{T}}$ is the only feasible point, hence guaranteed to be globally optimal in the problem.

(2p) b) Both constraints are active at $\boldsymbol{x}^*$; their respectively normals (writing them as "$\leq$" constraints) are $(2, 2)^{\mathrm{T}}$ and $(-2, -2)^{\mathrm{T}}$, respectively. They are not linearly independent, thus violating the LICQ; the problem also violates the Slater CQ, since no interior point exists. The vector $-\nabla f(x^*) = (-1, 0)^{\mathrm{T}}$ cannot be written as a nonnegative linear combination of the normals of the active constraints, so the KKT conditions are not satisfied.

Question 3

(The Frank–Wolfe method)

(2p) a) f is in C^1 and strictly convex on X and X is closed, convex and bounded, hence the problem has a unique optimal solution. Moreover, the Frank–Wolfe method converges to this point from any starting point. (The unconstrained minimum is $\frac{1}{15}(-8, 32)$.)

Starting at $\boldsymbol{x}_0 = (0, 0)^{\mathrm{T}}$, the algorithm proceeds as follows: $f(\boldsymbol{x}_0) = 0$; $\nabla f(\boldsymbol{x}_0) = (0, -2)^{\mathrm{T}}$; $\boldsymbol{y}_0 = (0, 1)^{\mathrm{T}}$ (for example); the lower bound $z(\boldsymbol{y}_0) = f(\boldsymbol{x}_0) + \nabla f(\boldsymbol{x}_0)^{\mathrm{T}}(\boldsymbol{y}_0 - \boldsymbol{x}_0) = -2$; $\boldsymbol{p}_0 = \boldsymbol{y}_0 - \boldsymbol{x}_0 = (0, 1)^{\mathrm{T}}$; $f(\boldsymbol{x}_0 + \alpha \boldsymbol{p}_0) = \frac{1}{2}\alpha^2 - 2\alpha$, which yields the unique minimum $\alpha = 1$ over the interval $\alpha \in [0, 1]$; $\boldsymbol{x}_1 = \boldsymbol{y}_0 = (0, 1)^{\mathrm{T}}$; $f(\boldsymbol{x}_1) = -3/2$; $\nabla f(\boldsymbol{x}_1) = (1/4, -1)^{\mathrm{T}}$; $\boldsymbol{y}_1 = \boldsymbol{x}_1 = (0, 1)^{\mathrm{T}}$; $z(\boldsymbol{x}_1) = f(\boldsymbol{x}_1) = -3/2$. The lower and upper bounds are equal, hence $\boldsymbol{x}_1 = (0, 1)^{\mathrm{T}} = \boldsymbol{x}^*$, with the optimal value $f^* = -3/2$.

(1p) b) The number of extreme points of X is 4; hence, the maximum number of iterations of the simplicial decomposition method needed is also 4.

Question 4

(convexity of functions)

(1p) a) See Theorem 3.40(a) in AEP05.

(2p) b) See Theorem 3.41(a) in AEP05.

(3p) Question 5

(linear programming duality and optimality)

(1p) a) $\boldsymbol{x}^* = (1, 2)^{\mathrm{T}}$; the set of optimal dual solutions is $\{\, \boldsymbol{y} \in \mathbb{R}^3 \mid \boldsymbol{y} = (-1 + t, -2 + t, -t)^{\mathrm{T}}, \quad t \in [0, 1] \,\}$.

(2p) b) The three primal BFSs $(x_1, x_2, s_1)^{\mathrm{T}}$, $(x_1, x_2, s_2)^{\mathrm{T}}$, and $(x_1, x_2, s_3)^{\mathrm{T}}$ correspond to the dual basic solutions $\boldsymbol{y} = (0, -1, -1)^{\mathrm{T}}$, $\boldsymbol{y} = (1, 0, -2)^{\mathrm{T}}$, and $\boldsymbol{y} = (-1, -2, 0)^{\mathrm{T}}$, out of which the second one is infeasible—recall that the dual variables are restricted to be non-positive! Hence, the primal BFSs $(x_1, x_2, s_1)^{\mathrm{T}}$ and $(x_1, x_2, s_3)^{\mathrm{T}}$ are optimal, but the BFS $(x_1, x_2, s_2)^{\mathrm{T}}$ is not.

(3p)
Question 6

(fixed points)

(1p) a) $\boldsymbol{x}_5 \approx 7.290000597780479485279914474913 7 \cdot 10^{-7}$; convergence is very rapid. Alternative (2) does not converge for any starting value $\boldsymbol{x}_0$.

(2p) b) With $g(x) = \frac{x^2+b}{-a}$ we can either establish the contraction property [hence utilize Banach's Theorem 4.34(a) in AEP05] or the convergence criterion that states that

$$|g'(x)| \leq \alpha < 1 \text{ holds on } S$$

(which is Exercise 4.9 in AEP05). Utilizing the latter, we obtain the condition that $2|\frac{x}{a}| \leq \alpha < 1$ holds on S, that is, that the value of a is "large enough" in comparison with x on S.

Question 7

(linear programming duality and matrix games)

(1p) a) Under the given conditions we have that

$$\begin{aligned}
z^* &= \text{minimum } \{ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}^n \} \\
&= \text{maximum } \{ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \mid \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}, \quad \boldsymbol{y} \geq \boldsymbol{0}^m \} \\
&= \text{maximum } \{ (-\boldsymbol{c})^{\mathrm{T}}\boldsymbol{y} \mid -\boldsymbol{A}\boldsymbol{y} \leq -\boldsymbol{b}, \quad \boldsymbol{y} \geq \boldsymbol{0}^n \} \\
&= \text{maximum } \{ (-\boldsymbol{c})^{\mathrm{T}}\boldsymbol{y} \mid \boldsymbol{A}\boldsymbol{y} \geq \boldsymbol{b}, \quad \boldsymbol{y} \geq \boldsymbol{0}^n \} \\
&= -z^*,
\end{aligned}$$

which implies that $z^* = 0$.

(2p) b) The self-dual skew symmetric LP problem sought is

$$\begin{aligned}
\text{minimize } \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}, \\
\text{subject to } \quad & \begin{pmatrix} \boldsymbol{0}^{m\times n} & -\boldsymbol{A}^{\mathrm{T}} \\ \boldsymbol{A} & \boldsymbol{0}^{n\times m} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \geq \begin{pmatrix} -\boldsymbol{c} \\ \boldsymbol{b} \end{pmatrix}, \\
& (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0}^n \times \boldsymbol{0}^m.
\end{aligned}$$
