Chalmers/GU
Mathematics

EXAM

TMA947/MAN280
APPLIED OPTIMIZATION

Date:	05–04–02
Time:	House V, morning
Aids:	Text memory-less calculator
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Axel Målqvist, tel. 0740-479626
Result announced:	05–03–29 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid methods and theory.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$
\begin{array}{lrrrl}
\text{minimize} & x_1 & +2x_2 & -x_3 & \\
\text{subject to} & x_1 & +2x_2 & -x_3 & \leq 1, \\
& 2x_1 & -x_2 & & \geq 1, \\
& x_1, & x_2, & x_3 & \geq 0.
\end{array} \tag{P}
$$

(2p) a) Solve the linear program (P) by using phase I and phase II of the simplex method.

Use the following identity to compute the necessary matrix inverses:

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.
$$

(1p) b) Use your answer from a) to either construct a feasible solution to the linear programming dual of (P), or to show that the linear programming dual of (P) is infeasible.

(3p)

Question 2

(application of the Levenberg–Marquardt algorithm)

Consider the problem to

$$
\operatorname*{minimize}_{x \in \mathbb{R}^n} \; f(\boldsymbol{x}) := \ln x_1 - \ln x_2 + \frac{1}{2}(x_1 - 1)^2 + \frac{1}{2}(x_2 - 1)^2. \tag{1}
$$

Let $\boldsymbol{x}_0 = (1, 1)^{\mathrm{T}}$ be the initial point chosen. Apply one iteration of Newton's method with the Levenberg–Marquardt modification of the Hessian. In the line search step choose the shift in the Levenberg–Marquardt modification of the Hessian to make sure that the unit step provides descent with respect to f.

(3p) Question 3

(on the SQP algorithm and the KKT conditions)

Consider the following nonlinear programming problem: find $\boldsymbol{x}^* \in \mathbb{R}^n$ that solves the problem to

$$
\begin{aligned}
\text{minimize } \; & f(\boldsymbol{x}), && && \text{(1a)}\\
\text{subject to } \; & g_i(\boldsymbol{x}) \leq 0, && i = 1, \dots, m, && \text{(1b)}\\
& h_j(\boldsymbol{x}) = 0, && j = 1, \dots, \ell, && \text{(1c)}
\end{aligned}
$$

where $f : \mathbb{R}^n \to \mathbb{R}$, g_i, and $h_j : \mathbb{R}^n \to \mathbb{R}$ are given functions in C^1 on $\mathbb{R}^n$.

We are interested in establishing that the classic SQP subproblem tells us whether an iterate $\boldsymbol{x}_k \in \mathbb{R}^n$ satisfies the KKT conditions or not, thereby establishing a natural termination criterion for the SQP algorithm.

Given the iterate $\boldsymbol{x}_k$, the SQP subproblem is to

$$
\begin{aligned}
\underset{p}{\text{minimize}} \; & \frac{1}{2}\boldsymbol{p}^{\mathrm{T}}\boldsymbol{B}_k\boldsymbol{p} + \nabla f(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p}, && && \text{(2a)}\\
\text{subject to } \; & g_i(\boldsymbol{x}_k) + \nabla g_i(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p} \leq 0, && i = 1, \dots, m, && \text{(2b)}\\
& h_j(\boldsymbol{x}_k) + \nabla h_j(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p} = 0, && j = 1, \dots, \ell, && \text{(2c)}
\end{aligned}
$$

where the matrix $\boldsymbol{B}_k \in \mathbb{R}^{n \times n}$ is symmetric and positive semi-definite.

Establish the following statement: the vector $\boldsymbol{x}_k$ is a KKT point in the problem (1) if and only if $\boldsymbol{p} = \boldsymbol{0}^n$ is a globally optimal solution to the SQP subproblem (2). In other words, the SQP algorithm terminates if and only if $\boldsymbol{x}_k$ is a KKT point.

Hint: Compare the KKT conditions of (1) and (2).

(3p) Question 4

(convexity)

Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. A version of *Jensen's inequality* can be stated as follows: *Let $g_1, \dots, g_k \in \mathbb{R}$ and $h_1, \dots, h_k \in \mathbb{R}_+$ be non-negative scalars such that*

$$\sum_{i=1}^{k} h_i = 1.$$

Then,

$$f\left(\sum_{i=1}^{k} h_i g_i\right) \leq \sum_{i=1}^{k} h_i f(g_i).$$

This result will be extended as follows. Assume that $f \in C^1$ is convex and let $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}_+$ be functions such that

$$\int_{\mathbb{R}} h(x)\,dx = 1 \quad \text{and} \quad \int_{\mathbb{R}} h(x)g(x)\,dx < \infty.$$

Your task is to show that

$$f\left(\int_{\mathbb{R}} h(x)g(x)\,dx\right) \leq \int_{\mathbb{R}} h(x)f(g(x))\,dx.$$

Hint: Utilize a C^1 characterization of convexity.

(3p) Question 5

(strong duality in linear programming)

Consider the following standard form of a linear program:

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{c}, \boldsymbol{x} \in \mathbb{R}^n$, and $\boldsymbol{b} \in \mathbb{R}^m$. State and prove the Strong Duality Theorem in linear programming.

Question 6

(Lagrangian duality)

Consider the problem to

$$\begin{aligned} \underset{x}{\text{minimize}} \;\; & f(\boldsymbol{x}) := \frac{1}{2}x_1^2 + x_1 + 4x_2^2 - 2x_2, \\ \text{subject to} \;\; & x_1 + x_2 \geq 4, \\ & x_1, \quad x_2 \geq 1. \end{aligned}$$

Suppose we Lagrangian relax the first constraint and consider the problem to maximize the Lagrangian dual function q over the set $\{ \mu \mid \mu \geq 0 \}$.

(1p) a) Derive an explicit form of the Lagrangian dual problem. In other words, provide an explicit formula of the Lagrangian dual function q.

(1p) b) Calculate the function q at the following three values of μ: 0, 5/2, and 5. Also, calculate the value of the primal objective function f at the following three primal feasible vectors: $(2, 2)^{\mathrm{T}}$, $(1, 3)^{\mathrm{T}}$, $(3, 1)^{\mathrm{T}}$. Based on these calculations, provide a non-empty and closed interval $[a, b]$ for which the primal optimal value f^* satisfies $f^* \in [a, b]$. (Note that the properties of the problem ensures that $f^* = q^*$.)

(1p) c) Derive an explicit form of the derivative q' of the Lagrangian dual function q. Calculate the value of q' at the following three values of μ: 0, 5/2, and 5. Based on these calculations, provide a non-empty and closed interval $[a, b]$ for which the dual optimal solution μ^* satisfies $\mu^* \in [a, b]$.

(3p) Question 7

(modelling)

You are asked to construct apartment store rooms in a basement according to Figure 1.

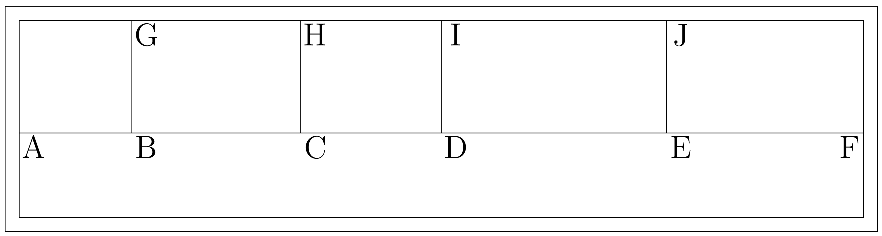

Figure 1: A sketch of the basement.

Each of the five store rooms should have a door of width 90 cm. The five doors have already been delivered. The rest of the walls are to be constructed with mesh panels of different widths. Observe that the panels are not divisible. The widths and costs of the mesh panels are given in Table 1.

Table 1: The data of the mesh panels.

Mesh panel	Width (cm)	Cost (EUR/cm)
1	20	0.57
2	30	0.38
3	70	0.20
4	80	0.19
5	100	0.16
6	120	0.15
7	150	0.14

Of course, the doors must be placed so that they can be opened; an infeasible as well as a feasible construction are illustrated in Figure 7.

The walls to be constructed are BG, CH, DI, EJ and AF, according to Figure 1. The lengths of BG, CH, DI and EJ are all 150 cm. The lengths of the rest of the sections are given in Table 2.

For the construction of the walls BG, CH, DI and EJ it is obviously cheapest to use the mesh panels of width 150 cm. However, the cheapest construction of

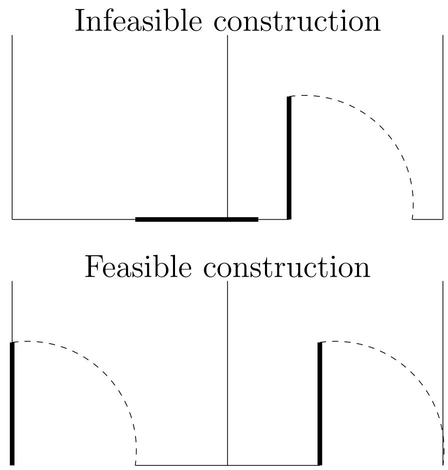

Figure 2: The upper figure illustrates an infeasible construction; the door to the left cannot be opened. The lower figure illustrates a feasible construction; both the doors can be opened.

Table 2: The lengths of the sections.

Section	Length (cm)
AB	150
BC	180
CD	160
DE	200
EF	190

the rest of the walls (i.e., between A and F) is not that obvious. Your task is to formulate an *integer linear program* for finding the cheapest construction of the wall AF.

Good luck!

Chalmers/GU
Mathematics

EXAM SOLUTION

TMA947/MAN280
APPLIED OPTIMIZATION

Date: 05–04–02

Examiner: Michael Patriksson

Question 1

(the simplex method)

(2p) a) The problem in standard form is to

$$\begin{array}{rrrrrrl} \text{minimize} & x_1 + 2\,x_2 - x_3 & & & & & \\ \text{subject to} & x_1 + 2\,x_2 - x_3 + x_4 & & & & & = 1, \\ & 2\,x_1 - \quad x_2 & & & & -x_5 & = 1, \\ & x_1, \quad x_2, & x_3, & x_4, & & x_5 & \geq 0. \end{array}$$

Introduce an artificial variable in the second constraint to get the phase I problem to

$$\begin{array}{rl} \text{minimize} \quad w = & a \\ \text{subject to} & x_1 + 2\,x_2 - x_3 + x_4 = 1, \\ & 2\,x_1 - x_2 - x_5 + a = 1, \\ & x_1, \ x_2, \ x_3, \ x_4, \ x_5, \ a \geq 0. \end{array}$$

Start with the basis $\boldsymbol{x}_B = (x_4, a)^{\mathrm{T}}$. The simplex method then gives that x_1 is the entering variable and a the leaving. Hence we have found a feasible solution for which $a = 0$, which means that $\boldsymbol{x}_B = (x_4, x_1)^{\mathrm{T}}$ is a feasible solution to the phase II problem. The reduced costs of the nonbasic variables $\boldsymbol{x}_N = (x_2, x_3, x_5)^{\mathrm{T}}$ become

$$\boldsymbol{c}_N^{\mathrm{T}} - \boldsymbol{c}_B^{\mathrm{T}} \boldsymbol{B}^{-1} \boldsymbol{N} = (5/2, -1, 1/2)^{\mathrm{T}},$$

which means that x_3 is the entering variable. Further, we have that

$$\begin{aligned} \boldsymbol{B}^{-1}\boldsymbol{b} &= (1/2, 1/2)^{\mathrm{T}}, \\ \boldsymbol{B}^{-1}\boldsymbol{N}_2 &= (-1, 0)^{\mathrm{T}}. \end{aligned}$$

Hence it follows that the phase II problem is unbounded, and we can draw the conclusion that the original problem (P) is unbounded.

(1p) b) Since (P) is unbounded it follows from weak duality that its linear programming dual is infeasible.

(3p) Question 2

(application of the Levenberg–Marquardt algorithm)

With a unit step the Levenberg–Marquardt algorithm is, for a given $\boldsymbol{x}_k$, to generate $\boldsymbol{x}_{k+1}$ through the formula

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \nabla^2 f(\boldsymbol{x}_k + \gamma_k \boldsymbol{I}^n)^{-1} \nabla f(\boldsymbol{x}_k),$$

where $\gamma_k \geq 0$ is the shift used in iteration k.

For the given problem and starting point,

$$f(\boldsymbol{x}_0) = 0;\ \nabla f(\boldsymbol{x}_0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix};\ \nabla^2 f(\boldsymbol{x}_0) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

With the shift γ_0 the next iterate therefore is

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/\gamma_0 \\ -1/(2+\gamma_0) \end{pmatrix}.$$

Inserting this into f yields that it is enough to set the value of γ_0 to something slightly larger than 1, while a choice of $\gamma_0 = 1$ would produce an undefined value of f (notice the presence of the logarithmic terms).

With $\gamma_0 = 2$ we obtain $\boldsymbol{x}_1 = (1/2, 5/4)^{\mathrm{T}}$ with $f(\boldsymbol{x}_1) \approx -0.76$.

(3p) Question 3

(on the SQP algorithm and the KKT conditions)

The result is based on a comparison between the KKT conditions of the original problem,

$$\begin{aligned}
&\text{minimize} \ f(\boldsymbol{x}), && && \text{(1a)}\\
&\text{subject to} \ g_i(\boldsymbol{x}) \leq 0, && i = 1, \dots, m, && \text{(1b)}\\
&\qquad\qquad h_j(\boldsymbol{x}) = 0, && j = 1, \dots, \ell, && \text{(1c)}
\end{aligned}$$

and those of the SQP subproblem,

$$\begin{aligned}
&\underset{p}{\text{minimize}} \ \frac{1}{2}\boldsymbol{p}^{\mathrm{T}}\boldsymbol{B}_k\boldsymbol{p} + \nabla f(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p}, && && \text{(2a)}\\
&\text{subject to} \ g_i(\boldsymbol{x}_k) + \nabla g_i(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p} \leq 0, && i = 1, \dots, m, && \text{(2b)}\\
&\qquad\qquad h_j(\boldsymbol{x}_k) + \nabla h_j(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p} = 0, && j = 1, \dots, \ell. && \text{(2c)}
\end{aligned}$$

We first note that the latter problem is a convex one (the matrix $\boldsymbol{B}_k$ was assumed to be positive semidefinite), and that the solution $\boldsymbol{p}_k$ is characterized by its KKT conditions, since the constraints are linear (so that Abadie's CQ is fulfilled). It remains to compare the two problems' KKT conditions. With $\boldsymbol{p}_k = \mathbf{0}^n$ they are in fact identical!

(3p) Question 4

(convexity)

We have the following convexity characterization:

$$f(y) \geq f(z) + f'(z)^{\mathrm{T}}(y - z).$$

The assertion follows by letting $y = g(x)$ and $z = \int_{\mathbb{R}} h(x)g(x)\,dx$, then multiply both the sides by $h(x)$, and finally integrate both sides over $\mathbb{R}$.

(3p) Question 5

(strong duality in linear programming)

See the notes for the proof.

Question 6

(Lagrangian duality)

(1p) a) We obtain that

$$q(\mu) = \begin{cases} 2\mu + 3\frac{1}{2}, & \text{if } \mu \leq 2, \\ -\frac{1}{2}(\mu - 1)^2 + 3\mu + 2, & \text{if } 2 \leq \mu \leq 6, \\ -\frac{1}{2}(\mu - 1)^2 - 4\frac{(2+\mu)}{8}^2 + 4\mu, & \text{if } \mu \geq 6. \end{cases}$$

(1p) b) $q(0) = 3\frac{1}{2}$; $q(\frac{5}{2}) = \frac{65}{8}$; $q(5) = 9$.
$f(2,2) = 16$; $f(1,3) = 31\frac{1}{2}$; $f(3,1) = 9\frac{1}{2}$.
Conclusion: $f^* \in [9, 9\frac{1}{2}]$.

(1p) c) From a) we obtain that

$$q(\mu) = \begin{cases} 2, & \text{if } \mu \leq 2, \\ -(\mu - 1) + 3, & \text{if } 2 \leq \mu \leq 6, \\ -(\mu - 1) - \frac{2+\mu}{8} + 4, & \text{if } \mu \geq 6. \end{cases}$$

$q'(0) = 2$; $q'(\frac{5}{2}) = \frac{3}{2}$; $q'(5) = -1$.

We note that the function q is concave and differentiable, and therefore its derivative is decreasing. According to the above, it must have a stationary point, hence the optimal solution, within the closed interval $[\frac{5}{2}, 5]$ which hence defines an interval wherein the optimum lies.

(3p) **Question 7**

(modelling)

For $d = 1, \ldots, 6$ and $m = 1, \ldots, 7$, introduce the integer variables

$$x_{dm} = \text{number of mesh panels of type } m \text{ used between door } d-1 \text{ and } d,$$

where "door" 0 is the wall on the left-hand side and "door" 6 is the wall on the right-hand side. Further, let c_m and w_m, respectively, be the cost and the width, respectively, of mesh panel m for $m = 1, \ldots, 7$, and let $l_1, \ldots, l_5$ denote the lengths of the sections AB, BC, CD, DE, and EF. Then the following integer linear program solves the problem:

$$\begin{aligned} \text{minimize} \quad & \sum_{m=1}^{7} \sum_{d=1}^{6} c_m x_{dm} \\ \text{subject to} \quad & \sum_{m=1}^{7} \sum_{d=1}^{k} w_m x_{dm} + 90k \leq \sum_{i=1}^{k} l_i, \quad k = 1, \ldots, 5, \\ & \sum_{m=1}^{7} \sum_{d=1}^{k+1} w_m x_{dm} + 90k \geq \sum_{i=1}^{k} l_i, \quad k = 1, \ldots, 4, \\ & \sum_{m=1}^{7} \sum_{d=1}^{k+1} w_m x_{dm} + 90k = \sum_{i=1}^{k} l_i, \quad k = 5, \\ & x_{dm} \in \mathbb{Z}_+^{6\times 7}. \end{aligned}$$
