MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2015-08–29 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Timo Hirscher, 0703-088304

Note:	Write your name and personal number on the cover. Write your code on every sheet you hand in. Only write on one page of each sheet. Do not use red pen. Do not answer more than one question per page. State your methodology carefully. Write legibly. Questions are not numbered by difficulty. Sort your solutions by the order of the questions. Mark on the cover the questions you have answered. Count the number of sheets you hand in and fill in the number on the cover. To pass requires 10 points.

1. Show that the BVP

$$\begin{cases} u''(x) + u(x) = \dfrac{u(x)}{2 + (u(x))^2}, \quad 0 \le x \le \dfrac{\pi}{2} \\ u(0) = u(\frac{\pi}{2}) = 0 \end{cases}$$

has a unique solution $u \in C^2([0,1])$.

(4p)

2. Set $E = \{\mathbb{x} = (x_1, x_2, \ldots, x_k, \ldots) \in l^2 : x_k \neq 0 \text{ only for finitely many } k\}$ with $\langle \cdot, \cdot \rangle_E = \langle \cdot, \cdot \rangle_{l^2}$. Moreover set

$$F = \{\mathbb{x} = (x_1, x_2, \ldots, x_k, \ldots) \in E : \Sigma_{k=1}^{\infty} \frac{1}{k} x_k = 0\}.$$

Show that

(a) F is a proper subspace of E

(b) F is a closed set in E

(c) $F^{\perp} = \{\mathbb{0}\}$

How does this result agree with Riesz representation theorem?

(4p)

3. Assume that $k : [0, \infty) \to \mathbb{R}$ is a continuous function with

$$\int_0^{\infty} |k(y)|\, dy < \infty.$$

Set

$$Kf(x) = \int_0^\infty k(x+y)f(y)\,dy, \quad x \in [0,\infty).$$

Show that K is a bounded linear operator on $L^2([0,\infty))$. Also show that K is compact.

(4p)

4. Formulate the following theorems:
   (a) Banach-Steinhaus theorem
   (b) Schauder´s fixed point theorem
   (c) Lax-Milgram theorem
   (d) Hilbert-Schmidt theorem

(5p)

5. Let A be a compact, self-adjoint operator on a Hilbert space. Show that $\|A\|$ or $-\|A\|$ is an eigenvalue to A. Moreover show that if λ is an eigenvalue to a bounded linear operator B then $|\lambda| \le \|B\|$.

(4p)

6. Set $\Omega = \{(x,y) \in \mathbb{R}^2 : |x| \le 1,\ |y| \le 1\}$. Assume $f, g : \Omega \to \mathbb{R}$ are continuous and satisfy

$$\begin{cases} f(-1,y) < 0, \quad f(1,y) > 0 & \text{all } |y| \le 1, \\ g(x,-1) < 0, \quad g(x,1) > 0 & \text{all } |x| \le 1. \end{cases}$$

Show[1] that there exists a $(\bar{x},\bar{y}) \in \Omega$ such that

$$f(\bar{x},\bar{y}) = 0 = g(\bar{x},\bar{y}).$$

(4p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK! PK

[1] Hint: Show that there exists an $\epsilon > 0$ such that $H : \Omega \to \Omega$ where $H(x,y) = (x,y) - \epsilon(f(x,y), g(x,y))$.