**MATEMATIK**
**Chalmers tekniska högskola och Göteborgs universitet**

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2013-01-17 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Dawan Mustata, 0703-088304

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| **Note:** | Write your name and personal number on the cover.<br>Write your code on every sheet you hand in.<br>Only write on one page of each sheet. Do not use red pen.<br>Do not answer more than one question per page.<br>State your methodology carefully. Write legibly.<br>Questions are not numbered by difficulty.<br>Sort your solutions by the order of the questions.<br>Mark on the cover the questions you have answered.<br>Count the number of sheets you hand in and fill in the number on the cover.<br>To pass requires 10 points. |

1. Show that the following boundary value problem

$$\begin{cases} u''(x) + u'(x) + \arctan(u(x^2)) = 0, & x \in [0,1], \\ u(0) = u(1) = 1, \\ u \in C^2([0,1]). \end{cases}$$

   has a unique solution.

(4p)

2. Set

$$Af(x) = \int_0^{2\pi} \sin^2(x-t) f(t)\, dt, \quad x \in [0, 2\pi].$$

   Show that $A$ is a linear bounded and compact operator on the Banach space $L^2([0,2\pi])$. Calculate the operator norm $\|A\|$.

(4p)

3. Let $M$ be a dense set in $[0,1]$ and let $f_n : [0,1] \to \mathbb{R}$, $n = 1, 2, 3, \ldots$, be a sequence of continuously differentiable functions that satisfies

$$\sup_{x\in[0,1]} |f_n'(x)| \le 1, \quad n = 1, 2, 3, \ldots$$

   Show that the sequence $(f_n)_{n=1}^{\infty}$ converges in $(C([0,1]), \|\cdot\|)$, where $\|f\| = \sup_{x\in[0,1]} |f(x)|$, if $\lim_{n\to\infty} f_n(x)$ exists for every $x \in M$.

(4p)

4. State and prove the Banach-Steinhaus theorem.

(5p)

5. Let $A \in \mathcal{B}(X, X)$, where $X$ is a Banach space, with $\|A\| < 1$. Show that $I + A$ is an invertible operator on $X$.

(4p)

6. Let $M$ be a subset of a Hilbert space $H$. Show that $(M^{\perp})^{\perp}$ is the smallest closed subspace of $H$ that contains $M$.

(4p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK! PK