MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet

Functional Analysis ENM, TMA401/ Applied Functional Analysis GU, MMA400,
Date: 2014-01–18 (4 hours)

Aids: Just pen, ruler and eraser.
Teacher on duty: Anna Persson, 0703-088304

Note: Write your name and personal number on the cover.
Write your code on every sheet you hand in.
Only write on one page of each sheet. Do not use red pen.
Do not answer more than one question per page.
State your methodology carefully. Write legibly.
Questions are not numbered by difficulty.
Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.
To pass requires 10 points.

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1. Show that the boundary value problem

$$\begin{cases} u''(x) + u(x) = \frac{1}{2}\cos(u(\frac{2}{\pi}x^2)), & x \in [0, \frac{\pi}{2}], \\ u'(0) = u'(\frac{\pi}{2}) = 0, \\ u \in C^2([0, \frac{\pi}{2}]). \end{cases}$$

has a unique solution u.

(4p)

2. Let A be a positive compact self-adjoint operator on a Hilbert space H with operator norm ≤ 1. Give an upper estimate for the operator norm of $3A^4 - 20A^3 + A^2$ (better than the trivial estimate 24).

(4p)

3. Let $(e_n)_{n=1}^{\infty}$ be an ON-basis in a Hilbert space H and set

$$\begin{cases} f_0 = e_1 \\ f_k = e_{2k+1} \quad k > 0 \\ f_k = e_{-2k} \quad k < 0 \end{cases}$$

Define S by $S(\Sigma_{k=-\infty}^{\infty} a_k f_k) = \Sigma_{k=-\infty}^{\infty} a_k f_{k+1})$. Show that S is a well-defined bounded linear mapping on H, calculate $\|S\|$ and show that S has no eigenvalues.

(4p)

P.T.O.

4. State and prove Riesz representation theorem.

(5p)

5. Let $k(x,y) \in C([0,1] \times [0,1])$ and define

$$Af(x) = \int_0^1 k(x,y)f(y)\,dy, \quad x \in [0,1].$$

Show that A is a compact operator on $L^2([0,1])$ and also on $C([0,1])$.

(5p)

6. Let $(X, \|\cdot\|)$ be a Banach space and let $T : X \to X$ be a mapping that satisfies

$$\|T(x) - T(y)\| \leq \psi(\|x - y\|), \quad x, y \in X,$$

where ψ is a real-valued continuous function on $\{t \in \mathbb{R} : t \geq 0\}$ with $\psi(t) < t$ for alla $t > 0$. Show[1] that T has a unique fixed point.

(4p)

For information on the announcement of results see the course homepage where also solutions to the problems will be presented.

GOOD LUCK! PK

[1] First show that $\|T^n(x) - T^{n-1}(x)\| \to 0$ as $n \to \infty$ for fixed $x \in X$. Then show that $(T^n(x))_{n=1}^{\infty}$ is a Cauchy sequence in X for fixed $x \in X$. To do this you can argue by contradiction, i.e. assume that there exists an $\epsilon > 0$ and sequences of integers $(m(k))_{k=1}^{\infty}, (n(k))_{k=1}^{\infty}$, $k = 1, 2, 3, \ldots$ with $m(k) > n(k) \geq k$ such that

$$a_k \equiv \|T^m(x) - T^n(x)\| \geq \epsilon, \ k = 1, 2, 3, \ldots.$$

Show that one may assume that $\|T^{m-1}(x) - T^n(x)\| < \epsilon$ for $k = 1, 2, 3, \ldots$. Try to obtain a contradiction.