MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet
Tentamen i
Funktionalanalys ENM, TMA401/ Tillämpad funktionalanalys GU, MMA400,
DATUM 2010-10-20, TID 8.30-13.30
Inga hjälpmedel, förutom penna och linjal, är tillåtna, ej heller räknedosa.
Telefonvakt: Magnus Goffeng, 0703-088304

Besökstider: ca 9.30 och 12.30

OBS:	Ange linje samt personnummer och namn på omslaget. Ange kod på *varje* inlämnat blad. Motivera dina svar väl. Det är i huvudsak beräkningarna och motiveringarna som ger poäng, inte svaret. Skriv tydligt. För godkänt krävs minst 10 poäng sammanlagt.

1. Prove the existence and uniqueness of solution to the following boundary value problem:

$$\begin{cases} -((1+x)u'(x))' = \arctan u(x), & 0 \le x \le 1 \\ u(0) = 1, u(1) = 0, & u \in C^2([0,1]) \end{cases}$$

(4p)

2. For $\varkappa = (x_1, x_2, \ldots, x_n, \ldots) \in l^2$ set $T(\varkappa) = (\frac{x_1}{1}, \frac{x_2}{2}, \ldots, \frac{x_n}{n}, \ldots)$. Show that T is a bounded linear operator on l^2 (with the standard norm on l^2), calculate $\|T\|$ and decide whether $\mathcal{R}(T)$, the range of T, is a closed set in l^2 or not.

(4p)

3. Consider the set

$$M = \{f \in L^2([-1,1]) : \int_{-1}^{1} f(t)\,dt = 0\}$$

in $L^2([-1,1])$ (with the standard inner product). Show that M is closed, find $M^\perp$ and calculate the distance of $g(t)$ to M for $g(t) = t^2$.

(4p)

P.T.O

4. State and prove the Hilbert-Schmidt theorem. Propositions that are used in the proof should be properly stated but need not be proven.

(5p)

5. Define the notion of a compact operator on a Hilbert space H. Show that AB is a compact operator on H if one of the operators is compact and the other is a bounded linear operator on H. Can the last statement be reversed, i.e. given two bounded linear operators A and B such that AB is compact, must one of the operators A, B be compact?

(4p)

6. Let H be a Hilbert space with a complete ON-sequence $(e_n)_{n=1}^{\infty}$ and let T be a bounded linear operator on H such that

$$\Sigma_{n=1}^{\infty} \|T(e_n)\|^2 < \infty.$$

Show that

$$\Sigma_{n=1}^{\infty} \|T(e_n)\|^2 = \Sigma_{n=1}^{\infty} \|T(f_n)\|^2$$

holds for every complete ON-sequence $(f_n)_{n=1}^{\infty}$.

(4p)

Information om när tentan är färdigrättad och tid för visning av tentan hos föreläsaren kommer att lämnas på kurshemsidan. När resultaten är registrerade i Ladok kommer ett e-brev.

LYCKA TILL! PK

Solutions to written exam TMA401/MMA400 (2010-10-20)

(1) Show existence and uniqueness for solution to

$$\begin{cases} -((1+x)u'(x))' = \arctan(u(x)) & x \in [0,1] \\ u(0) = 1, \quad u(1) = 0 \end{cases}$$

Solution: Set $u(x) = v(x) + 1 - x$. Then $v(x)$ satisfies

$$\begin{cases} -((1+x)(v'(x)-1))' = \arctan(v(x)+1-x) \\ v(0) = v(1) = 0 \end{cases} \qquad (*)$$

where the boundary conditions are homogeneous

Calculation for the Green's function for $L = -(1+x)D^2 - D$ with boundary conditions $R_1u = u(0)$, $R_2u = u(1)$:

$u_1(x) = 1$, $u_2(x) = \ln(1+x)$ is a basis for $N(L)$

$g(x,t) = (a_1(t)u_1(x) + a_2(t)u_2(x))\,\theta(x-t) + b_1(t)u_1(x) + b_2(t)u_2(x)$

where

$$\begin{cases} a_1(t) + a_2(t)\ln(1+t) = 0 \\ a_2(t)\cdot \frac{1}{1+t} = -\frac{1}{1+t} \end{cases} \quad \text{i.e.} \quad \begin{cases} a_1(t) = \ln(1+t) \\ a_2(t) = -1 \end{cases}$$

and

$$\begin{cases} b_1(t) = 0 \\ \ln\left(\frac{1+t}{2}\right) + b_1(t) + b_2(t)\ln 2 = 0 \end{cases} \quad \text{i.e.} \quad \begin{cases} b_1(t) = 0 \\ b_2(t) = -\frac{1}{\ln 2}\ln\left(\frac{1+t}{2}\right) \end{cases}$$

Hence we get

$$g(x,t) = \ln\left(\frac{1+t}{1+x}\right)\theta(x-t) - \ln\left(\frac{1+t}{2}\right)\frac{\ln(1+x)}{\ln 2}$$

The BVP (*) can be rewritten as

$$v(x) = \int_0^1 g(x,t)\left[\arctan(v(t)+1-t) - 1\right]dt, \quad x \in [0,1].$$

Set $T(v)(x) = \int_0^1 g(x,t)\left[\arctan(v(t)+1-t) - 1\right]dt$ for $v \in C([0,1])$

Here $T: C([0,1]) \longrightarrow C([0,1])$, where we equip $C([0,1])$ with the max-norm $\|\cdot\|$. $(C([0,1]), \|\cdot\|)$ is a Banach space

We note that (*) has a unique solution $v \in C^2([0,1])$ if T is a contraction on $C([0,1])$ by Banach's fixed point thm.

$$|T(v)(x) - T(\tilde{v})(x)| \leq \int_0^1 |g(x,t)|\,|\arctan(v(t)+1-t) -$$
$$- \arctan(\tilde{v}(t)+1-t)|\,dt \leq \int_0^1 |g(x,t)|\,|v(t) - \tilde{v}(t)|\,dt$$

by the mean value theorem we obtain

$$|T(v)(x) - T(\tilde{v})(x)| \leq \int_0^1 |g(x,t)|\,dt\, \|v - \tilde{v}\|$$

and so

$$\|T(v) - T(\tilde{v})\| \leq \int_0^1 |g(x,t)|\,dt\, \|v - \tilde{v}\|$$

But

$$\int_0^1 |g(x,t)|\,dt = \{ g(x,t) \geq 0\} = \int_0^1 g(x,t)(+1)\,dt \equiv j(x).$$

Here $j(x)$ satisfies $-(1+x)j'' - j' = 1$, $j(0) = j(1) = 0$

i.e. $j(x) = x - \frac{\ln(1+x)}{\ln 2}$ and $\max\limits_{x\in[0,1]} j(x) = j(\frac{1}{\ln 2} - 1) =$

$= \frac{1}{\ln 2} \ln(\frac{e \ln 2}{2}) < 1$ since $e \ln 2 < 4$.

So T is a contraction on the Banach space $(C([0,1]), \|\cdot\|)$ and the conclusion follows.

(2) $T(x) = (\frac{x_1}{1}, \frac{x_2}{2}, \ldots, \frac{x_n}{n}, \ldots)$ for $x = (x_1, x_2, \ldots, x_n, \ldots) \in l^2$

Show that $T \in \mathcal{B}(l^2, l^2)$, calculate $\|T\|$ and decide whether $\mathcal{R}(T)$ is closed in l^2

Solution: Clearly

$$\|T(x)\|_{l^2}^2 = \sum_{n=1}^{\infty} |\frac{x_n}{n}|^2 \leq \sum_{n=1}^{\infty} |x_n|^2 = \|x\|_{l^2}^2 \quad \text{for } x \in l^2$$

and $T(\alpha x + \beta y) = (\frac{\alpha x_1 + \beta y_1}{1}, \frac{\alpha x_2 + \beta y_2}{2}, \ldots, \frac{\alpha x_n + \beta y_n}{n}, \ldots) =$

$= \alpha(\frac{x_1}{1}, \frac{x_2}{2}, \ldots, \frac{x_n}{n}, \ldots) + \beta(\frac{y_1}{1}, \frac{y_2}{2}, \ldots, \frac{y_n}{n}, \ldots) = \alpha T(x) + \beta T(y)$

for all $x, y \in l^2$ and scalars α, β. Hence $T \in \mathcal{B}(l^2, l^2)$ with $\|T\| \leq 1$. Moreover $T(1,0,0,\ldots,0,\ldots) = (1,0,0,\ldots,0,\ldots)$ so $\|T\| = 1$.

$\mathcal{R}(T) \ni y$ if $(y_1, y_2, \ldots, y_n, \ldots) = (\frac{x_1}{1}, \frac{x_2}{2}, \ldots, \frac{x_n}{n}, \ldots)$ for some $x \in l^2$, so $x_n = n y_n$, for $n = 1, 2, \ldots$

Hence every y with finitely many nonzero y_n's belong to $\mathcal{R}(T)$. If $\mathcal{R}(T)$ where closed then $\mathcal{R}(T) = l^2$, since the set of y's with finitely many non zero y_n's is dense in l^2, BUT $y = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots) \in l^2 \setminus \mathcal{R}(T)$ since $x = (1\cdot 1, 2\cdot\frac{1}{2}, \ldots, n\cdot\frac{1}{n}, \ldots) = (1, 1, \ldots, 1, \ldots) \notin l^2$.

$\mathcal{R}(T)$ is not closed in ℓ^2.

(3) $M = \{ f \in L^2([-1,1]) : \int_{-1}^{1} f(t)\,dt = 0 \} \subset L^2([-1,1])$

Show that M is closed, find $M^\perp$ and calculate $\operatorname{dist}(g, M)$, where $g(t) = t^2$.

Solution: We observe that $M = \{\mathbb{1}\}^\perp$ where $\mathbb{1}(t) = 1$, $t \in [-1,1]$

This implies that M is closed and that

$$M^\perp = (\{\mathbb{1}\}^\perp)^\perp = \overline{\operatorname{Span}\{\mathbb{1}\}} = \operatorname{Span}\{\mathbb{1}\}.$$

Finally

$$\operatorname{dist}(g, M) = \inf_{f \in M} \| g - f \|_{L^2}$$

Let $e_1(t) = \frac{1}{\|\mathbb{1}\|_{L^2}} \mathbb{1}(t) = \frac{1}{\sqrt{2}}$, $t \in [-1,1]$. Then for $f \in M$

$$\| g - f \|_{L^2} = \| \underbrace{g - \langle g, e_1 \rangle e_1 - f}_{\in M} + \underbrace{\langle g, e_1 \rangle e_1}_{\in M^\perp} \|_{L^2} \geq$$

$$\geq \| \langle g, e_1 \rangle e_1 \|_{L^2} = | \langle g, e_1 \rangle |$$

Here

$$\langle g, e_1 \rangle = \frac{1}{\sqrt{2}} \int_{-1}^{1} t^2 \, 1 \, dt = \frac{1}{\sqrt{2}} \left[\frac{t^3}{3} \right]_{-1}^{1} = \frac{\sqrt{2}}{3}$$

So $\operatorname{dist}(g, M) = \frac{\sqrt{2}}{3}$