MATEMATIK
Chalmers tekniska högskola och Göteborgs universitet
Tentamen i
Funktionalanalys ENM, TMA401/ Tillämpad funktionalanalys GU, MMA400,
DATUM 2009-08-29, TID 8.30-13.30
Inga hjälpmedel, förutom penna och linjal, är tillåtna, ej heller räknedosa.
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Besökstider: ca 9.30 och 12.30

OBS:	Ange linje samt personnummer och namn på omslaget. Ange kod på *varje* inlämnat blad. Motivera dina svar väl. Det är i huvudsak beräkningarna och motiveringarna som ger poäng, inte svaret. Skriv tydligt. För godkänt krävs minst 20 poäng sammanlagt.

1. Show that there exists a unique C^2-function $u(x)$ defined on $[0, \frac{\pi}{2}]$ such that

$$\begin{cases} u''(x) + u(x) = \frac{1}{2}\sin u(\frac{1}{2}x^2), & x \in [0, \frac{\pi}{2}], \\ u'(0) = u'(\frac{\pi}{2}) = 0 \end{cases}$$

(4p)

2. Set

$$Tf(x) = \int_0^{1-x} f(y)\, dy, \quad f \in C([0,1])$$

for $x \in [0,1]$. Prove that

(a) T defines a linear bounded and compact[1] operator on $C([0,1])$ (with the uniform norm), and

(b) calculate[2] $\sigma(T)$ and the eigenvalues of T.

(5p)

3. Let $1 \le r < p < 2r$ and assume that the sequence $(x_1, x_2, \ldots)$ satisfies

$$\Sigma_{n=1}^{\infty} n|x_n|^p < \infty.$$

Show that $(x_1, x_2, \ldots) \in l^r$.

(3p)

[1]The definition of a compact operator on a Banach space is word by word the same as the definition of a compact operator on a Hilbert space.
[2]You may assume that $\sigma(T) \setminus \{0\} \subset \sigma_p(T)$ for a compact operator on a Banach space which indeed is true.

4. Formulate and prove the Lax-Milgram theorem.

(5p)

5. Let H be a complex Hilbert space and let $T \in \mathcal{B}(H,H)$. Define T^*, also show that it exists, and that $T^* \in \mathcal{B}(H,H)$ with $\|T^*\| = \|T\|$. Also show that $T \mapsto T^*$ is a linear continuous operator on $\mathcal{B}(H,H)$.

(4p)

6. Let H be a complex Hilbert space and let V be a dense subspace. Show that for each $x \in H$, $\{x\}^{\perp} \bigcap V$ is dense in $\{x\}^{\perp}$.

(4p)

Information om när tentan är färdigrättad och tid för visning av tentan hos föreläsaren kommer att lämnas på kurshemsidan. När resultaten är registrerade i Ladok kommer ett e-brev.

LYCKA TILL! PK

1) Show that

$$\begin{cases} u''(x)+u(x) = \frac{1}{2}\sin\left(u\left(\frac{1}{2}x^2\right)\right) & x\in[0,\frac{\pi}{2}] \\ u'(0)=u'(\frac{\pi}{2})=0 \end{cases}$$

has a unique C^2-solution

Solution. (I) Calculate the green's function $g(x,t)$ for $Lu \equiv u''+u$, $u'(0)=u'(\frac{\pi}{2})=0$:

$g(x,t) = (a_1(t)\cos x + a_2(t)\sin x)\Theta(x-t) + b_1(t)\cos x + b_2(t)\sin x$

where $a_1(t)\cos t + a_2(t)\sin t = 0$, $-a_1(t)\sin t + a_2(t)\cos t = 1$,

i.e. $a_1(t) = -\sin t$, $a_2(t) = \cos t$, and

$g_x'(0,t) = g_x'(\frac{\pi}{2},t) = 0$, $0<t<\frac{\pi}{2}$, i.e. $b_1(t) = \sin t$, $b_2(t)=0$

We get

$$g(x,t) = \sin(x-t)\Theta(x-t) + \sin t\cos x = \begin{cases}\sin t\cdot\cos x , & x<t \\ \cos t\,\sin x , & x>t\end{cases}$$

(II) Set $Tu(x) = \int_0^{\frac{\pi}{2}} g(x,t)\left(\frac{1}{2}\sin\left(u\left(\frac{1}{2}t^2\right)\right)\right)dt$, $u\in C([0,\frac{\pi}{2}])$

If $T: C([0,\frac{\pi}{2}]) \to C([0,\frac{\pi}{2}])$ is a contraction then the BVP has a unique solution.

$$|Tu(x) - Tv(x)| = \frac{1}{2}\left|\int_0^{\frac{\pi}{2}} g(x,t)\left(\sin\left(u\left(\tfrac{1}{2}t^2\right)\right) - \sin\left(v\left(\tfrac{1}{2}t^2\right)\right)\right)dt\right| \le$$
$$\le \frac{1}{2}\int_0^{\frac{\pi}{2}} |g(x,t)|\,dt\,\|u-v\| = \frac{1}{2}\|u-v\| \quad , x\in[0,\tfrac{\pi}{2}]$$

where $\|h\| = \max_{x\in[0,\frac{\pi}{2}]}|h(x)|$. Hence $\|Tu-Tv\| \le \frac{1}{2}\|u-v\|$, $u,v\in C([0,\frac{\pi}{2}])$, and T is a contraction on the Banach space $(C([0,\frac{\pi}{2}]),\|\cdot\|)$. T has a unique fixed point by Banach's fixed point theorem, and so the BVP has a unique solution in C^2

2) $Tf(x) = \int_0^{1-x} f(y)\,dy$, $f\in C([0,1])$.

Show a) T linear, bounded and compact operator on $C([0,1])$

b) calculate $\sigma(T)$ and the eigenvalues $(=\sigma_p(T))$

Solution: a) linearity of T: trivial and omitted

T bounded: $|Tf(x)| = |\int_0^{1-x} f(y)\,dy| \leq \|f\|$, where $\|f\| = \max_{x\in[0,1]} |f(x)|$, and hence $\|T\| \leq 1$.

T compact: Let $(f_n)_{n\geq 1}^{\infty}$ be a bounded sequence in $C([0,1])$. Hence $\|f_n\| \leq M$ all n for some M. By Arzela-Ascoli theorem it suffices to show that $A = \{Tf_n : n = 1,2,\ldots\}$ is bounded and equicontinuous.

A bounded: $\|Tf_n\| \leq \|T\|\|f_n\| \leq \|f_n\| \leq M$ all n

A equicontinuous: Fix $\varepsilon > 0$. WLOG we assume $x < y$, and

$$|Tf_n(x) - Tf_n(y)| = |\int_{1-y}^{1-x} f_n(t)\,dt| \leq M|x-y|$$

So $|Tf_n(x) - Tf_n(y)| < \varepsilon$ for all n provided $|x-y| < \frac{\varepsilon}{M}$.

b) Calculate the eigenvalues of T: We see that $\lambda = 0$ is no eigenvalue. Assume $\lambda \neq 0$ is an eigenvalue, i.e.

$$\lambda g(x) = Tg(x) = \int_0^{1-x} g(y)\,dy, \quad x\in[0,1], \quad \text{for some } g\neq 0$$

$g \in C([0,1])$ implies $Tg \in C^1([0,1])$ and so

$\lambda g'(x) = -g(1-x)$, $x\in[0,1]$. Moreover $g(1) = 0$.

But $g\in C^1([0,1])$ then implies $g\in C^2([0,1])$. Differentiate once more. We get

$$\lambda g''(x) = \frac{g(x)}{\lambda}, \quad x\in[0,1], \quad \text{and} \quad g(1) = g'(0) = 0$$

Hence $g(x) = A\cos(\frac{x}{\lambda})$ satisfying $g(1) = 0$. This gives

$\lambda_k = \dfrac{1}{\frac{\pi}{2} + \pi k}$, $k\in\mathbb{Z}$. Check if all these λ_k's are eigenvalues. We calculate

$$Tg_k(x) = \int_0^{1-x} \cos(\tfrac{t}{\lambda_k})\,dt = \lambda_k[\sin(\tfrac{t}{\lambda_k})]_0^{1-x} =$$

$$= \lambda_k \sin((\tfrac{\pi}{2}+\pi k)(1-x)) = \lambda_k[\sin(\tfrac{\pi}{2}+\pi k)\cos((\tfrac{\pi}{2}+\pi k)x) -$$

$$- \cos(\tfrac{\pi}{2}+\pi k)\sin((\tfrac{\pi}{2}+\pi k)x)] = \lambda_k(-1)^k g_k(x).$$

Hence $\lambda = \lambda_{2l}$, $l\in\mathbb{Z}$ are the eigenvalues of T, i.e.

$\sigma_p(T) = \{\lambda_{2l} : l\in\mathbb{Z}\}$

Calculate $\sigma(T)$: We know $\sigma(T)$ closed and

$\sigma(T) \setminus \{0\} \subset \sigma_p(T)$. This yields $\sigma(T) = \{0\} \cup \sigma_p(T)$.

3) $1 \le r < p < 2r$ and $\sum_{n=1}^{\infty} n|x_n|^p < \infty$. Show that $(x_1, x_2, \ldots, x_n, \ldots) \in \ell^r$

Solution: To show: $\sum_{n=1}^{\infty} |x_n|^r < \infty$. We apply Hölder's inequality to

$$\sum_{n=1}^{\infty} |x_n|^r = \sum_{n=1}^{\infty} n^{-\frac{r}{p}} \cdot n^{\frac{r}{p}} |x_n|^r \le \{ r < p \} \le$$

$$\le \underbrace{\left(\sum_{n=1}^{\infty} \left(n^{-\frac{r}{p}}\right)^{\frac{p}{p-r}} \right)^{\frac{p-r}{p}}}_{= \Big(\underbrace{\sum_{n=1}^{\infty} n^{-\frac{r}{p-r}}}_{<\infty \text{ since } p<2r} \Big)^{\frac{p-r}{p}}} \cdot \underbrace{\left(\sum_{n=1}^{\infty} \left(n^{\frac{r}{p}} |x_n|^r\right)^{\frac{p}{r}} \right)^{\frac{r}{p}}}_{= \left(\sum_{n=1}^{\infty} n|x_n|^p \right)^{\frac{r}{p}}} < \infty$$

4) & 5) See textbook

6) H complex Hilbert space and V dense subspace. Show that for each $x \in H$, $\{x\}^{\perp} \cap V$ dense in $\{x\}^{\perp}$.

Proof: Fix $x \in H$ and $y \in \{x\}^{\perp}$. Assume WLOG $\|x\| = 1$.
To show: There exists a sequence $(z_n)_{n=1}^{\infty}$ in V s.th $z_n \to y$ in H. Pick sequences $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ in V s.th $x_n \to x$ in H and $y_n \to y$ in H.

Set $z_n = \langle x_n, x \rangle y_n - \langle y_n, x \rangle x_n$, $n = 1, 2, \ldots$

Here $z_n \in \{x\}^{\perp}$ and $z_n \in V$, since V subspace.

Moreover $\langle x_n, x \rangle \to \|x\|^2 = 1$, $n \to \infty$ and $\langle y_n, x \rangle \to \langle y, x \rangle = 0$, $n \to \infty$. Hence $z_n \to y$ in H.