

Föreläsning 1

Introductory example

$$\begin{cases} f'' + f = g & \text{on } [0, 1] = I \\ f(0) = 1, \quad f'(0) = 1 \end{cases} \quad (+)$$

① $g=0, \quad f(x) = A\cos(x) + B\sin(x), \quad x \in I, A, B \in \mathbb{R}$

② g arbitrary

- Method of variation of constants

Set $f(x) = A(x)\cos(x) + B(x)\sin(x)$

• Differentiate

$$f'(x) = \underbrace{A'(x)\cos(x) + B'(x)\sin(x)}_{(*)} - A(x)\sin(x) + B(x)\cos(x)$$

Assume (part of method) that $(*) = 0$.

• Differentiate $f'(x)$

$$f''(x) = -A'(x)\sin(x) + B'(x)\cos(x) - \underbrace{A(x)\cos(x) - B(x)\sin(x)}_{= -f(x)}$$

$$g(x) = f''(x) + f(x) = -A'(x)\sin(x) + B'(x)\cos(x)$$

Now

$$\begin{cases} A'(x)\cos(x) + B'(x)\sin(x) = 0 \\ -A'(x)\sin(x) + B'(x)\cos(x) = g(x) \end{cases}, \quad x \in I$$

with $A(0)=1, B(0)=0$.

We get

$$\begin{cases} A'(x) = -g(x)\sin(x) \\ A(0) = 1 \\ B'(x) = g(x)\cos(x) \\ B(0) = 0 \end{cases}$$

This implies

$$A(x) = A(0) + \int_0^x A'(t) dt = 1 - \int_0^x g(t)\sin(t) dt$$

$$B(x) = B(0) + \int_0^x B'(t) dt = \int_0^x g(t)\cos(t) dt$$

\implies

$$\begin{aligned} f(x) &= \cos(x) - \int_0^x g(t)\sin(t)\cos(x) dt + \int_0^x g(t)\cos(t)\sin(x) dt \\ &= \cos(x) + \int_0^x \underbrace{(\cos(t)\sin(x) - \cos(x)\sin(t))}_{=\sin(x-t)} g(t) dt \quad (*) \end{aligned}$$

which satisfies (+).

Special case:

$$g(x) = k(x)f(x), \quad x \in I$$

where k is a known continuous function on I .

Insert $g(x)$ in $(*)$ and we obtain

$$f(x) = \cos(x) + \int_0^x \sin(x-t)k(t)f(t)dt, \quad x \in I \quad (xx)$$

- Observe that f appear in both LHS and RHS.
- (xx) is a reformulation of $(+)$ with $g = kf$.

Set

$$f_1(x) = \cos(x) + \int_0^x \sin(x-t)k(t)f_0(t)dt$$

$$f_2(x) = \cos(x) + \int_0^x \sin(x-t)k(t)f_1(t)dt$$

$$\vdots$$
$$f_m(x) = \cos(x) + \int_0^x \sin(x-t)k(t)f_{m-1}(t)dt, \quad m=1,2,\dots$$

Hope: f_m tends to some continuous function f

on I , denoted $f_m \rightarrow f$. "Tends to" has to be made precise.

$$f_{m+1}(x) = \cos(x) + \int_0^x \sin(x-t)k(t)f_m(t)dt$$

$\downarrow \qquad \parallel \qquad \downarrow$

$$f(x) = \cos(x) + \int_0^x \sin(x-t)k(t)f(t)dt$$

Set

$$\begin{cases} u(x) = \cos(x) \\ Kv(t) = \int_0^x \sin(x-t) k(t)v(t) dt \\ \text{for } v \in C(I) \end{cases}$$

We have

$$(\exists) \begin{cases} f_0 \in C(I) \\ f_{m+1} = u + Kf_m, \quad m=0,1,2,\dots \end{cases}$$

Facts from previous calculus classes.

Definition: $v_m \in C(I), \quad m=1,2,\dots$

We say that $(v_m)_{m=1}^\infty$ converges uniformly on $I = [0,1]$ if

$$\max_{x \in I} |v_n(x) - v_m(x)| \rightarrow 0 \quad \text{as } n,m \rightarrow \infty$$

$$\text{i.e. } \forall \varepsilon > 0 \quad \exists N : n,m \geq N \Rightarrow \max_{x \in I} |v_n(x) - v_m(x)| < \varepsilon$$

Lemma:

Suppose that $(v_m)_{m=1}^\infty$ converges uniformly on I .
Then there exists $v \in C(I)$ such that

$$\max_{x \in I} |v_m(x) - v(x)| \rightarrow 0 \quad \text{as } m \rightarrow \infty$$

Back to (1)

Notations:

- $K(Kv) = K^2v$, $v \in C(I)$
- $K^{n+1}v = K(K^n v)$, $n = 1, 2, \dots$

We have

$$\begin{aligned}f_0 &\in C(I) \\f_1 &= u + Kf_0 \\f_2 &= u + K(u + Kf_0)\end{aligned}$$

Note that K has the linear property
 $K(v+w) = \underbrace{K(v)}_{Kv} + \underbrace{K(w)}_{Kw}$
for $v, w \in C(I)$

Then

$$f_2 = u + K(u + Kf_0) = u + Ku + K^2f_0$$

$$f_3 = u + Kf_2 = u + Ku + K^2u + K^3f_0$$

In general

$$f_n = u + Ku + K^2u + \dots + K^{n-1}u + K^n f_0, n=1, 2, \dots$$

Assume $n > m$, then

$$f_n - f_m = K^m u + \dots + K^{n-1}u + K^n f_0 - K^m f_0$$

Set $\|v\| = \max_{x \in I} |v(x)|$, for $v \in C(I)$

Note

$$\begin{aligned}\|v+w\| &\leq \|v\| + \|w\| \\\|-v\| &= \|v\|\end{aligned}\quad \text{for } v, w \in C(I)$$

We have

$$\begin{aligned}\|f_n - f_m\| &= \|K^m u + \dots + K^{n-1} u + K^n f_0 - K^m f_0\| \leq \\ &\leq \|K^m u\| + \dots + \|K^{n-1} u\| + \|K^n f_0\| + \underbrace{\|-K^m f_0\|}_{= \|K^m f_0\|}\end{aligned}$$

Assumption: $\sum_{k=1}^{\infty} \|K^k v\| < \infty, \forall v \in C(I) \quad (\times \times \times)$

Under this assumption

$$\|f_n - f_m\| \rightarrow 0 \quad \text{as } n, m \rightarrow \infty$$

Since

- $\sum_{k=1}^{\infty} \|K^k u\| < \infty \quad (u(x) = \cos(x) \in C(I))$
- $\sum_{k=1}^{\infty} \|K^k f_0\| < \infty \quad (f_0 \in C(I))$

Conclusion: $(f_n)_{n=1}^{\infty}$ converges uniformly on I .

By previously mentioned lemma, there exists $f \in C(I)$ such that

$$\max_{x \in I} |f_n(x) - f(x)| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

$$\text{i.e. } \|f_n - f\| \xrightarrow{n \rightarrow \infty} 0$$

Back to "Hope":

" f_n tends to f " denoted $f_n \rightarrow f$ shall be interpreted as $\|f_n - f\| \rightarrow 0, n \rightarrow \infty$.

$$\begin{aligned} f_{n+1}(x) &= u(x) + \left[\underset{\downarrow}{Kf_n(x)} \right], \quad x \in I \\ &\qquad\qquad\qquad \parallel \\ f(x) &= u(x) + \left[\underset{\cdots}{Kf(x)} \right]. \end{aligned}$$

For $x \in I$

$$\begin{aligned} |Kf_n(x) - Kf(x)| &= \left| \int_0^x \sin(x-t) k(t) f_n(t) dt - \int_0^x \sin(x-t) k(t) f(t) dt \right| \leq \\ &\leq \int_0^x |\sin(x-t)| |k(t)| (f_n(t) - f(t)) dt = \\ &= \int_0^x |\sin(x-t)| |k(t)| \underbrace{|f_n(t) - f(t)|}_{\leq \|f_n - f\|} dt \leq \\ &\leq \int_0^x |\sin(x-t)| |k(t)| dt \|f_n - f\| \end{aligned}$$

In particular

$$\begin{aligned} \|Kf_n - Kf\| &\leq \max_{x \in I} \int_0^x \underbrace{|\sin(x-t)|}_{\leq 1} \underbrace{|k(t)|}_{\leq \max_{t \in I} |k(t)| < \infty} dt \|f_n - f\| \leq \\ &\leq \|k\| \|f_n - f\| \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

We have, provided that $(\times \times \times)$ holds, shown

$$\begin{aligned} f_{n+1} &= u + Kf_n \\ \downarrow &\qquad\qquad\qquad \downarrow \\ f &= u + Kf \end{aligned}$$

Let us try to prove $(\times \times \times) \left(\sum_{k=1}^{\infty} \|K^k v\| < \infty, \forall v \in C(I) \right)$

For $v \in C(I)$ arbitrary and $x \in I$

$$\begin{aligned} |Kv(x)| &= \left| \int_0^x \sin(x-t) k(t)v(t) dt \right| \leq \int_0^x \underbrace{|\sin(x-t)|}_{\leq 1} \underbrace{|k(t)|}_{\leq \|k\|} |v(t)| dt \leq \\ &\leq \int_0^x \underbrace{|v(t)|}_{\leq \|v\|} dt \cdot \|k\| \leq \|k\| \|v\| x, \quad x \in I \end{aligned}$$

In particular $\|Kv\| \leq \|k\| \|v\|$

$$|K^2 v| \leq \int_0^x |Kv(t)| dt \cdot \|k\| \leq \int_0^x \|k\| \|v\| t dt \cdot \|k\| = \|k\|^2 \|v\| \frac{x^2}{2}$$

In particular $\|K^2 v\| \leq \|k\|^2 \|v\| \frac{1}{2}$

By induction we get

$$\begin{aligned} |K^n v(x)| &\leq \|k\|^n \|v\| \frac{x^n}{n!}, \quad x \in I \\ \Rightarrow \|K^n v\| &\leq \|k\|^n \|v\| \frac{1}{n!} \end{aligned}$$

So

$$\sum_{l=1}^{\infty} \|K^l v\| \leq \sum_{l=1}^{\infty} \|k\|^l \|v\| \frac{1}{l!} = \|v\| \sum_{l=1}^{\infty} \frac{\|k\|^l}{l!} \leq e^{\|k\|} \|v\| < \infty$$

Conclusion: (xxx) holds true.

So far shown that

$$f = u + Kf$$

has a solution $f \in C(I)$.

Question: Is the solution unique?

Assume $f, \tilde{f} \in C(I)$ such that

$$\begin{cases} f = u + kf \\ \tilde{f} = u + k\tilde{f} \end{cases}$$

Set

$$v = f - \tilde{f} \in C(I)$$

$$v = (u + kf) - (u + k\tilde{f}) = kf - k\tilde{f} = k(f - \tilde{f}) = Kv$$

We have

$$v = Kv \Rightarrow Kv = K(Kv) = K^2v$$

$$\Rightarrow v = Kv = \dots = K^n v, \quad n = 1, 2, \dots$$

We know

$$\sum_{n=1}^{\infty} \|K^n v\| < \infty, \quad \forall v \in C(I)$$

Apply this to $\hat{v} = v$.

$$\text{Then } \sum_{n=1}^{\infty} \underbrace{\|K^n v\|}_{\substack{= \|v\| \\ \text{for all } n}} < \infty \Rightarrow \|v\| = 0$$

i.e. $v(x) = 0, \forall x \in I$.

We have

$$f(x) = \tilde{f}(x), \quad \forall x \in I$$

Answer to question: Yes!

We have more or less proved following theorem.

Theorem 1:

Set $I = [0, 1]$ and suppose $u \in C(I)$ and $k \in C(I \times I)$

Consider

$$f(x) = u(x) + \int_0^x k(x, t) f(t) dt, \quad x \in I \quad (1).$$

Then (1) has a unique solution $f \in C(I)$.

By using the same technology we can prove

Theorem 2:

Set $I = [0, 1]$ and $u \in C(I)$ and $k \in C(I \times I)$
suppose

and

$$\max_{(x,t) \in I \times I} |k(x, t)| < 1.$$

Consider

$$f(x) = u(x) + \int_0^x k(x, t) f(t) dt, \quad x \in I \quad (2)$$

Then (2) has a unique solution $f \in C(I)$.

Remark:

(1) is called a Volterra integral equation

(2) is called a Fredholm integral equation

Different notions used in the example:

Vector space: $C(I)$ with the operations

addition: $(v+w)(x) = v(x) + w(x)$, $v, w \in C(I)$

Multiplication by scalars: $(\lambda v)(x) = \lambda v(x)$, $v \in C(I), \lambda \in \mathbb{R}$

$x \in I$

Note that $v+w, \lambda v \in C(I)$.

Norm on $C(I)$

$$\|v\| = \max_{x \in I} |v(x)|$$

With norm given, we can talk about convergence and continuity.

Cauchy sequence in $C(I)$

In our example $\|f_n - f_m\| \rightarrow 0$, $n, m \rightarrow \infty$.

We say that the sequence $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Banach space $C(I)$ with the max-norm $\|\cdot\|$.

Lemma says every Cauchy sequence converges.

i.e. $\|v_n - v_m\| \rightarrow 0$, $n, m \rightarrow \infty$

$$\Rightarrow \exists v \in C(I) : \|v_n - v\| \xrightarrow{n \rightarrow \infty} 0.$$

K linear mapping: $C(I) \rightarrow C(I)$

$$K(v+w) = Kv + Kw \quad , \quad v, w \in C(I), \lambda \in \mathbb{R}$$

$$K(\lambda v) = \lambda Kv$$

K bounded linear

$$\|Kv\| \leq M\|v\|, \quad \forall v \in C(I)$$

when M independent of v .

Definition: $\|K\| = \inf \{M > 0 : \|Kv\| \leq M\|v\|, \forall v \in C(I)\}$

Fixed point results

Our example:

$$f = u + Kf \equiv T(f)$$

$$f_0 \in C(I)$$

Formed the sequence of iterates $(f_n)_{n=0}^{\infty}$

$$f_n = T(f_{n-1}), \quad n=1,2,\dots$$

If $\|T(v) - T(w)\| \leq c\|v-w\|$, for all $v,w \in C(I)$
and for some $c < 1$.

Then there is a unique $v \in C(I)$ such that

$$v = T(v).$$

(This is Banach's fixed point theorem)

Green's function

Our example

$$L = \left(\frac{d}{dx}\right)^2 + 1 \quad \text{differential operator}$$

Boundary conditions $v(0) = v'(0) = 0$.

$$\text{Then } f(x) = \underbrace{\int_0^x g(x,t) h(t) dt}_{\text{Green's function}}$$

Vector spaces

Definition:

We say that E is a real vector space if it is a nonempty set with the operations

- addition: $E \times E \rightarrow E$ $(x, y) \mapsto x + y$
- M. B. S : $\mathbb{R} \times E \rightarrow E$ $(\lambda, x) \mapsto \lambda x$

satisfying the axioms:

- 1) $x + y = y + x, \quad \forall x, y \in E$
- 2) $(x + y) + z = x + (y + z), \quad \forall x, y, z \in E$
- 3) For all $x, y \in E$ there exists $z \in E$ such that $x + z = y$.
- 4) $\alpha(\beta x) = (\alpha\beta)x, \quad \forall \alpha, \beta \in \mathbb{R}, \quad \forall x \in E$
- 5) $\alpha(x+y) = \alpha x + \alpha y, \quad \forall \alpha \in \mathbb{R}, \quad \forall x, y \in E$
- 6) $(\alpha + \beta)x = \alpha x + \beta x, \quad \forall \alpha, \beta \in \mathbb{R}, \quad \forall x \in E$
- 7) $1x = x, \quad \forall x \in E$

Remark: E is a complex vector space if all \mathbb{R} above are replaced by \mathbb{C} .

Remark: $\exists! 0 \in E : x + 0 = x, \quad \forall x \in E$

Fix $x \in E$. By 3) there exists 0_x such that $x + 0_x = x$.

Fix $y \in E$. Want to show that $y + 0_x = y$. By 3) there exists $z \in E$ such that $x + z = y$

$$y + 0_x = (x + z) + 0_x = (x + 0_x) + z = x + z = y.$$

Assume $x + \mathbb{O}_1 = x$, $x + \mathbb{O}_2 = x$, $\forall x \in E$.

We want to show that $\mathbb{O}_1 = \mathbb{O}_2$ (implying the uniqueness of the zero vector).

$$\mathbb{O}_1 = \mathbb{O}_1 + \mathbb{O}_2 = \mathbb{O}_2 + \mathbb{O}_1 = \mathbb{O}_2.$$

2) $\forall x \in E \exists! (-x) \in E : x + (-x) = \mathbb{O}$.

3) $\mathbb{O}x = \mathbb{O}$, $\forall x \in E$

$$(-1)x = -x, \quad \forall x \in E.$$

Examples of real vector spaces

1) \mathbb{R} , with standard addition and M.B.S.

2) \mathbb{R}^n , $n = 1, 2, \dots$

$$\text{addition: } (x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

$$\text{M.B.S: } \lambda(x_1, x_2, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$$

3) $\mathbb{R}^\infty = \{(x_1, x_2, \dots, x_n, \dots) : x_n \in \mathbb{R}, n = 1, 2, \dots\}$

4) $1 \leq p < \infty$

$$\ell^p = \left\{ (x_1, \dots, x_n, \dots) \in \mathbb{R}^\infty : \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} < \infty \right\}$$

with the same addition and M.B.S as \mathbb{R}^∞

To check:

$$1) x, y \in \ell^p \Rightarrow x+y \in \ell^p$$

$$2) x \in \ell^p \Rightarrow \lambda x \in \ell^p$$

1) Assume $x = (x_1, \dots, x_n, \dots)$, $y = (y_1, \dots, y_n, \dots)$

$$\begin{cases} x \in l^p & \sum_{n=1}^{\infty} |x_n|^p < \infty \\ y \in l^p & \sum_{n=1}^{\infty} |y_n|^p < \infty \end{cases}$$

$$x+y = (x_1+y_1, \dots, x_n+y_n, \dots) \stackrel{?}{\in} l^p$$

We note that

$$|x_n+y_n| \leq |x_n| + |y_n| \leq 2 \max(|x_n|, |y_n|)$$

$$\Rightarrow |x_n+y_n|^p \leq 2^p \max(|x_n|^p, |y_n|^p) \leq 2^p (|x_n|^p + |y_n|^p)$$

$$\begin{aligned} \Rightarrow \sum_{n=1}^{\infty} |x_n+y_n|^p &\leq \sum_{n=1}^{\infty} 2^p (|x_n|^p + |y_n|^p) = \\ &= \underbrace{2^p \sum_{n=1}^{\infty} |x_n|^p}_{< \infty} + \underbrace{2^p \sum_{n=1}^{\infty} |y_n|^p}_{< \infty} < \infty. \end{aligned}$$

$$2) \quad \sum_{n=1}^{\infty} |\lambda x_n|^p \leq \sum_{n=1}^{\infty} |\lambda|^p |x_n|^p = |\lambda|^p \sum_{n=1}^{\infty} |x_n|^p < \infty.$$

$\Rightarrow l^p$ vector space.

5) Function spaces, say real valued functions on I.

addition: $(f+g)(x) = f(x) + g(x)$

M.B.S : $(\lambda f)(x) = \lambda f(x)$

for functions f and g.

6) $C(I)$, addition and M.B.S as in 5)

• f, g continuous on I $\Rightarrow f+g$ continuous on I

• f cont, $\lambda \in \mathbb{R} \Rightarrow \lambda f$ cont on I.

7) $P(I)$, polynomials on I.

8) $P_k(I)$, polynomials of degree at most k on I.

Föreläsning 2

Hölder's inequality:

Theorem:

Assume $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Let (x_1, \dots, x_n, \dots) and (y_1, \dots, y_n, \dots) be sequences of complex numbers.

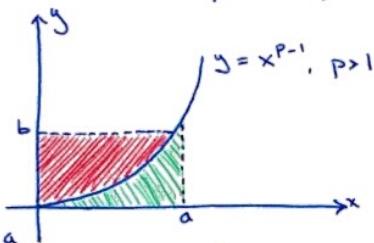
$$\Rightarrow \sum_{n=1}^{\infty} |x_n y_n| \leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |y_n|^q \right)^{1/q}.$$

Remark: Here the LHS can be $=\infty$, but then the RHS is also $=\infty$.

Proof:

Step 1: $ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \forall a, b > 0$.

Proof:



$$\text{Area}_{\text{red}} = \int_0^a x^{p-1} dx = \frac{a^p}{p}$$

$$\text{Note: } y = x^{p-1} \text{ gives } x = y^{\frac{1}{p-1}} = \left\{ P = \frac{1}{1-\frac{1}{q}} \right\} = y^{\frac{1}{\frac{1}{1-\frac{1}{q}} - 1}} = \\ = y^{\frac{q}{q-1} - 1} = y^{q-1}$$

$$\text{Area}_{\text{green}} = \int_0^b y^{q-1} dy = \frac{b^q}{q}$$

$$\text{We get } ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Step 2: It is enough to consider the case $LHS > 0$ and $RHS < \infty$ (the other cases are trivial).

There exists integer N such that

$$0 < \sum_{n=1}^N |x_n|^p, \quad \sum_{n=1}^N |y_n|^q < \infty.$$

Set

$$\begin{cases} a = \frac{|x_k|}{\left(\sum_{n=1}^N |x_n|^p\right)^{1/p}}, & k = 1, 2, \dots, N \\ b = \frac{|y_k|}{\left(\sum_{n=1}^N |y_n|^q\right)^{1/q}}, & k = 1, 2, \dots, N \end{cases}$$

Insert into $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$:

$$\frac{|x_k y_k|}{\left(\sum_{n=1}^N |x_n|^p\right)^{1/p} \left(\sum_{n=1}^N |y_n|^q\right)^{1/q}} \leq \frac{|x_k|^p}{p \sum_{n=1}^N |x_n|^p} + \frac{|y_k|^q}{q \sum_{n=1}^N |y_n|^q}, \quad k = 1, 2, \dots, N$$

• Sum over k from 1 to N .

$$\Rightarrow \sum_{k=1}^N |x_k y_k| \leq \left(\sum_{n=1}^N |x_n|^p \right)^{1/p} \left(\sum_{n=1}^N |y_n|^q \right)^{1/q}.$$

Let $N \rightarrow \infty$, first in RHS and then in LHS. □

Minkowski's inequality.

Theorem: Assume $1 \leq p < \infty$ and $x, y \in l^p$

$$\rightarrow \|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

$$\left(\|x\|_{l^p} = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \right)$$

Proof: $p=1$

$$\begin{aligned} \|x + y\|_{l^1} &= \|(x_1, \dots, x_n, \dots) + (y_1, \dots, y_n, \dots)\|_{l^1} = \\ &= \|(x_1 + y_1, \dots, x_n + y_n, \dots)\| = \sum_{n=1}^{\infty} |x_n + y_n| \leq \sum_{n=1}^{\infty} (|x_n| + |y_n|) = \\ &= \sum_{n=1}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = \|x\|_{l^1} + \|y\|_{l^1}. \end{aligned}$$

$1 < p < \infty$

$$\begin{aligned} \|x + y\|_p^p &= \sum_{n=1}^{\infty} |x_n + y_n|^p = \sum_{n=1}^{\infty} \underbrace{|x_n + y_n|}_{\leq |x_n| + |y_n|} |x_n + y_n|^{p-1} \leq \\ &\leq \sum_{n=1}^{\infty} |x_n| |x_n + y_n|^{p-1} + \sum_{n=1}^{\infty} |y_n| |x_n + y_n|^{p-1}; \end{aligned}$$

$$\begin{aligned} \sum_{n=1}^{\infty} |x_n| |x_n + y_n|^{p-1} &\leq \left\{ \text{Holder's inequality} \right\} \leq \underbrace{\left(\sum_{n=1}^{\infty} |x_n|^p \right)}^{p} \cdot \underbrace{\left(\sum_{n=1}^{\infty} |x_n + y_n|^q \right)}_{= \|x\|_{l^p}}^{q/p} \\ &= \left\{ (p-1)q = (p-1) \frac{1}{1 - \frac{1}{p}} = p \right\} = \|x\|_{l^p} \left(\sum_{n=1}^{\infty} |x_n + y_n|^p \right)^{1/q} = \|x\|_{l^p} \end{aligned}$$

We have

$$\|x + y\|_p^p \leq (\|x\|_{l^p} + \|y\|_{l^p}) \|x + y\|_{l^p}^{p/q}$$

If $\|x + y\|_p \neq 0$, then

$$\|x + y\|_p^{p - \frac{p}{q}} \leq \|x\|_{l^p} + \|y\|_{l^p} \quad \text{with} \quad p - \frac{p}{q} = p \left(1 - \frac{1}{q} \right) = p \cdot \frac{1}{p} = 1. \quad \square$$

Remark: $f \in C([0,1])$

$$\|f\|_{L^p} = \left(\int_0^1 |f(t)|^p dt \right)^{1/p}, \quad 1 \leq p < \infty.$$

Claim:

$$(x) \|fg\|_{L^1} = \int_0^1 |f(t)g(t)| dt \leq \|f\|_{L^p} \cdot \|g\|_{L^{p'}} , \quad \frac{1}{p} + \frac{1}{p'} = 1$$

$$(xx) \|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

(x): Corresponding Hölder's

(xx): Corresponding Minkowski's

This can be proven with the same technique as we used with ℓ^p with

$$\sum_{n=1}^{\infty} \text{ replaced by } \int_0^1 dt$$

- E is a real/complex vector space

$$x_1, \dots, x_n \in E$$

$$\lambda_1, \dots, \lambda_n \in \mathbb{F} \quad (\mathbb{F} \text{ arbitrary scalar field})$$

- We say that

$$\lambda_1 x_1 + \dots + \lambda_n x_n$$

is a linear combination of x_1, \dots, x_n .

- We say x_1, \dots, x_n are linearly independent if

$$\alpha_1 x_1 + \dots + \alpha_n x_n = 0$$

$$\Rightarrow \alpha_1 = \dots = \alpha_n = 0.$$

- $A \subset E$. A is linearly independent if every linear combination of vectors in A is linearly independent.

Example:

$$E = \mathcal{P}([0,1])$$

$$A = \{P_k : P_k(x) = x^k, x \in [0,1], k=0,1,2,\dots\}$$

- A is linearly independent.

• Let's consider

$$\alpha_0 P_0 + \dots + \alpha_n P_n = 0$$

$$\text{i.e. } \alpha_0 P_0(x) + \dots + \alpha_n P_n(x) = 0(x), x \in [0,1]$$

$$\text{i.e. } \alpha_0 + \alpha_1 x + \dots + \alpha_n x^n = 0, x \in [0,1]$$

$$x=0 \Rightarrow \alpha_0 = 0$$

$$\alpha_1 x + \dots + \alpha_n x^n = 0$$

- Differentiate $\Rightarrow \alpha_1 = 0$, repeat and get that

$$\alpha_0 = \alpha_1 = \dots = \alpha_n = 0.$$

□

- $B \subset E$

- $\text{Span } B$ = set of all linear combination of elements in $B = \left\{ \sum_{k=1}^n \lambda_k x_k : x_k \in B, \lambda_k \in F, k=1,2,\dots,n \right\}$

where n is a positive integer

Remark: $\sum_{k=1}^n \lambda_k x_k \in E$

$\sum_{k=1}^{\infty} \lambda_k x_k$ has no meaning

$C \subset E$ is called a basis for E if

- 1) C is linearly independent
- 2) $\text{Span } C = E$

Example:

$$E = \mathbb{P}(\{0,1\})$$

$$A = \{P_k : k=0,1,\dots\}$$

Claim: A is a basis for E .

Example:

$$E = \ell^2$$

$$A = \{x_k : k=1,2,\dots\}$$

$$x_k = (0, 0, \dots, 0, \underset{\substack{\uparrow \\ \text{Position } k}}{1}, 0, \dots)$$

Claim: A is linearly independent, since

$$\underbrace{\alpha_1 x_1}_{(\alpha_1, 0, \dots)} + \dots + \underbrace{\alpha_n x_n}_{(0, 0, \dots, 0, \alpha_n, \dots)} = 0 \Rightarrow (\alpha_1, \dots, \alpha_n) = (0, 0, \dots, 0)$$

$$\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0.$$

Question: is A a basis for ℓ^2 ?

We note: if $x \in \text{Span } A$, then

$$x = (x_1, \dots, x_n, 0, 0, 0, \dots) \text{ for some positive } n.$$

i.e. x has finitely many non-zero positions.

Consider: $x = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots)$

$$\|x\|_{\ell^2} = \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} < \infty$$

$$\text{So } x \in \ell^2 \setminus \text{Span } A$$

Remark: Every vector space has basis (if we are allowed to use axioms of choice/Zorn's lemma).

Assume x_1, \dots, x_n is a basis for E . Then every basis for E must contain n different elements.
 $n = \dim E$ is well-defined.

NORMS

- E vector space
- We say that a mapping $\|\cdot\|: E \rightarrow [0, \infty)$ is a norm on E if
 - 1) $\|x\| = 0 \Rightarrow x = 0$
 - 2) $\|\lambda x\| = |\lambda| \|x\|, \forall x \in E, \forall \lambda \in F$
 - 3) $\|x+y\| \leq \|x\| + \|y\|, \forall x, y \in E$

Remark: $\|0\| = \|0 \cdot 0\| = \underbrace{\|0\|}_{=0} \|0\| = 0$

Example: $1 \leq p < \infty$

$\|x\|_p$ is a norm on ℓ^p . Need to check 1, 2 and 3.

$$1) \|0\|_{\ell^p} = \left(\sum_{n=1}^{\infty} |0_n|^p \right)^{1/p} \Rightarrow x_n = 0, n=1,2,\dots \Rightarrow x = (x_1, \dots) = (0, \dots) = 0$$
$$2) \|\lambda x\|_{\ell^p} = \left(\sum_{n=1}^{\infty} |\lambda x_n|^p \right)^{1/p} = \left(|\lambda|^p \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} = |\lambda| \|x\|_{\ell^p}$$

$$3) \|x+y\|_{\ell^p} \stackrel{\text{Minkowski's}}{\leq} \|x\|_{\ell^p} + \|y\|_{\ell^p}$$

Example: $E = C([0,1]) \ni f$

$$\|f\| = \max_{t \in [0,1]} |f(t)| \in [0, \infty)$$

Check 1, 2 and 3.

$$1) \|f\| = 0 \Rightarrow |f(t)| = 0, \forall t \in [0,1] \Rightarrow f = \emptyset.$$

$$2) \|\lambda f\| = \max_{t \in [0,1]} |\underbrace{(\lambda f)(t)}_{\lambda f(t)}| = |\lambda| \max_{t \in [0,1]} |f(t)| = |\lambda| \|f\|$$

$$3) \|f+g\| = \max_{t \in [0,1]} |\underbrace{(f+g)(t)}_{f(t)+g(t)}| \leq \max_{t \in [0,1]} (|f(t)| + |g(t)|) \leq \\ \leq \max_{t \in [0,1]} |f(t)| + \max_{t \in [0,1]} |g(t)| = \|f\| + \|g\|$$

Example: $E = C([0,1]) \ni f$

$$\|f\|_L = \int_0^1 |f(t)| dt \text{ defines a norm on } E.$$

Check 1, 2, 3.

$$3) \|f+g\|_L = \int_0^1 |f+g| dt \leq \int_0^1 |f| dt + \int_0^1 |g| dt = \|f\|_L + \|g\|_L$$

$$2) \|\lambda f\|_L = \int_0^1 |\underbrace{(\lambda f)(t)}_{\lambda f(t)}| dt = |\lambda| \int_0^1 |f(t)| dt = |\lambda| \|f\|_L$$

$$1) 0 = \|f\|_L - \int_0^1 |f(t)| dt \Rightarrow f(t) = 0, \forall t \in [0,1]$$

Since f is continuous.

i.e. $f = \emptyset$.

Equivalence of norms:

- E vector space with norms $\|\cdot\|$ and $\|\cdot\|_*$.
- We say that $\|\cdot\|, \|\cdot\|_*$ are equivalent if there exists $\alpha, \beta > 0$ such that

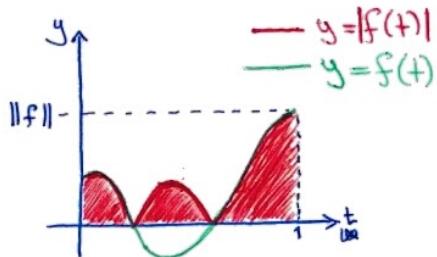
$$\alpha \|x\|_* \leq \|x\| \leq \beta \|x\|_*, \quad \forall x \in E.$$

Example:

$$E = C([0,1])$$

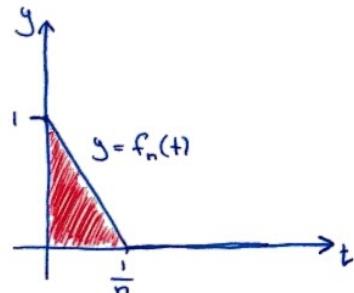
$$\|f\| = \max_{t \in [0,1]} |f(t)|$$

$$\|f\|_* = \|f\|_{L^1} = \text{area}$$



Question: Are these two norms equivalent?

$$\|f\|_* = \int_0^1 |f(t)| dt \leq \|f\|$$



$$\|f_n\| = 1$$

$$\|f_n\|_* = \frac{1}{2n} \quad (\text{area})$$

$$\text{We want } \|f_n\|_* \geq \alpha \|f_n\|$$

$$\Rightarrow \frac{\|f_n\|_*}{\|f_n\|} \geq \alpha > 0$$

$$\text{But } \frac{\|f_n\|_*}{\|f_n\|} = \frac{1}{2n} \xrightarrow{n \rightarrow \infty} 0$$

Answer: No! Not equivalent.

Theorem:

E vector space with finite dimension ($\dim E < \infty$)

\Rightarrow All norms on E are equivalent.

(Actually \Leftrightarrow but need Zorn's lemma to prove)

Proof:

Assume that $n = \dim E$ (n positive integer) and let x_1, x_2, \dots, x_n be a basis for E .

For every $x \in E$

$$x = \alpha_1(x)x_1 + \dots + \alpha_n(x)x_n \text{ where } \alpha_1(x), \dots, \alpha_n(x) \text{ unique}$$

Set

$$\|x\|_* = |\alpha_1(x)| + \dots + |\alpha_n(x)|, \quad x \in E.$$

Claim: $\|x\|_*$ defines a norm on E (to prove this, simply check norm conditions 1, 2, 3 as usual).

Fix an arbitrary norm $\|\cdot\|$ on E .

Claim: $\|\cdot\|, \|\cdot\|_*$ are equivalent.

Note: for $x \in E$

$$\begin{aligned} \|x\| &= \|\alpha_1(x)x_1 + \dots + \alpha_n(x)x_n\| \leq |\alpha_1(x)|\|x_1\| + \dots + |\alpha_n(x)|\|x_n\| \leq \\ &\leq \max_{k=1,2,\dots,n} \|x_k\| \underbrace{(|\alpha_1(x)| + \dots + |\alpha_n(x)|)}_{=\|x\|_*}, \end{aligned}$$

$$\text{Set } \beta = \max_{k=1,2,\dots,n} \|x_k\|.$$

$$\text{Then } \|x\| \leq \beta \|x\|_*, \quad \forall x \in E.$$

Remains to prove there exists $\alpha > 0$ such that

$$\alpha \|x\|_* \leq \|x\|, \quad \forall x \in E.$$

(Following not part of proof)

- Let E be a vector space with norm $\|\cdot\|$ and $(v_n)_{n=1}^{\infty}$ a sequence in E . We say that $(v_n)_{n=1}^{\infty}$ converges in $(E, \|\cdot\|)$ if

$$\exists v \in E : \|v_n - v\| \xrightarrow{n \rightarrow \infty} 0$$

Notation: $v_n \rightarrow v$ in $(E, \|\cdot\|)$

Note: if $\|\cdot\|, \|\cdot\|_*$ are equivalent, then

$$v_n \rightarrow v \text{ in } (E, \|\cdot\|) \iff v_n \rightarrow v \text{ in } (E, \|\cdot\|_*)$$

(Back to proof):

- We argue by contradiction.
- Assume there is no $\alpha > 0$ such that $\alpha \|x\|_* \leq \|x\|, \forall x \in E$.
- For $k = 1, 2, \dots$ there are $y_k \in E$ such that

$$\frac{1}{k} \|y_k\|_* > \|y_k\| \quad (\times \times)$$

- We have

$$y_k = \alpha_1^{(k)} x_1 + \dots + \alpha_n^{(k)} x_n, \quad \alpha_1^{(k)}, \dots, \alpha_n^{(k)} \text{ unique scalars, } k = 1, 2, \dots$$

$(\times \times)$:

$$k \|y_k\| < |\alpha_1^{(k)}| + \dots + |\alpha_n^{(k)}|$$

We have

$$k \|y_k\| < 1, \quad k=1, 2, \dots$$

which implies that $y_k \rightarrow 0$ in $(E, \|\cdot\|)$.

If

$$\alpha_1^{(k)} \rightarrow \bar{\alpha}_1, \quad k \rightarrow \infty$$

$$\alpha_2^{(k)} \rightarrow \bar{\alpha}_2, \quad k \rightarrow \infty$$

⋮

$$\alpha_n^{(k)} \rightarrow \bar{\alpha}_n, \quad k \rightarrow \infty$$

then set $\bar{y} = \bar{\alpha}_1 x_1 + \dots + \bar{\alpha}_n x_n$.

$$\begin{aligned} \|y_k - \bar{y}\| &= \|(\alpha_1^{(k)} - \bar{\alpha}_1)x_1 + \dots + (\alpha_n^{(k)} - \bar{\alpha}_n)x_n\| \leq \\ &\leq |\alpha_1^{(k)} - \bar{\alpha}_1| \underbrace{\|x_1\|}_{\substack{\longrightarrow 0 \\ k \rightarrow \infty}} + \dots + |\alpha_n^{(k)} - \bar{\alpha}_n| \underbrace{\|x_n\|}_{\substack{\longrightarrow 0 \\ k \rightarrow \infty}} \xrightarrow{k \rightarrow \infty} 0 \end{aligned}$$

$$\|\bar{y}\| = \|\bar{y} + y_k - y_k\| \leq \underbrace{\|\bar{y} - y_k\|}_{\substack{\longrightarrow 0 \\ k \rightarrow \infty}} + \underbrace{\|y_k\|}_{\substack{\longrightarrow 0 \\ k \rightarrow \infty}}$$

$$\text{So } \|\bar{y}\| = 0 \Rightarrow \bar{y} = 0$$

But

$$|\bar{\alpha}_1| + |\bar{\alpha}_2| + \dots + |\bar{\alpha}_n| = 1$$

and this contradicts the fact that

x_1, \dots, x_n is a basis!

$$\left(\boxed{\alpha_{1,1}^{(k)}} \alpha_2^{(k)}, \dots, \alpha_n^{(k)} \right) \quad k=1,2,\dots$$

where

$$|\alpha_{1,1}^{(k)}| + \dots + |\alpha_n^{(k)}| = 1$$

$$(x): \quad |\alpha_{1,1}^{(k)}| \leq 1, \quad k=1,2,\dots$$

By Bolzano-Weierstrass theorem there exists a convergent subsequence $(\alpha_{1,1}^{(k)})_{k=1}^{\infty}$ of $(\alpha_{1,1}^{(k)})_{k=1}^{\infty}$

$$\text{Set } \bar{\alpha}_1 = \lim_{k \rightarrow \infty} \alpha_{1,1}^{(k)}$$

Consider

$$(\alpha_{1,1}^{(k)}, \alpha_{2,1}^{(k)}, \dots, \alpha_{n,1}^{(k)}) , \quad k=1,2,\dots$$

We have that

$$|\alpha_{2,1}^{(k)}| \leq 1, \quad k=1,2,\dots$$

$$\text{Set } \bar{\alpha}_2 = \lim_{k \rightarrow \infty} \alpha_{2,2}^{(k)}, \quad \text{etc.}$$

etc.

Föreläsning 3 (Topology of normed spaces, ~~and~~ Banach spaces)

A norm generates a distance function (metric)

$$d(x, y) := \|x - y\|, \quad x, y \in E.$$

Example:

$$\mathbb{R}^n, \quad \|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}, \quad \|x - y\|_2 - \text{Euclidean distance}.$$

$$C([a, b]), \quad d(f, g) = \|f - g\| = \max_{x \in [a, b]} |f(x) - g(x)|$$

Let $x \in E$ and $r > 0$. Define

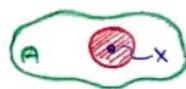
$$B(x, r) = \{y \in E : \|x - y\| < r\} \quad (\text{open ball})$$

$$\bar{B}(x, r) = \{y \in E : \|x - y\| \leq r\} \quad (\text{closed ball})$$

Definition:

A subset $A \subset E$ ($(E, \|\cdot\|)$) is called **open** if any point x of A is inner, i.e.

$$\exists r > 0 : B(x, r) \subset A$$



A subset $A \subset E$ is called **closed** if $\underbrace{E \setminus A}_{\text{its complement}}$ is open.

- Open balls are open sets.

- Closed balls are closed sets.

- $C([0, 1]), \|f\|_\infty = \max_{x \in [0, 1]} |f(x)|.$

Let f be fixed.

$$\{g \in C([0, 1]) : f(x) < g(x), \forall x \in [0, 1]\} \quad \text{open set in } C([0, 1])$$

$$\{g \in C([0, 1]) : f(x) \leq g(x), \forall x \in [0, 1]\} \quad \text{closed set in } C([0, 1])$$

Properties (open sets)

- Any union of open sets is an open set.
- Any finite intersection of open sets is open.
- \emptyset and E are both open and closed.

(Topological spaces satisfy these conditions \Rightarrow normed spaces are topological spaces).

Convergence in normed spaces

Definition: let $(E, \|\cdot\|)$ be a normed space, and let x_n be a sequence in E . We say that x_n converges to $x \in E$ if

$$\|x_n - x\| \rightarrow 0, \quad n \rightarrow \infty.$$

Statement: $A \subset E$ is closed iff any convergent sequence in A has a limit in A , i.e.

$$x_n \rightarrow x, \quad x_n \in A \Rightarrow x \in A.$$

Proof: \Rightarrow

Assume A is closed and $x_n \rightarrow x$, $x_n \in A$, but $x \notin A$.

If A is closed, then $E \setminus A$ is open, and hence

$$\exists r > 0 : B(x, r) \subset E \setminus A$$

Hence $\|x_n - x\| \geq r, \forall n \Rightarrow$ contradiction!

So $x_n \rightarrow x \Rightarrow x \in A$ if A closed.

\Leftarrow Assume that for any convergent $x_n \in A$: $x_n \rightarrow x$, we have $x \in A$.
Assume A is not closed, then $E \setminus A$ is not open.

Therefore $\exists x \in E \setminus A$ which is not inner

$\Rightarrow \forall B(x, \frac{1}{n})$ contains points outside $E \setminus A$, i.e.

$$\exists x_n \in B(x, \frac{1}{n}), \quad x_n \in A.$$

We get a sequence $x_n \in A$: $\|x_n - x\| < \frac{1}{n}$ and hence
 $x_n \rightarrow x$. Contradiction! \blacksquare

Definition: $A \subset E$

Closure of A is the minimal closed subset containing A . Notation: $\text{cl } A$ (or \bar{A}).

Proposition: $\text{cl } A = \text{the set of all limit points of } A = \{x \in E \mid \exists x_n \in A : x_n \rightarrow x\}$

Proof left as exercise.

Definition: $A \subset E$ is dense in E if $\text{cl } A = E$
 $\Leftrightarrow \forall x \in E, \forall \varepsilon > 0 \exists y \in A : \|x - y\| < \varepsilon$.

Examples:

1) $\mathbb{Q} \subset \mathbb{R}$. \mathbb{Q} is dense in \mathbb{R} .

2) The Weierstrass theorem says that the set of all polynomials are dense in $(C([0,1]), \|\cdot\|_\infty)$.

$\forall f \in C([0,1]), \forall \varepsilon > 0 \exists p \text{-polynomial} : \max_{x \in [0,1]} |f(x) - p(x)| < \varepsilon$.

Example: $(C_0, \|\cdot\|_\infty)$

$C_0 = \{\underline{x} = (x_1, x_2, \dots) : x_k \xrightarrow{k \rightarrow \infty} 0\}, \|\underline{x}\|_\infty = \sup |x_i|$

$(C_0, \|\cdot\|_\infty)$ is a normed space

$C_F = \{\underline{x} = (x_1, x_2, \dots) : \underset{x_i \neq 0}{\text{only finite}}\} \subset C_0$

Statement: C_F is dense in C_0 .

Proof: $\forall \underline{x} \in C_0, \forall \varepsilon > 0$ must find $\underline{y} \in C_F$ such that $\|\underline{y} - \underline{x}\| < \varepsilon$.

$\underline{x} \in C_0 \Rightarrow x_k \xrightarrow{k \rightarrow \infty} 0 \Rightarrow \forall \varepsilon > 0 \exists K : |x_k| < \varepsilon \quad \forall k \geq K$.

let now $\underline{y} = (x_1, x_2, \dots, x_K, 0, 0, \dots) \in C_F$

Then $\|\underline{x} - \underline{y}\|_\infty = \|(0, 0, \dots, 0, x_{K+1}, x_{K+2}, \dots)\|_\infty = \sup_{k > K} |x_k| < \varepsilon$. \square

Definition: $(E, \|\cdot\|)$ is called separable if contains countable dense subset.

Example: $(\mathbb{R}, |\cdot|)$ is separable as \mathbb{Q} is countable and dense in \mathbb{R} . $(\mathbb{R}^n, \|\cdot\|_2)$ separable as $\mathbb{Q}^n \dots$

Compact sets in $(E, \|\cdot\|)$

Definition: A subset $A \subset E$ is compact if any sequence $x_n \in A$ has a subsequence convergent to element $x \in A$.

Example: A bounded and closed subset in \mathbb{R} ($\mathbb{R}^n, \mathbb{C}^n$) is compact.

A sequence x_n of a bounded set is bounded. From analysis one knows that it has a subsequence that is convergent. If a subset is closed, then the limit point is in the set.

Lemma: If S is compact in $(E, \|\cdot\|)$, then S is closed and bounded.

(S bounded means that $S \subset B(0, R)$ for some $R > 0$).

Proof: Let $S \subset E$ be compact. Assume that S is not bounded. Then for any $n > 0$ there exists points in S which are outside $B(0, n)$, i.e.

$$\exists x_n \in S : \|x_n\| > n.$$

Then x_n can not have a convergent subsequence since if $x_n \rightarrow x$, then $n_k < \|x_{n_k}\| = \|x_{n_k} - x + x\| \leq \|x_{n_k} - x\| + \|x\| \rightarrow \|x\|$. Hence S must be bounded.

S must be closed, because if $\underset{n \rightarrow \infty}{\lim} x_n = x$, then any subsequence $\rightarrow x$. From definition of compactness and uniqueness of the limit we have $x \in S$. \square

• In general: S bounded and closed $\not\Rightarrow S$ is compact.

Example: $E = C([0, 1])$, $S = \{g \in C([0, 1]) : \|g\|_\infty \leq 1\}$ closed and bounded, but not compact.

Take $x_n(t) = t^n$, $x_n \in S$

x_n does not have a subsequence convergent to a cont. funct.

If $x_n \rightarrow x$ in $\|\cdot\|_\infty$, then

$$\forall t \in [0, 1] \quad |x_n(t) - x(t)| \leq \max_{t \in [0, 1]} |x_n(t) - x(t)| \longrightarrow 0$$

Lemma: (Riesz's lemma)

If X is a proper closed subspace of a normed space $(E, \|\cdot\|)$, then

$$\forall \varepsilon \in (0,1) \exists x_\varepsilon \in E, \|x_\varepsilon\|=1 : \|x_\varepsilon - x\| \geq \varepsilon, \forall x \in X.$$

Proof: Let $z \in E \setminus X$ (X proper $\Rightarrow E \setminus X \neq \emptyset$)

$$\text{Set } d := \inf \|z - x\|, x \in X.$$

As X is closed, $d > 0$ (otherwise z is the limit point and hence in X (contradiction)).

Fix $\varepsilon \in (0,1)$.

$$\exists x_0 \in X : d \leq \|z - x_0\| < \frac{d}{\varepsilon}.$$

$$\text{Let } x_\varepsilon = \frac{z - x_0}{\|z - x_0\|}.$$

We have $\|x_\varepsilon\|=1$ and

$$\begin{aligned} \|x - x_\varepsilon\| &= \left\| x - \frac{z - x_0}{\|z - x_0\|} \right\| = \left\| \frac{x\|z - x_0\| - z + x_0}{\|z - x_0\|} \right\| = \\ &= \left\| \frac{\underbrace{\varepsilon z}_{\varepsilon \in \mathbb{R}} + x_0 - z}{\|z - x_0\|} \right\| \geq \frac{d}{\left(\frac{d}{\varepsilon}\right)} = \varepsilon. \quad \square \end{aligned}$$

Theorem: $(E, \|\cdot\|)$

$\dim E < \infty \Leftrightarrow$ any compact set $A \subset E$ is closed and bounded.

Proof: \Rightarrow if $\dim E < \infty \Rightarrow A$ is compact iff A is bounded and closed. (left as exercise)

\Leftarrow Enough to prove following:

if $\dim E = \infty$, then the unit ball $S = \{x \in E : \|x\| \leq 1\}$ is not compact.

Let $x_1 \in S$, consider $X = \text{Span}\{x_1\} \leftarrow$ proper closed subspace of E .

Hence, by Riesz's lemma $\exists x_2, \|x_2\|=1 : \|x_2 - x_1\| \geq \frac{1}{2}$.

Consider $\text{Span}\{x_1, x_2\}$ - proper closed subspace of E .

By Riesz's lemma, $\exists x_3 \in E, \|x_3\|=1 : \|x_3 - x_1\| \geq \frac{1}{2}$.

Continuing in the same fashion we get $x_n, \|x_n\|=1 : \|x_n - x_m\| \geq \frac{1}{2}, \forall n, m, n \neq m$.

Clearly, $x_n \in S$ has no convergent subsequence.

Hence, S is not compact. \square

Cauchy Sequences in $(E, \|\cdot\|)$

Definition: $x_n \in E$ is called Cauchy if

$$\forall \epsilon > 0 \exists N : \|x_n - x_m\| < \epsilon, n, m \geq N.$$

Example: $(C_F, \|\cdot\|_\infty)$, $\|x\|_\infty = \sup_{k \in \mathbb{N}} |x_k|$, $x = (x_1, x_2, \dots)$

$$x_n = \left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, \dots\right)$$

x_n is Cauchy as $\limsup_{n > m} \|x_n - x_m\|_\infty = \|(0, 0, \dots, \frac{1}{m+1}, \dots, \frac{1}{n}, 0, 0, \dots)\|_\infty = \frac{1}{m+1}$.

Observe that x_n is convergent in $(C_0, \|\cdot\|_\infty)$, but not in C_F .

$$\underset{\substack{x_n \\ \in C_0}}{\longrightarrow} (1, \frac{1}{2}, \dots, \frac{1}{n}, \dots) \in C_0 \setminus C_F.$$

Statement: A convergent sequence is always Cauchy.
(Proof left as exercise)

Definition: A normed vector space is called complete if any Cauchy sequence in E is convergent in E .

Example: C_F is not complete.

Definition: A complete, normed space is called Banach space.

Examples: $\circ (\mathbb{R}, |\cdot|), (\mathbb{C}, |\cdot|)$ - Banach spaces.

$\circ (l^2, \|\cdot\|_{l^2})$ - normed space

$$l^2 = \left\{ (x_1, x_2, \dots) : \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\}$$

Check if $(l^2, \|\cdot\|_{l^2})$ is complete (and hence a Banach space).

Let $\underline{x}_n = (x_1^n, x_2^n, \dots)$ be a Cauchy sequence in ℓ^2 .

Must show that it has a limit in ℓ^2 .

- Steps:
- 1) Find a candidate for limit \underline{a}
 - 2) Show that $\underline{a} \in \ell^2$.
 - 3) $\|\underline{x}_n - \underline{a}\| \rightarrow 0$ as $n \rightarrow \infty$

Step 1: $\underline{x}_1 = (x_1^1, x_2^1, \dots)$

$$\underline{x}_2 = (x_1^2, x_2^2, \dots)$$

$$\underline{x}_n = (x_1^n, x_2^n, \dots)$$

For each k , consider sequence $\{x_k^n\}$ (k th coordinates in each \underline{x}_n)

Each sequence is Cauchy, as

$$|x_k^n - x_k^m| < \left(\sum_{i=1}^{\infty} |x_i^n - x_i^m|^2 \right)^{1/2} = \|\underline{x}_n - \underline{x}_m\|_{\ell^2} < \epsilon, \quad \forall n, m \geq N.$$

As $(\mathbb{C}, \|\cdot\|)$ complete, $\{x_k^n\}_n$ has a limit $a_k \in \mathbb{C}$.

Candidate for limit of \underline{x}_n is

$$\underline{a} = (a_1, a_2, \dots, a_n)$$

Step 2: Write $\underline{a} = \underline{x}_n - (\underline{x}_n - \underline{a})$

In order to show that $\underline{a} \in \ell^2$, it is enough to see that $\underline{x}_n - \underline{a} \in \ell^2$ for some n .

$$\underline{x}_n \text{ Cauchy} \Rightarrow \forall \epsilon > 0 \exists N: \forall n, m \geq N \quad \|\underline{x}_n - \underline{x}_m\|_{\ell^2} < \epsilon$$

Consider for some $M > 0$

$$\sum_{i=1}^M |x_i^n - x_i^m|^2 \leq \sum_{i=1}^{\infty} |x_i^n - x_i^m|^2 = \|\underline{x}_n - \underline{x}_m\|_{\ell^2}^2 < \epsilon^2$$

Let $m \rightarrow \infty$.

We get

$$\sum_{i=1}^M |x_i^n - a_i|^2 \leq \epsilon^2$$

This holds for any $M \in \mathbb{N}$, hence

$$\sum_{i=1}^{\infty} |x_i^n - a_i|^2 \leq \epsilon^2 \Rightarrow \underline{x}_n - \underline{a} \in \ell^2$$

and moreover $\|\underline{x}_n - \underline{a}\| \xrightarrow{n \rightarrow \infty} 0$ □

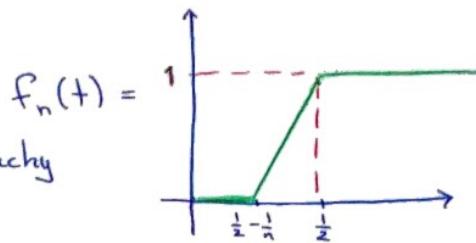
Another example: $(C([a,b]), \|\cdot\|_\infty)$ is a Banach space.

Remark: $E = C([a,b])$, $\|f\|_{L^2} = \left(\int_a^b |f(t)|^2 dt \right)^{1/2}$ ← norm on space of continuous functions.

One can prove that

$(C([a,b]), \|\cdot\|_{L^2})$ is not a Banach space.

Exercise: $[a,b] = [0,1]$



Show that f_n is Cauchy in $(C([0,1]), \|\cdot\|_{L^2})$ but

$f_n \not\rightarrow f \in C([0,1])$

Convergent and absolutely convergent series

Definition: The series $\sum_{n=1}^{\infty} x_n$ in E is called **convergent** if $\left\{ \sum_{n=1}^m x_n \right\}_{m=1}^{\infty}$ ← the sequence of partial sums is convergent in E .

If $\sum_{n=1}^{\infty} \|x_n\| < \infty$, then we say that $\sum_{n=1}^{\infty} x_n$ **converges absolutely**.

Theorem: A normed space E is complete iff every absolutely convergent series converges in E .

Proof: \Rightarrow Suppose E is complete and $\sum_{n=1}^{\infty} \|x_n\| < \infty$.

Let $S_N = \sum_{n=1}^N x_n \in E$.

$$\|S_N - S_M\| = \left\| \sum_{n=M+1}^N x_n \right\| \leq \sum_{n=M+1}^N \|x_n\| \leq \sum_{n=N+1}^{\infty} \|x_n\| \xrightarrow{N \rightarrow \infty} 0$$

Hence S_N is Cauchy. As E is complete, S_N has a limit in E , i.e.

$\sum_{n=1}^{\infty} x_n$ converges in E .

Assume that every absolutely convergent series is convergent in E. Must see that E is complete.
 Let $\{x_n\}$ be Cauchy. Must prove $\{x_n\}$ has limit in E.



$$\forall k \exists n_k : \|x_n - x_{n_k}\| < \frac{1}{2^k} \quad \forall n, m \geq n_k$$

Can assume $\{n_k\}$ is increasing sequence.
 Write

$$\begin{aligned} x_{n_k} &= (x_{n_k} - x_{n_{k-1}}) + (x_{n_{k-1}} - x_{n_{k-2}}) + \dots + (x_{n_1} - x_{n_0}) = \\ &= \sum_{l=1}^{\infty} (x_{n_l} - x_{n_{l-1}}) \\ \sum_{l=1}^{\infty} \|x_{n_l} - x_{n_{l-1}}\| &\leq \sum_{l=1}^{\infty} \frac{1}{2^l} < \infty \quad \Rightarrow \sum_{l=1}^{\infty} (x_{n_l} - x_{n_{l-1}}) \text{ absolutely convergent} \end{aligned}$$

By assumption

$$\sum_{l=1}^{\infty} (x_{n_l} - x_{n_{l-1}}) \text{ is absolutely convergent in } E.$$

~~and thus the sequence $\{x_{n_k}\}$ is Cauchy~~
 and thus the sequence $\{x_{n_k}\}$ converges to an element $x \in E$. Consequently

$$\|x_n - x\| \leq \|x_n - x_{n_k}\| + \|x_{n_k} - x\| \rightarrow 0$$

because x_n is Cauchy. □

Föreläsning 4

Mappings between normed spaces

- $(E, \|\cdot\|), (E_2, \|\cdot\|)$ normed spaces
- $T: E_1 \rightarrow E_2$ (not necessarily linear) is called **continuous** at $x_0 \in E_1$ if
 $x_n \rightarrow x_0$ in $(E_1, \|\cdot\|) \Rightarrow T(x_n) \rightarrow T(x_0)$ in $(E_2, \|\cdot\|)$.
It is called **continuous** if it is continuous at $x_0 \in E_1$,
 $\forall x_0 \in E_1$.
- We say $T: E_1 \rightarrow E_2$ is **linear** if
 $T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2), \forall \lambda_1, \lambda_2 \in \mathbb{F}, \forall x_1, x_2 \in E_1$.
- $T: E_1 \rightarrow E_2$ linear is called **bounded** if
 $\exists M > 0: \|T(x)\|_2 \leq M \|x\|_1, \forall x \in E_1$.
- If T is bounded linear $E_1 \rightarrow E_2$, define
 $\|T\| = \|T\|_{E_1 \rightarrow E_2} = \inf \{M > 0 : \|T(x)\|_2 \leq M \|x\|_1, \forall x \in E_1\}$

Lemma:

$$\|T\| = \sup_{\substack{x \in E_1 \\ x \neq 0}} \frac{\|T(x)\|_2}{\|x\|_1} = \sup_{\substack{x \in E_1 \\ \|x\|_1=1}} \|T(x)\|_2.$$

Proposition: Assume $T: E_1 \rightarrow E_2$ linear

Then these ~~are~~ are equivalent:

- 1) T continuous at $0 \in E_1$.
- 2) T continuous at $x_0 \in E_1$ for some $x_0 \in E_1$.
- 3) T continuous at $x_0 \in E_1, \forall x_0$.
- 4) T is bounded

Proof: (1 \Leftrightarrow 4)

Assume T is continuous at $0 \in E_1$, i.e.
 $x_n \rightarrow 0$ in $(E_1, \|\cdot\|) \Rightarrow T(x_n) \rightarrow T(0) \underset{\in E_2}{=} 0 \in E_2$

by linearity

Proof T is bounded, i.e.

$$\exists M > 0 : \|T(x)\|_2 \leq M \|x\|_1, \quad \forall x \in E_1. \quad (x)$$

Assume (x) does not hold. For $n=1,2,\dots$

$$\exists x_n \in E_1 : \|T(x_n)\|_2 > n \|x_n\|_1.$$

Set

$$z_n = \frac{1}{n \|x_n\|_1} x_n, \quad n=1,2,\dots$$

(note that $\|x_n\|_1 > 0$, all n).

Note:

$$\|z_n\|_1 = \left\| \frac{1}{n \|x_n\|_1} x_n \right\|_1 = \frac{1}{n \|x_n\|_1} \|x_n\|_1 = \frac{1}{n} \xrightarrow{n \rightarrow \infty} 0$$

But

$$\|T(z_n)\|_2 = \left\| \frac{1}{n \|x_n\|_1} T(x_n) \right\|_2 = \frac{1}{n \|x_n\|_1} \|T(x_n)\|_2 > 1, \quad \forall n.$$

Hence

$T(z_n) \not\rightarrow 0$ in $(E_2, \|\cdot\|_2)$. Contradiction.



Assume T bounded. For some $M > 0$

$$\|T(x)\|_2 \leq M \|x\|_1, \quad \forall x \in E_1.$$

To show: T continuous at $0 \in E_1$, i.e.

$$x_n \rightarrow 0 \text{ in } (E_1, \|\cdot\|_1) \Rightarrow T(x_n) \rightarrow T(0) = 0 \text{ in } (E_2, \|\cdot\|_2).$$

From $\|T(x_n)\|_2 \leq M \|x_n\|_1 \rightarrow 0$,

so $T(x_n) \rightarrow 0$ in $(E_2, \|\cdot\|_2)$.



Example: $E_1 = E_2 = C([0,1]) \ni f$

• $\|\cdot\|_1 = \|\cdot\|_2$ max-norms

• $\|f\| = \max_{x \in [0,1]} |f(x)|$

$$T(f)(x) = \int_0^x \min(x,y) f(y) dy, \quad f \in C([0,1]), \quad x \in [0,1]$$

Show: 1) $T(f) \in C([0,1])$ for $f \in C([0,1])$

2) T linear

3) T bounded

4) Calculate $\|T\|$

1) Fix $f \in C([0,1])$ arbitrary. Fix $x \in [0,1]$.

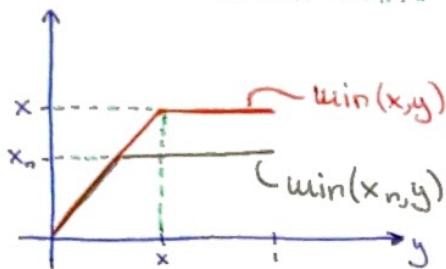
Show $T(f)$ is continuous at x .

Consider a sequence $(x_n)_{n=1}^{\infty}$ in $[0,1]$ such that

$$x_n \rightarrow x \text{ in } (\mathbb{R}, \|\cdot\|)$$

To show: $T(f)(x_n) \rightarrow T(f)(x)$ in $(\mathbb{R}, \|\cdot\|)$.

$$\begin{aligned} |T(f)(x_n) - T(f)(x)| &= \left\{ \text{assume } x_n \leq x \right\} = \left| \int_0^x \min(x_n, y) f(y) dy - \int_0^x \min(x, y) f(y) dy \right| \\ &\leq \left| \int_0^x (\min(x_n, y) - \min(x, y)) f(y) dy \right| + \left| \int_0^x \min(x, y) f(y) dy \right| \leq \\ &\leq \underbrace{\int_0^x |\min(x_n, y) - \min(x, y)| |f(y)| dy}_{\leq |x_n - x|} + \underbrace{\int_0^x \min(x, y) |f(y)| dy}_{\leq 1} \leq \|f\| \\ &\leq |x_n - x| \|f\| \end{aligned}$$



If $x_n > x$ we get a similar calculation.

Conclusion:

$$T(f)(x_n) \rightarrow T(f)(x) \text{ in } (\mathbb{R}, \|\cdot\|)$$

2) T linear

Fix $f_1, f_2 \in C([0,1])$ and λ_1, λ_2 scalars.

$$\begin{aligned} T(\lambda_1 f_1 + \lambda_2 f_2)(x) &= \int_0^x \min(x, y) (\lambda_1 f_1 + \lambda_2 f_2)(y) dy = \\ &= \lambda_1 \int_0^x \min(x, y) f_1(y) dy + \lambda_2 \int_0^x \min(x, y) f_2(y) dy = \lambda_1 T(f_1)(x) + \lambda_2 T(f_2)(x), \\ &\quad x \in [0,1]. \end{aligned}$$

3) T bounded

Fix $f \in C([0,1])$. For $x \in [0,1]$

$$|T(f)(x)| = \left| \int_0^x \min(x, y) f(y) dy \right| \stackrel{\geq 0}{\leq} \int_0^x \min(x, y) |f(y)| dy \stackrel{\leq \|f\|}{\leq} \int_0^x \min(x, y) dy \leq \|f\|$$

Clearly, $\max_{x \in [0,1]} \int_0^x \min(x, y) dy \leq 1$. This gives

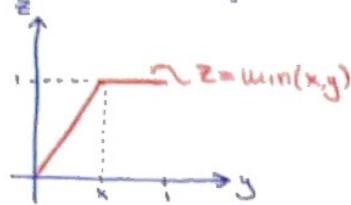
$$\|T(f)\| = \max_{x \in [0,1]} |T(f)(x)| \leq 1 \cdot \|f\|, \quad \forall f \in C([0,1])$$

Equality if f is constant function

Equality if f has constant sign

4) Calculate $\|T\|$

Calculate $\int_0^x \min(x,y) dy$, $x \in [0,1]$



Case 1: $1-x \leq x$

$$\int_0^x \min(x,y) dy = \left[\frac{1}{2} y^2 \right]_0^x = \frac{1}{2} (1-x)^2$$

Case 2: $x < 1-x$, i.e. $x < \frac{1}{2}$.

$$\int_0^x \min(x,y) dy = \int_0^x y dy + \int_x^{1-x} x dy = \frac{1}{2} x^2 + x(1-2x) = x - \frac{3}{2} x^2$$

$$\text{Claim: } \|T\| = \max \left(\max_{x \in [0,1]} \frac{1}{2} (1-x)^2, \max_{x \in [0,1]} \left(x - \frac{3}{2} x^2 \right) \right) = \frac{1}{6}.$$

Note: $\|T(f)\| \leq \|T\| \|f\|$

$$\|T(1)\| = \|T\| \cdot \|1\| \quad \text{where } 1(x) = x, x \in [0,1].$$

Example: $E_1 = C([0,1])$ with max-norm.

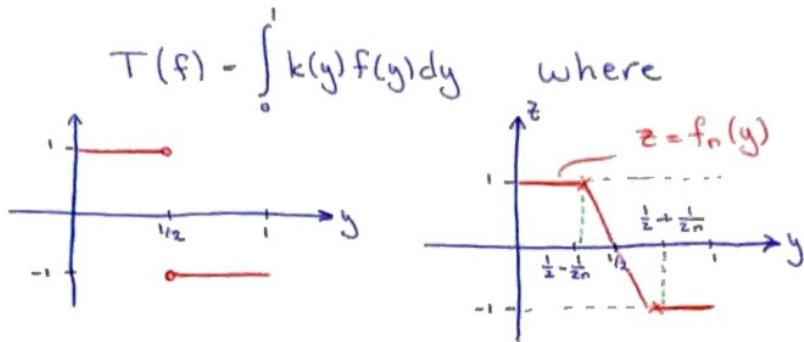
$E_2 = \mathbb{R}$ with absolute value norm.

$T: E_1 \rightarrow E_2$

$$T(f) = \int_0^1 f(y) dy - \int_{1/2}^1 f(y) dy, \quad f \in E_1$$

$$\begin{aligned} |T(f)| &= \left| \int_0^1 f(y) dy - \int_{1/2}^1 f(y) dy \right| \leq \left| \int_0^{1/2} f(y) dy \right| + \left| \int_{1/2}^1 f(y) dy \right| \leq \underbrace{\int_0^{1/2} |f(y)| dy}_{\leq \|f\|} + \underbrace{\int_{1/2}^1 |f(y)| dy}_{\leq \|f\|} \leq 1 \cdot \|f\| \end{aligned}$$

Hence T is bounded and $\|T\| \leq 1$



$$T(f_n) \geq 1 \cdot \left(\frac{1}{2} - \frac{1}{2n} + \frac{1}{2} - \frac{1}{2n} \right) = 1 - \frac{1}{n}, \quad n = 1, 2, \dots$$

note $k(y)f_n(y) \geq 0$ for $y \in [0, 1]$

Hence $\|T\| \geq 1 - \frac{1}{n}$ for $n = 1, 2, \dots$

Conclusion: $\|T\| = 1$

Here $|T(f)| \leq \underbrace{\|T\|}_{=1} \|f\|, \forall f \in C([0, 1])$

T_1, T_2 bounded linear mappings $(E_1, \|\cdot\|_1) \rightarrow (E_2, \|\cdot\|_2)$, λ scalar.

Set $\begin{aligned} \circ (T_1 + T_2)(x) &= T_1(x) + T_2(x), \quad x \in E_1, \\ \circ (\lambda T_1)(x) &= \lambda T_1(x), \quad x \in E_1. \end{aligned}$

Claim: 1) $T_1 + T_2, \lambda T_1$ are both linear mappings $(E_1, \|\cdot\|_1) \rightarrow (E_2, \|\cdot\|_2)$
 2) $T_1 + T_2, \lambda T_1$ are bounded linear mappings $(E_1, \|\cdot\|_1) \rightarrow (E_2, \|\cdot\|_2)$.

$\mathcal{B}(E_1, E_2)$ denotes the vector space of all bounded linear mappings $(E_1, \|\cdot\|_1) \rightarrow (E_2, \|\cdot\|_2)$.

3) $\|T\| = \inf \{M > 0 : \|T(x)\|_2 \leq M \|x\|_1, \forall x \in E_1\}$ defines a norm on $\mathcal{B}(E_1, E_2)$.

Proof: ① $\|T\| = 0 \Rightarrow \|T(x)\|_2 = 0, \forall x \in E_1 \Rightarrow T(x) = 0 \in E_2$.

$\Rightarrow T = 0 \in \mathcal{B}(E_1, E_2)$

② $T \in \mathcal{B}(E_1, E_2)$, λ scalar.

$$\|\lambda T\| = \inf \{M > 0 : \|\underbrace{(\lambda T)(x)}_{=\lambda T(x)}\|_2 \leq M \|x\|_1, \forall x \in E_1\} =$$

$$= \inf \{M > 0 : |\lambda| \|T(x)\|_2 \leq M \|x\|_1, \forall x \in E_1\} =$$

$$\begin{aligned}
&= \left\{ \text{if } \lambda \neq 0 \right\} = \inf \left\{ M > 0 : \|T(x)\|_2 \leq \frac{\boxed{M}}{\boxed{|\lambda|}} \|x\|_1, \forall x \in E_1 \right\} = \\
&= |\lambda| \inf \left\{ \tilde{M} > 0 : \|T(x)\|_2 \leq \tilde{M} \|x\|_1, \forall x \in E_1 \right\}.
\end{aligned}$$

③ $T_1, T_2 \in \mathcal{B}(E_1, E_2)$

$$\begin{aligned}
\|T_1 + T_2\| &= \inf \left\{ M > 0 : \left\| \underbrace{(T_1 + T_2)(x)}_{= T_1(x) + T_2(x)} \right\|_2 \leq M \|x\|_1, \forall x \in E_1 \right\} \leq \\
&\leq \inf \left\{ M_1 + M_2 > 0 : \|T_1(x)\|_2 \leq M_1 \|x\|_1, \|T_2(x)\|_2 \leq M_2 \|x\|_1, \forall x \in E_1 \right\} = \\
&= \|T_1\| + \|T_2\|
\end{aligned}$$

Conclusion: $(\mathcal{B}(E_1, E_2), \|\cdot\|_{E_1 \rightarrow E_2})$ is a normed space.

Is this a Banach space?

- Yes, if $(E_2, \|\cdot\|_2)$ is a Banach space.

Proof: Assume $(T_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{B}(E_1, E_2)$

where $(E_2, \|\cdot\|_2)$ is a Banach space. Fix $x \in E_1$.

$$\|T_n(x) - T_m(x)\|_2 = \|(T_n - T_m)(x)\|_2 \leq \underbrace{\|T_n - T_m\|}_{\substack{n,m \rightarrow \infty}}_{E_1 \rightarrow E_2} \cdot \|x\|_1 \xrightarrow{n,m \rightarrow \infty} 0$$

$(T_n(x))_{n=1}^{\infty}$ is a Cauchy sequence in $(E_2, \|\cdot\|_2)$, which is a Banach space, and hence it converges in $(E_2, \|\cdot\|_2)$.

Call the limit $T(x) \in E_2, \forall x \in E_1$. Need to show that

- 1) $T: E_1 \rightarrow E_2$ is linear
- 2) $T: E_1 \rightarrow E_2$ is bounded
- 3) $\|T_n - T\|_{E_1 \rightarrow E_2} \xrightarrow{n \rightarrow \infty} 0$

1) Observe

$$\begin{aligned}
T_n(\lambda_1 x_1 + \lambda_2 x_2) &= \{T_n \text{ linear}\} = \underbrace{\lambda_1 T_n(x_1)}_{\substack{\rightarrow T(x_1) \\ \rightarrow \lambda_1 T(x_1)}} + \underbrace{\lambda_2 T_n(x_2)}_{\substack{\rightarrow T(x_2) \\ \rightarrow \lambda_2 T(x_2)}} \\
&\quad \downarrow \text{in } (E_2, \|\cdot\|_2) \\
T(\lambda_1 x_1 + \lambda_2 x_2) &\quad \rightarrow \underbrace{\lambda_1 T(x_1) + \lambda_2 T(x_2)}
\end{aligned}$$

2) T bounded

Fix $\epsilon > 0$. Then

$$\exists N : \|T_n - T_m\|_{E_1 \rightarrow E_2} < \epsilon, \text{ for } n, m \geq N.$$

So for $x \in E_1$,

$$\|T_n(x) - T_m(x)\|_2 \leq \|T_n - T_m\|_{E_1 \rightarrow E_2} \|x\|_1 < \epsilon \|x\|_1, \quad m, n \geq N$$

(xx)

Let $m \rightarrow \infty$

$$\|T_n(x) - T(x)\|_2 \leq \epsilon \|x\|_1, \quad n \geq N$$

So

$$\begin{aligned} \|T(x)\|_2 &\leq \|T(x) - T_N(x)\|_2 + \|T_N(x)\|_2 \leq \epsilon \|x\|_1 + \|T_N\|_{E_1 \rightarrow E_2} \|x\|_1 = \\ &= (\epsilon + \|T_N\|_{E_1 \rightarrow E_2}) \|x\|_1, \quad \forall x \in E_1. \end{aligned}$$

3) $\|T_n - T\|_{E_1 \rightarrow E_2} \xrightarrow{n \rightarrow \infty} 0 \quad (\text{By (xx)})$

Banach-Steinhaus theorem (uniform boundedness principle)

- $(E_1, \|\cdot\|_1)$ is a Banach space.
- $(E_2, \|\cdot\|_2)$ normed space.
- $\mathcal{F} \subseteq \mathcal{B}(E_1, E_2)$

"This is a good theorem!"
- Peter

Assume $\sup_{T \in \mathcal{F}} \|T(x)\|_2 < \infty, \quad \forall x \in E_1.$

$$\Rightarrow \sup_{T \in \mathcal{F}} \|T\|_{E_1 \rightarrow E_2} < \infty$$

Remark: The implication \Leftarrow is easy to prove.

Remark: If \mathcal{F} is a finite set, the theorem is trivial.

Proof: The proof is done in two steps.

Step 1: We prove the theorem under the assumption

$$(x) \exists x_0 \in E_1, \exists r > 0, \exists M > 0, \forall x \in \bar{B}(x_0, r), \forall T \in \mathcal{T}: \|T(x)\|_2 \leq M.$$

Fix an arbitrary $T \in \mathcal{T}$. For $x \in E_1$ with $\|x\|_1 \leq r$ we have

$$\begin{aligned} \|T(x)\|_2 &= \|T(x_0 + x - x_0)\|_2 = \|T(x_0 + x) - T(x_0)\|_2 \leq \\ &\leq \|T(x_0 + x)\|_2 + \|T(x_0)\|_2 \leq 2M, \end{aligned}$$

where we used the linearity of T . Hence, for $x \neq 0$ it follows that

$$2M \geq \|T\left(\frac{r}{\|x\|_1} x\right)\|_2 = \frac{r}{\|x\|_1} \|T(x)\|_2$$

where the linearity of T is used. Thus

$$\|T(x)\|_2 \leq \frac{2M}{r} \|x\|_1, \forall x \in E_1.$$

We conclude that

$$\sup_{T \in \mathcal{T}} \|T\| < \infty.$$

Step 2: Remains to prove that the assumption (x) is true. We argue by contradiction. So assume that the negation of the statement: (x) is true. Hence

$$(xx) \forall x_0 \in E_1, \forall r > 0, \forall M > 0, \exists x \in \bar{B}(x_0, r), \exists T \in \mathcal{T} : \|T(x)\|_2 > M.$$

This statement is equivalent to

$$(xxx) \forall x_0 \in E_1, \forall r > 0, \forall M > 0, \exists x \in \bar{B}(x_0, r), \exists T \in \mathcal{T} : \|T(x)\|_2 > M$$

Since

$$B(x_0, r_1) \subset B(x_0, r_2) \subset \bar{B}(x_0, r_2) \subset B(x_0, r_3), \forall r_1 < r_2 < r_3$$

The idea is to find a Cauchy sequence $(x_n)_{n=1}^{\infty}$ in E_1 (which converges since E_1 is a Banach space) and a sequence $(T_n)_{n=1}^{\infty}$ in \mathcal{T} such that $T_n(x_n) > n$ for $n = 1, 2, 3, \dots$ and also $T_n(x) > n$ for the limit x . This yields a contradiction to the hypothesis of the theorem.

From (xxx) it follows that there exists a $x_1 \in B(0,1)$ and $T_1 \in \mathcal{F}$ such that $\|T_1(x_1)\|_2 > 1$. Since T_1 is bounded linear and hence continuous there exists $0 < r_1 < \frac{1}{2}$ such that $\|T_1(x)\|_2 > 1$ for all $x \in B(x_1, r_1)$ and $\overline{B}(x_1, r_1) \subset B(x_1, r_1)$.

In the same way it follows from (xxx) that there exists a $x_2 \in B(x_1, r_1)$ and T_2 such that $\|T_2(x)\|_2 > 2$. Moreover, since T_2 is bounded linear it follows that there exists $0 < r_2 < (\frac{1}{2})^2$ such that $\|T_2(x)\|_2 > 2$ for all ~~$x \in B(x_2, r_2)$~~ $x \in B(x_2, r_2)$ and $\overline{B}(x_2, r_2) \subset B(x_1, r_1)$. Proceed inductively. We obtain a sequence $(x_n)_{n=1}^{\infty}$ in E_1 and a sequence $(T_n)_{n=1}^{\infty}$ in \mathcal{F} and $(r_n)_{n=1}^{\infty}$ in $(0, \infty)$ such that for $n=1, 2, \dots$

- $\|T_n x\|_2 > n$ for all $x \in B(x_n, r_n)$
- $\overline{B}(x_n, r_n) \subset B(x_{n-1}, r_{n-1})$
- $0 < r_n < (\frac{1}{2})^n$

We conclude that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in E_1 since for $n > m$

$$\|x_n - x_m\|_1 < r_m < \left(\frac{1}{2}\right)^m \xrightarrow{n, m \rightarrow \infty} 0$$

Since $(E_1, \|\cdot\|_1)$ is a Banach space the sequence $(x_n)_{n=1}^{\infty}$ converges. Call the limit x . Here $x \in \overline{B}(x_n, r_n) \subset B(x_{n-1}, r_{n-1})$ for all n and hence

$$\|T_{n-1}(x)\|_2 > n-1, \quad n = 2, 3, \dots$$

and so

$$\sup_{T \in \mathcal{F}} \|T(x)\|_2 \geq \sup_{n=1, 2, \dots} \|T_n(x)\|_2 = \infty$$

This is a contradiction to the hypothesis and the assumption (xxx) is false.



The Banach-Steinhaus theorem will be used several times in the course. Here we just give a corollary to the theorem.

Corollary:

Let E_1 be a Banach space and E_2 a normed space. Moreover let $(T_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{B}(E_1, E_2)$ such that $T(x) = \lim_{n \rightarrow \infty} T_n(x)$ exists for all $x \in E_1$.

Then

$$T \in \mathcal{B}(E_1, E_2).$$

To sketch the proof, note that T inherits the property of being linear from that for all T_n . Since ~~all~~ $(T_n(x))_{n=1}^{\infty}$ converges in E_2 for all $x \in E_1$, it is a bounded sequence for all x and hence by the Banach-Steinhaus theorem we have that $\sup_n \|T_n\| < \infty$. This implies that T is a bounded linear mapping since for $x \in E_1$,

$$\|T(x)\| = \left\| \lim_{n \rightarrow \infty} T_n(x) \right\| = \lim_{n \rightarrow \infty} \|T_n(x)\| \leq \sup_n \|T_n\| \|x\|.$$

Föreläsning 5

Fixed point theory

Example: Consider

$$f(x) + 5 \int_0^x \min(x,y) f(y) dy = g(x), \quad x \in [0,1] \quad (\star)$$

where $g \in C([0,1])$.

Claim: There exists a unique solution $f \in C([0,1])$ to (\star) .

Idea:

$$f(x) = g(x) - 5 \int_0^{x-x} \min(x,y) f(y) dy.$$

Set $\tilde{T}(f)(x) = \text{RHS } (\star)$, $x \in [0,1]$. To find a solution to (\star) is the same as finding $f \in C([0,1])$ such that

$$f = \tilde{T}(f).$$

Clearly $\tilde{T}: C([0,1]) \rightarrow C([0,1])$.

(Continued later)

Banach's fixed point theorem:

- $(E, \|\cdot\|)$ Banach space
- $T: E \rightarrow E$ (no assumption on linearity) is a contraction on E , i.e. there exists $C < 1$ such that

$$\|T(x) - T(\tilde{x})\| \leq C \|x - \tilde{x}\|, \quad \forall x, \tilde{x} \in E.$$

Then there exists a unique $\bar{x} \in E$: $\bar{x} = T(\bar{x})$. (\bar{x} is a fixed point).

Proof:

Uniqueness:

Assume $T(\bar{x}) = \bar{x}$ and $T(\tilde{x}) = \tilde{x}$. Then

$$\|\bar{x} - \tilde{x}\| = \|T(\bar{x}) - T(\tilde{x})\| \leq C \|\bar{x} - \tilde{x}\|.$$

Thus $\|\bar{x} - \tilde{x}\| = 0$, i.e. $\bar{x} = \tilde{x}$

Existence:

Pick an arbitrary element $x_0 \in E$.

Set $x_{n+1} = T(x_n)$, $n = 0, 1, 2, \dots$

Claim: $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence in $(E, \|\cdot\|)$

Note:

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|T(x_n) - T(x_{n-1})\| \leq C \|x_n - x_{n-1}\| \leq \dots \leq \\ &\leq C^n \|x_1 - x_0\|, \quad n = 1, 2, \dots\end{aligned}$$

For $n > m$

$$\begin{aligned}\|x_n - x_m\| &= \|x_n - x_{n-1} + x_{n-1} - \dots + x_{m+1} - x_m\| \leq \\ &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \dots + \|x_{m+1} - x_m\| \leq \\ &\leq (C^{n-1} + C^{n-2} + \dots + C^m) \|x_1 - x_0\| \leq \frac{C^m}{1-C} \|x_1 - x_0\| \xrightarrow{n,m \rightarrow \infty} 0.\end{aligned}$$

Hence $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence in $(E, \|\cdot\|)$. $(E, \|\cdot\|)$ is a Banach space, so $(x_n)_{n=0}^{\infty}$ converges in $(E, \|\cdot\|)$. Call the limit \bar{x} .

Claim: \bar{x} is a fixed point for T .

$$\begin{aligned}\|\bar{x} - T(\bar{x})\| &= \|\bar{x} - x_{n+1} + x_{n+1} - T(\bar{x})\| \leq \|\bar{x} - x_{n+1}\| + \underbrace{\|x_{n+1} - T(\bar{x})\|}_{T(x_n)} \leq \\ &\leq \underbrace{\|\bar{x} - x_{n+1}\|}_{\rightarrow 0} + C \underbrace{\|x_n - \bar{x}\|}_{\rightarrow 0} \xrightarrow{n \rightarrow \infty} 0. \quad \square\end{aligned}$$

Remark: ① $x_n \rightarrow \bar{x}$, $n \rightarrow \infty$ independent of the choice of x_0 .

② Fix $z \in E$

$$\begin{aligned}\|\bar{x} - z\| &= \|T(\bar{x}) - T(z) + T(z) - z\| \leq \|T(\bar{x}) - T(z)\| + \|T(z) - z\| \leq \\ &\leq C \|\bar{x} - z\| + \|T(z) - z\|\end{aligned}$$

Hence $\|\bar{x} - z\| \leq \frac{1}{1-C} \|T(z) - z\|$

Continuation of example:

$(C([0,1]), \|\cdot\|)$ with $\|f\| = \max_{x \in [0,1]} |f(x)|$ is a Banach space!

To apply Banach's fixed point theorem, we need \tilde{T} to be a contraction!

Fix $f_1, f_2 \in C([0,1])$, for $x \in [0,1]$.

$$\begin{aligned} |(\tilde{T}(f_1) - \tilde{T}(f_2))(x)| &= \left| 5 \int_0^x \min(x,y) f_2(y) dy - 5 \int_0^x \min(x,y) f_1(y) dy \right| = \\ &= \left| 5 \int_0^x \min(x,y) (f_2(y) - f_1(y)) dy \right| \leq 5 \int_0^x \min(x,y) \underbrace{|f_2(y) - f_1(y)|}_{\leq \|f_2 - f_1\|} dy \leq \\ &\leq 5 \int_0^x \min(x,y) dy \|f_2 - f_1\| \leq \frac{5}{6} \|f_2 - f_1\| \end{aligned}$$

Hence $\|\tilde{T}(f_1) - \tilde{T}(f_2)\| \leq \frac{5}{6} \|f_2 - f_1\|$.

We conclude that \tilde{T} is a contraction with $c = \frac{5}{6}$ (for example). By Banach's fixed point theorem \tilde{T} has a unique fixed point. Finally (x) has a unique solution $f \in C([0,1])$

Banach's fixed point theorem (generalization) ↑
fixed point

- $(E, \|\cdot\|)$ Banach space
- $T: F \rightarrow F$ where F is a closed set in E .
- N positive integer.

Assume $T^N = \underbrace{T \circ T \circ \dots \circ T}_{N \text{ times}}$ is a contraction on F , i.e.

$$\exists c < 1 : \|T^N(x) - T^N(\tilde{x})\| \leq c \|x - \tilde{x}\|, \forall x, \tilde{x} \in F$$

Then T has a unique fixed point \bar{x} , i.e. $\bar{x} = T(\bar{x}) \in F$.

Proof:

[N=1] Fix $x_0 \in F$ and consider $(x_n)_{n=1}^\infty$ where $x_{n+1} = T(x_n)$, $n = 0, 1, 2, \dots$

Here $(x_n)_{n=1}^\infty$ is a Cauchy sequence and hence converges in E since this is a Banach space.

Call the limit \bar{x} .

Note: $F \ni x_n \rightarrow \bar{x}$ in E and F is closed

$\Rightarrow \bar{x} \in F$. The rest of the argument is the same as before.

$\boxed{N > 1}$: By previous result we know that T^N has a unique fixed point $\bar{x} \in \mathbb{F}$, i.e. $\bar{x} = T^N(\bar{x})$

Claim: \bar{x} is a fixed point for T

$$\|T(\bar{x}) - \bar{x}\| = \|T(T^N(\bar{x})) - T^N(\bar{x})\| = \|T^N(T(\bar{x})) - T^N(\bar{x})\| \leq \|T(\bar{x}) - \bar{x}\| \cdot c$$

This gives $\|T(\bar{x}) - \bar{x}\| = 0$, i.e. $\bar{x} = T(\bar{x})$.

Existence of a fixed point for T , done.

Uniqueness: Assume $\bar{x} = T(\bar{x})$ and $\tilde{x} = T(\tilde{x})$

Then

$$\bar{x} = T(\bar{x}) = T^2(\bar{x}) = \dots = T^N(\bar{x})$$

$$\tilde{x} = T(\tilde{x}) = T^2(\tilde{x}) = \dots = T^N(\tilde{x})$$

But T^N has a unique fixed point.

$$\Rightarrow \bar{x} = \tilde{x} \quad \square$$

Remark:

① $T: (0, 1] \rightarrow (0, 1]$ where $T(x) = \frac{x}{2}$. Clearly T is a contraction on $(0, 1]$ but has no fixed point.

Note that $(0, 1]$ is not a closed interval.

② $T: [0, \infty) \rightarrow [0, \infty)$ where $T(x) = x + \frac{1}{x}$. Clearly $[0, \infty)$ is a closed set in \mathbb{R} but T has no fixed point.

Claim: T is not a contraction, but close to being a contraction.

$$|T(x) - T(\tilde{x})| < |x - \tilde{x}| \text{ for } x, \tilde{x} \in [1, \infty), x \neq \tilde{x}$$

Note: $|T(x) - T(\tilde{x})| = \underbrace{|T'(t)|}_{(1-t)^{-2}} |x - \tilde{x}|$ for some t between x and \tilde{x} .
 $(1-t)^{-2} < 1$ for $t \in [1, \infty)$

Example: $(E, \|\cdot\|)$ Banach space

K compact set in E

$T: K \rightarrow K$ where $\|T(x) - T(\tilde{x})\| < \|x - \tilde{x}\|$, $\forall x, \tilde{x} \in K$, $x \neq \tilde{x}$

Show: T has a unique fixed point in K .

Uniqueness:

Assume $\bar{x} = T(\bar{x})$, $\tilde{x} = T(\tilde{x})$ and $\bar{x} \neq \tilde{x}$, $\bar{x}, \tilde{x} \in K$.
Then

$$\|\bar{x} - \tilde{x}\| = \|T(\bar{x}) - T(\tilde{x})\| < \|x - \tilde{x}\|. \text{ Contradiction.}$$

We have $\bar{x} = \tilde{x}$

Existence:

To show: ~~there exists~~ there exists $x \in K$ such that $x = T(x)$, i.e.

$$\|T(x) - x\| = 0.$$

Set $d = \inf_{x \in K} \|T(x) - x\|$

Let $(x_n)_{n=1}^{\infty}$ be a sequence in K such that

$$\|T(x_n) - x_n\| \rightarrow d \text{ as } n \rightarrow \infty$$

K compact implies that there exists a subsequence $(\tilde{x}_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $(\tilde{x}_n)_{n=1}^{\infty}$ converges in K .

Call the limit \bar{x} , $\bar{x} \in K$.

We know $\tilde{x}_n \rightarrow \bar{x}$ in K

$$\|T(\tilde{x}_n) - \tilde{x}_n\| \rightarrow d$$

Question: $T(\tilde{x}_n) \rightarrow T(\bar{x})$ in K ?

Since $\|T(x) - T(\tilde{x})\| \leq \|x - \tilde{x}\|$ for all $x, \tilde{x} \in K$ we have

$\tilde{x}_n \rightarrow \bar{x}$ in K implies $T(\tilde{x}_n) \rightarrow T(\bar{x})$ in K .

Hence ~~we have~~ $\|T(\bar{x}) - \bar{x}\| \leftarrow \|T(\tilde{x}_n) - \tilde{x}_n\| \rightarrow d$, $n \rightarrow \infty$

Question: Is $d = 0$?

If $d > 0$, then $\bar{x} \neq T(\bar{x})$, $\bar{x}, T(\bar{x}) \in K$

$$\|T(\bar{x}) - T(T(\bar{x}))\| < \|\bar{x} - T(\bar{x})\| = d = \inf_{x \in K} \|x - T(x)\|$$

Contradiction.

This gives $d = 0$ and $\bar{x} = T(\bar{x})$.

Example:

Consider

$$f(x) = \int_0^x k(x,y) h(y, f(y)) dy + g(x), \quad x \in [0,1] \quad (\star)$$

where $g \in C([0,1])$, $k \in C([0,1] \times [0,1])$ and
 $h: [0,1] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies
 \uparrow continuous $\exists M > 0: |h(x, z_1) - h(x, z_2)| \leq M|z_1 - z_2|, \forall x \in [0,1], z_1, z_2 \in \mathbb{R}.$

Claim: (\star) has a unique solution $f \in C([0,1])$.

For $f \in C([0,1])$ set

$$T(f)(x) = \int_0^x k(x,y) h(y, f(y)) dy + g(x), \quad x \in [0,1]$$

Here $T(f) \in C([0,1])$

Want to show: $T: C([0,1]) \rightarrow C([0,1])$ has a unique fixed point.

Start with the Banach space $(C([0,1]), \| \cdot \|_\infty)$.

Check if T contraction on $C([0,1])$.

Fix $f_1, f_2 \in C([0,1])$

$$T(f_1)(x) - T(f_2)(x) = \int_0^x k(x,y) (h(y, f_1(y)) - h(y, f_2(y))) dy$$

k is continuous on the compact set $[0,1] \times [0,1]$ so

$$\sup_{(x,y) \in [0,1] \times [0,1]} |k(x,y)| = N < \infty$$

We obtain

$$\begin{aligned} |(T(f_1) - T(f_2))(x)| &\leq \int_0^x \underbrace{|k(x,y)|}_{\leq N} \underbrace{|h(y, f_1(y)) - h(y, f_2(y))|}_{\leq M|f_1(y) - f_2(y)|} dy \leq \\ &\leq \int_0^x NM dy \|f_1 - f_2\| \leq NM \|f_1 - f_2\| \end{aligned}$$

This yields $\|T(f_1) - T(f_2)\| \leq NM \|f_1 - f_2\|$

If $NM < 1$ then T contraction.

Trick: For $a > 0$ set

$$\|f\|_a = \max_{x \in [0,1]} e^{-ax} |f(x)| \quad \text{for } f \in C([0,1])$$

Claim: $\|\cdot\|_a$ defines a norm on $C([0,1])$. Easy check

Claim: $\|\cdot\|, \|\cdot\|_a$ are equivalent

Follows from

$$e^a \|f\| \leq \|f\|_a \leq \|f\| \quad \text{for all } f \in C([0,1]).$$

Claim: $(C([0,1]), \|\cdot\|_a)$ is a Banach space.

This follows from $\|\cdot\|, \|\cdot\|_a$ equivalent and $C([0,1], \|\cdot\|)$ is a Banach space.

Claim: T is a contraction on $(C([0,1]), \|\cdot\|_a)$ for $a > 0$ large enough.

For $f_1, f_2 \in C([0,1])$ and $x \in [0,1]$ we have

$$\begin{aligned} |(T(f_1) - T(f_2))(x)| &\leq \int_0^x NM |(f_1 - f_2)(y)| dy = \\ &= \int_0^x NM e^{ay} \cdot \underbrace{e^{-ay}}_{\leq \|f_1 - f_2\|_a} |(f_1 - f_2)(y)| dy \leq NM \underbrace{\int_0^x e^{ay} dy}_{=\frac{1}{a}(e^{ax}-1)} \cdot \|f_1 - f_2\|_a \end{aligned}$$

So

$$e^{ax} |(T(f_1) - T(f_2))(x)| \leq \frac{NM}{a} (1 - e^{-ax}) \|f_1 - f_2\|_a$$

$$\Rightarrow \|T(f_1) - T(f_2)\|_a \leq \frac{NM}{a} \|f_1 - f_2\|_a$$

For $a > NM$ T is a contraction on $(C([0,1]), \|\cdot\|_a)$.

Banach's fixed point theorem implies that there is a unique solution $f \in C([0,1])$ to $(*)$.

Exercises:

Chapter 1

⑤ Prove that the solution to

$$y'' + y = 0$$

forms a vector space.

Remains to prove that if $y'' + y = 0$ and $z'' + z = 0$, $\lambda \in \mathbb{R}$, then

$$(y+z)'' + y + z = 0$$

$$(\lambda y)'' + \lambda y = 0$$

⑬ Show that $\{e^{nx}, n=1,2,\dots\}$ is linearly independent.

Assume

$$a_1 e^x + a_2 e^{2x} + \dots + a_n e^{nx} = 0, \quad \forall x \in \mathbb{R}$$

To show: $a_1 = a_2 = \dots = a_n = 0$

Set $t = e^x$

$$a_1 t + a_2 t^2 + \dots + a_n t^n = 0 \quad \text{all } t > 0.$$

(Differentiate, set $t=0$, repeat. Done this before).

⑭ Show that: $C^\infty(\mathbb{R})$ is infinite dimensional.

$\mathcal{P}(\mathbb{R})$ is an infinite dimensional vector space.

$\mathcal{P}(\mathbb{R}) \subset C^\infty(\mathbb{R})$ implies $C^\infty(\mathbb{R})$ is infinite dimensional.

⑮ $(E_1, \|\cdot\|_1), (E_2, \|\cdot\|_2)$ normed spaces

$T: E_1 \rightarrow E_2$ continuous

$S \subset E_1$ compact

$\Rightarrow T(S)$ compact set in E_2 .

Proof: Fix an arbitrary sequence $(y_n)_{n=1}^\infty$ in $T(S) = \{T(x) : x \in S\}$

For each n fix a $x_n \in S$ such that $T(x_n) = y_n$

Consider $(x_n)_{n=1}^\infty$. This is a sequence in the convergent set S

in E_1 . There exists a convergent subsequence $(\tilde{x}_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ in S , i.e. $\tilde{x}_n \rightarrow \tilde{x} \in S$ in E_1 .

Consider $(T(\tilde{x}_n))_{n=1}^\infty$ subsequence of $(y_n)_{n=1}^\infty$. Here $(T(\tilde{x}_n))_{n=1}^\infty$ converges in $T(S)$ in E_2 since

$$\tilde{x}_n \rightarrow \tilde{x} \text{ in } E_1 \Rightarrow T(\tilde{x}_n) \rightarrow T(\tilde{x}) \text{ in } T(S) \text{ in } E_2,$$

since T is continuous.

⑯ $L: C([0,1]) \rightarrow C([0,1])$

$$L(f)(x) = \int_0^x f(t) dt$$

Show that L is continuous.

Proof: Bounded and linear \Rightarrow continuous.

Exercise:

Show that

$$\lim_{j \rightarrow \infty} \sum_{n=1}^{\infty} \frac{x_n}{j+n} = 0 \quad \text{for all } x = (x_1, \dots, x_n, \dots) \in l^2$$

Solution: $x \in l^2 \Rightarrow \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2} < \infty$

We have by Hölder's inequality

$$\begin{aligned} \sum_{n=1}^{\infty} \left| \frac{x_n}{j+n} \right| &\leq \left(\sum_{n=1}^{\infty} \frac{1}{(j+n)^2} \right)^{1/2} \cdot \underbrace{\|x\|_{l^2}}_{< \infty} \\ \sum_{n=1}^{\infty} \frac{1}{(j+n)^2} &\leq \int_j^{\infty} \frac{1}{t^2} dt = \left[-\frac{1}{t} \right]_j^{\infty} = \frac{1}{j} \xrightarrow{j \rightarrow \infty} 0 \end{aligned}$$

(II) section 6.3

Two norms on l^1

$$\|x\|_* = 2 \left| \sum_{n=1}^{\infty} x_n \right| + \sum_{n=2}^{\infty} \left(1 + \frac{1}{n} \right) |x_n|$$

$$\|x\| = \|x\|_{l^1}$$

Claim: $\|\cdot\|_*$ defines a norm on l^1 . (Easy to check)

Claim: $\|\cdot\|_*$, $\|\cdot\|$ are equivalent norms on l^1 .

Fix $x \in l^1$

$$\|x\|_* \leq 2 \sum_{n=1}^{\infty} |x_n| + \frac{3}{2} \sum_{n=2}^{\infty} |x_n| \leq \frac{7}{2} \sum_{n=1}^{\infty} |x_n| = \frac{7}{2} \|x\|$$

$$\|x\| = \sum_{n=1}^{\infty} |x_n| = |x_1| + \sum_{n=2}^{\infty} |x_n| \leq |x_1| + \|x\|_*$$

Moreover

$$\begin{aligned} |x_1| &= \left| x_1 + \sum_{n=2}^{\infty} |x_n| - \sum_{n=2}^{\infty} |x_n| \right| \leq \underbrace{\left| \sum_{n=1}^{\infty} x_n \right|}_{\leq \|x\|_{l^1}} + \underbrace{\left| \sum_{n=2}^{\infty} x_n \right|}_{\leq \|x\|_*} \\ &\leq \|x\|_{l^1} \leq \|x\|_* \end{aligned}$$

Hence

$$\|x\| \leq 2\|x\|_* + \|x\|_* = 3\|x\|_*$$

\Rightarrow ~~both~~ $\|\cdot\|$, $\|\cdot\|_*$ are equivalent norms on l^1 .

$(l^1, \|\cdot\|)$ is a Banach space $\Rightarrow (l^1, \|\cdot\|_*)$ is a Banach space.

Föreläsning 6

Theorem:

- $(E, \|\cdot\|)$ Banach space
- $(Y, \|\cdot\|_*)$ normed space

$T: E \times Y \rightarrow E$ where

- 1) $\exists c > 1 : \|T(x, y) - T(\tilde{x}, y)\| \leq c \|x - \tilde{x}\|, \forall x, \tilde{x} \in E, \forall y \in Y$
- 2) $T_x: Y \rightarrow E$ where $T_x(y) = T(x, y)$ is continuous $\forall x \in E$

\Rightarrow For every $y \in Y$ there exists a unique $g(y) \in E$ such that $g(y) = T(g(y), y)$ and $g: Y \rightarrow E$ is continuous.

Proof: The existence of a unique element $g(y) \in E$ for every $y \in Y$ follows from Banach fixed point theorem.

Assume $y_n \rightarrow \tilde{y}$ in $(Y, \|\cdot\|_*)$. Remains to show that $g(y_n) \rightarrow g(\tilde{y})$ in $(E, \|\cdot\|)$.

$$\begin{aligned} \|g(y_n) - g(\tilde{y})\| &= \|T(g(y_n), y_n) - T(g(\tilde{y}), \tilde{y})\| = \dots \\ &= \|T(g(y_n), y_n) - T(g(\tilde{y}), \tilde{y}) + T(g(\tilde{y}), y_n) - T(g(\tilde{y}), \tilde{y})\| \leq \\ &\leq \|T(g(y_n), y_n) - T(g(\tilde{y}), y_n)\| + \|T(g(\tilde{y}), y_n) - T(g(\tilde{y}), \tilde{y})\| \\ &\quad \text{1) } \underbrace{\leq c \|g(y_n) - g(\tilde{y})\|}_{n \rightarrow \infty} \quad \text{2) } \underbrace{\xrightarrow{n \rightarrow \infty} 0} \end{aligned}$$

We obtain

$$\|g(y_n) - g(\tilde{y})\| \leq \frac{1}{1-c} \|T(g(\tilde{y}), y_n) - T(g(\tilde{y}), \tilde{y})\| \xrightarrow{n \rightarrow \infty} 0 \quad \square$$

Brouwer's fixed point theorem

K compact (=closed & bounded) convex subset of \mathbb{R}^n .

$T: K \rightarrow K$ continuous

$\Rightarrow T$ has a fixed point, i.e. there exists $\bar{x} \in K$ such that $T(\bar{x}) = \bar{x}$.

Remark: No uniqueness. Consider the case $T = \text{Id}_k$.

Perron's theorem:

A real valued $n \times n$ -matrix with positive entries
 $A = \{a_{ij}\}_{i,j=1}^n$ all $a_{ij} > 0$

\Rightarrow The mapping

$$\mathbb{R}^n \ni x \mapsto Ax \in \mathbb{R}^n$$

has an eigenvalue $\lambda > 0$ with an eigenvector with positive entries, i.e. there exists $x > 0$ and $\tilde{x} \in \mathbb{R}^n$ such that $A\tilde{x} = \lambda\tilde{x}$ and all entries in \tilde{x} are positive.

Proof: Use Brouwer's fixed point theorem.

$$\text{Set } K = \{(x_1, x_2, \dots, x_n) : x_k > 0, \forall k, \sum_{k=1}^n x_k = 1\}$$

Claim: K is a closed, bounded, convex set on \mathbb{R}^n

compact since K is in the finite dimensional vector space \mathbb{R}^n

$$(E: \dim E < \infty \Leftrightarrow \forall F \subset E : F \text{ compact} \Leftrightarrow F \text{ closed})$$

$$\text{Set } T(x_1, \dots, x_n) = \frac{1}{\|Ax\|_{\ell^1}} A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \forall (x_1, \dots, x_n) \in K$$

Claim: ~~continuous~~ $T: K \rightarrow K$ is continuous.

Since $x_k \rightarrow x$ wrt ℓ^1 -norm. (in K)

To show: $T(x_k) \rightarrow T(x)$ in K wrt ℓ^1 -norm

$$x = (x_1, x_2, \dots, x_n)$$

$$x_k = (x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}) \quad k = 1, 2, \dots$$

$$\begin{aligned} \|T(x_k) - T(x)\|_{\ell^1} &= \left\| \frac{1}{\|Ax_k\|_{\ell^1}} Ax_k - \frac{1}{\|Ax\|_{\ell^1}} Ax \right\|_{\ell^1} \leq \\ &\leq \left\| \frac{1}{\|Ax_k\|_{\ell^1}} Ax_k - \frac{1}{\|Ax_k\|_{\ell^1}} A\tilde{x}_k \right\| + \left\| \frac{1}{\|Ax\|_{\ell^1}} A\tilde{x}_k - \frac{1}{\|Ax\|_{\ell^1}} Ax \right\| = \\ &= \left| \frac{1}{\|Ax_k\|_{\ell^1}} - \frac{1}{\|Ax\|_{\ell^1}} \right| \left\| A\tilde{x}_k \right\| + \left| \frac{1}{\|Ax\|_{\ell^1}} \right| \left\| A(\tilde{x}_k - x) \right\|_{\ell^1} \leq \end{aligned}$$

$$\begin{aligned} \|A(x - x_k)\|_{\ell^1} &= \sum_{i=1}^n \left| \sum_{j=1}^n a_{ij}(x_j - x_j^{(k)}) \right| \leq \\ &\leq \sum_{i=1}^n \sum_{j=1}^n a_{ij} |x_j - x_j^{(k)}| \leq n \cdot \max_{i,j} a_{ij} \|x - x_k\|_{\ell^1} \xrightarrow[k \rightarrow \infty]{\text{as}} 0 \end{aligned}$$

$Ax_k \rightarrow Ax$ in ℓ^1 .

This implies $\|Ax_k\|_{\ell^1} \rightarrow \|Ax\|_{\ell^1}$ in \mathbb{R} .

Brouwer's fixed point theorem implies that T has a fixed point $\bar{x} \in K$

$$\begin{aligned} \bar{x} &= (\bar{x}_1, \dots, \bar{x}_n) \\ \bar{x} &= T(\bar{x}) = \frac{1}{\|A\bar{x}\|_{\ell^1}} A\bar{x} \end{aligned}$$

then

$$A\bar{x} = \|A\bar{x}\|_{\ell^1} \bar{x} \quad \text{where} \quad \|A\bar{x}\|_{\ell^1} > 0$$

and \bar{x} has all entries > 0

□

Schauder's fixed point theorem

- $(E, \|\cdot\|)$ Banach space
 - K compact, convex set in E
 - $T: K \rightarrow K$ continuous
- $\Rightarrow T$ has a fixed point in K .

Example: $S = \{f \in C([0,1]) : f(0) = 0, f(1) = 1, \|f\| = \max_{x \in [0,1]} |f(x)| \leq 1\}$

$T: S \rightarrow S$ defined by

$$T(f)(x) = f(x^2), \quad x \in [0,1]$$

Claim: S is closed, bounded, convex in $C([0,1])$

- $T: S \rightarrow S$ is continuous
- T has no fixed point in S .

S bounded: $f \in S \Rightarrow \|f\| \leq 1$

S closed: $S \ni f_n \rightarrow f$ in $C([0,1])$. To show: $f \in S$

Note: $\max_{x \in [0,1]} |f_n(x) - f(x)| \xrightarrow{n \rightarrow \infty} 0$

$$|f(0)| = |f_n(0) - f(0)| \rightarrow 0, n \rightarrow \infty$$

$$|1 - f(1)| = |f_n(1) - f(1)| \rightarrow 0, n \rightarrow \infty$$

$$\text{so } f(1) = 1.$$

$$\text{For } x \in [0,1] \quad |f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| \leq \underbrace{\lim_{n \rightarrow \infty} |f - f_n|}_{\leq 1} + \underbrace{\|f_n\|}_{\leq 1}$$

Conclusion: $f \in S$

S convex: $f, \tilde{f} \in S, \lambda \in [0,1]$

To show: $\lambda f + (1-\lambda)\tilde{f} \in S$

Trivial since

$$(\lambda f + (1-\lambda)\tilde{f})(0) = 0$$

$$(\lambda f + (1-\lambda)\tilde{f})(1) = 1$$

$$\|\lambda f + (1-\lambda)\tilde{f}\| \leq |\lambda| \|f\| + |1-\lambda| \|\tilde{f}\| \leq 1$$

T: $S \rightarrow S$ continuous:

Assume $f_n \rightarrow f$ in S in max-norm, i.e.

$$\max_{x \in [0,1]} |f_n(x) - f(x)| \rightarrow 0, n \rightarrow \infty$$

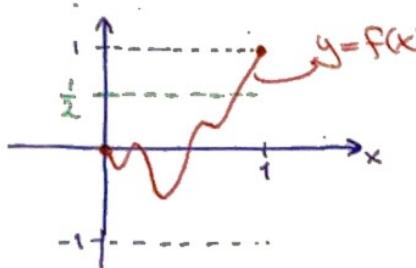
To show: $T(f_n) \rightarrow T(f)$ in S in max-norm

$$\begin{aligned} \|T(f_n) - T(f)\| &= \max_{x \in [0,1]} |T(f_n)(x) - T(f)(x)| = \max_{x \in [0,1]} |f_n(x^2) - f(x^2)| \\ &= \|f_n - f\| \rightarrow 0, n \rightarrow \infty \end{aligned}$$

T has no fixed point:

If $f \in S$ is a fixed point for T , then $f(x^2) = T(f)(x) = f(x)$

To show: there can be no such $f \in S$



$$\text{Set } a = \inf \{x \in [0,1] : f(x) = \frac{1}{2}\}$$

$\neq \emptyset$ since f continuous.

$a \in (0,1)$ since if $a=0$ then there exists a sequence $a_n = \{x \in [0,1] : f(x) = \frac{1}{2}\}$ such that $a_n \rightarrow a$ in \mathbb{R} , as $n \rightarrow \infty$.

Contradiction since $\frac{1}{2}f(a_n) \xrightarrow{n \rightarrow \infty} f(a) = f(0) = 0$ since f is continuous.

But $0 < a^2 < a$ and $f(a^2) = f(a) = \frac{1}{2}$
Contradiction.

If we believe in Schauder, then we can conclude that $S \subset C([0,1])$ is not compact.

Arzela-Ascoli theorem:

Assume K is a compact set in \mathbb{R}^n (e.g. $K = [0,1]$ in $\mathbb{R}_{\geq 0}$) and ~~is closed~~ $S \subset C(K)$, where $C(K)$ is equipped with the max-norm.

Then S is relatively compact in $C(K)$ iff

- { 1) S is uniformly bounded
 - 2) S is equicontinuous
-

Definition:

① S uniformly bounded if

$$\sup_{f \in S} \|f\| < \infty$$

② S is equicontinuous if

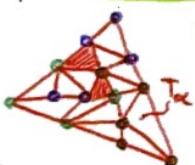
$$\forall \varepsilon > 0 \exists \delta > 0 : \forall f \in S |x - \tilde{x}| < \delta, x, \tilde{x} \in K \Rightarrow |f(x) - f(\tilde{x})| < \varepsilon.$$

③ S is relatively compact in $C(K)$ if

for every sequence $(f_n)_{n=1}^{\infty}$ in S there exists a converging subsequence in $C(K)$.

• Can show: S is relatively compact in $C(K)$ iff $d(S) = \overline{S}$ is compact in $C(K)$.

Sperner's lemma:



Big triangle T

$$T = \bigcup_{\alpha \in A} T_\alpha$$

$\{T_\alpha\}_{\alpha \in A}$ is triangulation of T ,

i.e. for any pair T_α, T_β in the triangulation

$$T_\alpha \cap T_\beta = \begin{cases} \emptyset \text{ or} \\ \text{common vertex or} \\ \text{common side or} \\ T_\alpha = T_\beta \end{cases}$$

\Rightarrow there must exist a triangle T_α with all vertices colored differently. (red in drawn triangle)

Proof of Schauder's fixed point theorem

Lemma: Assume $(x_n)_{n=1}^{\infty}$ sequence in K such that

$$\|T(x_n) - \underset{x_n}{\cancel{T(x_n)}}\| \rightarrow 0, n \rightarrow \infty$$

$\Rightarrow T$ has a fixed point in K .

Proof: Consider $(T(x_n))_{n=1}^{\infty}$ in K .

K compact \Rightarrow there exists a $z \in K$ and a subsequence $(T(\tilde{x}_n))_{n=1}^{\infty}$ of $(T(x_n))_{n=1}^{\infty}$ such that $T(\tilde{x}_n) \xrightarrow{n \rightarrow \infty} z$ in K .

Then $\|T(\tilde{x}_n) - \tilde{x}_n\| \xrightarrow{n \rightarrow \infty} 0$

So $\tilde{x}_n \rightarrow z, n \rightarrow \infty$.

But T continuous implies $T(\tilde{x}_n) \xrightarrow{n \rightarrow \infty} T(z)$

Conclusion: $z = T(z)$, so z is a fixed point.

Lemma: K compact set in E , $\varepsilon > 0$,

\Rightarrow there exists a finite set $x_1, x_2, \dots, x_N \in K$ such that for all $x \in K$

$$\min_{k=1, \dots, N} \|x - x_k\| < \varepsilon$$

i.e.

$$K \subset \bigcup_{k=1}^N B(x_k, \varepsilon)$$

Proof: Assume there is no such finite sequence x_1, \dots, x_N . Then there exists a sequence $(x_n)_{n=1}^{\infty}$ such that $\|x_k - x_l\| \geq \varepsilon$ for $k \neq l$.

Clearly, $(x_n)_{n=1}^{\infty}$ has no convergent subsequence.

This contradicts K being compact. \square Lemma

Fix positive integer n .

Apply previous lemma with $\varepsilon = \frac{1}{n}$.

Then there exists a finite set x_1, \dots, x_N such that

$$K \subset \bigcup_{k=1}^N B(x_k, \frac{1}{n})$$

Set $K_n = \text{set of all convex combinations of } x_1, \dots, x_N = \left\{ \sum_{k=1}^n \lambda_k x_k : \lambda_k \geq 0 \forall k, \sum_{k=1}^n \lambda_k = 1 \right\}$

K_n is a closed, bounded set in $\text{Span}(K)$ finite dimensional.
Also K_n convex. Want $T_n : K_n \rightarrow K_n$ where T_n close to T .

Set $f_k(x) = \max(0, \frac{1}{n} - \|x - x_k\|), x \in K, k=1, 2, \dots, N$

For each $x \in K$ there exists k such that $f_k(x) > 0$.

$$\text{Set } P_n(x) = \frac{f_1(x)x_1 + f_2(x)x_2 + \dots + f_N(x)x_N}{f_1(x) + f_2(x) + \dots + f_N(x)} \quad x \in K$$

$P_n(x)$ is a convex combination of x_1, \dots, x_N for every $x \in K$

So $P_n(x) \in K_n$ for every $x \in K$

Claim: $\|P_n(x) - x\| \leq \frac{1}{n}$ for all $x \in K$

Set T_n to be defined

$$T_n = P_n T : K_n \rightarrow K_n$$

Here, T_n is continuous.

K_n compact, convex in finite dimensional space.

We can apply Brouwer's fixed point theorem, which implies that T_n has a fixed point in K_n , i.e. there exists $x_n \in K_n$ such that

$$x_n = T_n(x_n) = P_n T(x_n)$$

But then

$$\|x_n - T(x_n)\| \leq \underbrace{\|x_n - P_n T(x_n)\|}_{=0} + \underbrace{\|P_n T(x_n) - T(x_n)\|}_{\leq \frac{1}{n}}$$

The first lemma above implies that T has a fixed point in K .

Föreläsning 7

Inner product spaces

Example:

$$\mathbb{C}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{C}\}$$

$$x, y \in \mathbb{C}^n \quad x = (x_1, \dots, x_n) \quad y = (y_1, \dots, y_n)$$

$$\langle x, y \rangle \stackrel{\text{def}}{=} \sum_{i=1}^n x_i \bar{y}_i \in \mathbb{C}$$

We have a map

$$\begin{aligned} \mathbb{C}^n \times \mathbb{C}^n &\longrightarrow \mathbb{C} \\ (x, y) &\longmapsto \langle x, y \rangle \end{aligned} \quad \begin{array}{l} \text{inner product of } x, y \\ (\text{scalar product}) \end{array}$$

Properties:

- $x \neq 0 \Rightarrow \langle x, x \rangle = \sum_{i=1}^n x_i \bar{x}_i = \sum_{i=1}^n |x_i|^2 > 0.$
- $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$
 $x, y \in \mathbb{C}^n, \lambda \in \mathbb{C}$
- $\langle x, y \rangle = \overline{\sum_{i=1}^n x_i \bar{y}_i} = \sum_{i=1}^n \bar{y}_i \bar{x}_i = \overline{\langle y, x \rangle}$
 $x, y \in \mathbb{C}^n$

In particular $\langle x, \lambda y \rangle = \overline{\lambda} \langle x, y \rangle$

$$\bullet \langle x+y, z \rangle = \underbrace{\langle x, z \rangle + \langle y, z \rangle}_{\lambda \in \mathbb{C}} \quad x, y, z \in \mathbb{C}^n$$

Definition:

An inner product space V is a vector space with inner product, which is a map

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$$

satisfying

- $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle \quad \forall \lambda \in \mathbb{C}, x, y \in V$
- $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle, \quad \forall x, y, z \in V$
- $\langle x, y \rangle = \overline{\langle y, x \rangle}, \quad \forall x, y \in V$
- $\langle x, x \rangle \geq 0, \quad \forall x \in V, x \neq 0.$

• Can we generalize \mathbb{C}^n ?

$$\mathbb{C}^\infty = \{(x_1, x_2, \dots) : x_i \in \mathbb{C}\}$$

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i \leftarrow \text{not necessarily convergent.}$$

Example: $\ell^2 = \{(x_1, x_2, \dots) : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$

We have

$$\left| \sum_{i=1}^{\infty} x_i \bar{y}_i \right| \leq \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2} \left(\sum_{i=1}^{\infty} |y_i|^2 \right)^{1/2}$$

Cauchy-Schwarz inequality

Now if $x \in \ell^2, y \in \ell^2$

$$\sum_{i=1}^{\infty} |x_i \bar{y}_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2} \left(\sum_{i=1}^{\infty} |y_i|^2 \right)^{1/2} < \infty$$

\downarrow

$\sum_{i=1}^{\infty} x_i \bar{y}_i$ converges absolutely and hence convergent.

- The following $\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i$ is well-defined for vectors $x, y \in \ell^2$
- Easy to check that $\langle x, y \rangle, x, y \in \ell^2$ satisfies axioms for inner prod.
 $\Rightarrow (\ell^2, \langle \cdot, \cdot \rangle)$ is inner product space.

Example: $C([0,1])$

$$\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} dt \quad \leftarrow \text{inner product}$$

$f, g \in C([0,1])$

$$\langle \lambda f, g \rangle = \int_0^1 \lambda f(t) \overline{g(t)} dt = \lambda \int_0^1 f(t) \overline{g(t)} dt = \lambda \langle f, g \rangle$$

$$\langle f, f \rangle = \int_0^1 f(t) \overline{f(t)} dt = \int_0^1 |f(t)|^2 dt > 0.$$

:

Norms induced by inner products

- \mathbb{R}^3

$$\text{Euclidean norm on } \mathbb{R}^3, \| (x_1, x_2, x_3) \| = \sqrt{x_1^2 + x_2^2 + x_3^2} = \left(\sum_{i=1}^3 |x_i|^2 \right)^{1/2}.$$

Let V be an inner product space with $\langle \cdot, \cdot \rangle$ -inner product.
 Let for $x \in V$ $\|x\|_{\text{def}} = \langle x, x \rangle^{1/2}$

Statement: $(*)$ $x \mapsto \|x\|$ is a norm on V .

Recall: $\|x\|$ is norm function if

- $\|x\| > 0 \quad \forall x \in V, x \neq 0$
- $\|\lambda x\| = |\lambda| \|x\|, \quad \forall x \in V, \lambda \in \mathbb{C}$
- $\|x+y\| \leq \|x\| + \|y\|, \quad \forall x, y \in V$

Theorem: (Cauchy-Schwartz inequality)

For any $x, y \in V$ (inner product space)

$$|\langle x, y \rangle| \leq \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2}$$

(The equality holds iff x and y are linearly dependent)

Proof: Assume x, y linearly dependent. Can assume $x = \lambda y$ for $\lambda \in \mathbb{C}$.

$$|\langle x, y \rangle| = |\langle \lambda y, y \rangle| = |\lambda| |\langle y, y \rangle|$$

$$\langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2} = \langle \lambda y, \lambda y \rangle^{1/2} \langle y, y \rangle^{1/2} = |\lambda| \langle y, y \rangle^{1/2} \langle y, y \rangle^{1/2} = |\lambda| |\langle y, y \rangle|$$

$$\Rightarrow |\langle x, y \rangle| = \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2}$$

Assume x, y linearly independent. Then $x + \lambda y \neq 0$ for any $\lambda \in \mathbb{C}$.
By an axiom (for inner product):

$$0 < \langle x + \lambda y, x + \lambda y \rangle = \langle x, x \rangle + \lambda \langle y, x \rangle + \bar{\lambda} \langle x, y \rangle + |\lambda|^2 \langle y, y \rangle$$

Pick $\lambda = -\frac{\langle x, y \rangle}{\langle y, y \rangle}$ (obs! $y \neq 0$ as x, y linearly independent) Holds for any $\lambda \in \mathbb{C}$

We have

$$\begin{aligned} 0 &< \langle x, x \rangle - \underbrace{\frac{\langle x, y \rangle \langle y, x \rangle}{\langle y, y \rangle}}_{= |\langle x, y \rangle|^2} - \underbrace{\frac{\langle x, y \rangle \langle x, y \rangle}{\langle y, y \rangle}}_{= |\langle x, y \rangle|^2} + \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle^2} \cancel{\langle y, y \rangle} = \\ &= \langle x, x \rangle - \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} \end{aligned}$$

$$\Rightarrow \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} < \langle x, x \rangle \Rightarrow |\langle x, y \rangle|^2 < \langle x, x \rangle \langle y, y \rangle \quad \square$$

Proof of statement (x): (Axiom 1&2 left as exercise)

Let $x, y \in V$

$$\begin{aligned} \|x+y\|^2 &\stackrel{\text{def}}{=} \langle x+y, x+y \rangle \stackrel{\text{linearity}}{=} \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle = \\ &= \langle x, x \rangle + 2\Re \langle x, y \rangle + \langle y, y \rangle \leq \langle x, x \rangle + 2|\langle x, y \rangle| + \langle y, y \rangle \leq \\ &\leq \langle x, x \rangle + 2\langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2} + \langle y, y \rangle = \\ &= \left(\underbrace{\langle x, x \rangle^{1/2}}_{\|x\|} + \underbrace{\langle y, y \rangle^{1/2}}_{\|y\|} \right)^2 \\ \|x+y\|^2 &\leq (\|x\| + \|y\|)^2 \Rightarrow \|x+y\| \leq \|x\| + \|y\| \end{aligned}$$

Theorem: (The parallelogram law)

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space.

$$\text{Let } \|x\| = \sqrt{\langle x, x \rangle}$$

Then

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad \forall x, y \in V$$

Proof left as exercise.

Statement: ℓ^p has inner product $\langle \cdot, \cdot \rangle_p$ such that

$$\|x\|_p = \sqrt{\langle x, x \rangle_p} \text{ iff } p=2.$$

Proof: Enough to show that $\|\cdot\|_p$ -norm does not satisfy the parallelogram law for some $x, y \in \ell^p$ if $p \neq 2$.

Take, for example $x = (1, 0, 0, \dots)$, $y = (0, 1, 0, \dots)$

$$\|x+y\|_p^2 = \|(1, 1, 0, 0, \dots)\|_p^2 = 2^{2/p} \quad \|x\|_p^2 = 1$$

$$\|x-y\|_p^2 = \|(1, -1, 0, 0, \dots)\|_p^2 = 2^{2/p} \quad \|y\|_p^2 = 1$$

$$\|x+y\|_p^2 + \|x-y\|_p^2 = 2 \cdot 2^{2/p} = 2(\|x\|_p^2 + \|y\|_p^2) = 2 \cdot 2$$

$$\text{iff } p=2.$$

Exercise: Show that $(C([0,1]), \|\cdot\|_\infty)$ is not an inner product space, i.e. $\|\cdot\|_\infty$ is not coming from an inner product on $C([0,1])$.

Remark: Whenever a norm satisfies the parallelogram law $\forall x, y \in V$ then there exists an inner product on V such that $\|x\| = \sqrt{\langle x, x \rangle}$

Theorem: (The polarization identity)

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Then

$$4\langle x, y \rangle = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2.$$

Proof left as exercise.

Orthogonality, orthogonal systems, Bessel's inequality

Definition: Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. We say that $x, y \in V$ are orthogonal ($x \perp y$) if $\langle x, y \rangle = 0$.

Let $M \subset V$. Define $M^\perp = \{x \in V : x \perp y \text{ for any } y \in M\}$

\uparrow
orthogonal complement
to M .

Proposition: If $M \subset V$.

Then M^\perp is a subspace of.

Proof: $x, y \in M^\perp$, $\lambda, \mu \in \mathbb{C}$. We must check that $\lambda x + \mu y \in M^\perp$

Take $z \in M$

$$\begin{aligned} \langle \lambda x + \mu y, z \rangle &= \lambda \underbrace{\langle x, z \rangle}_{=0} + \mu \underbrace{\langle y, z \rangle}_{=0} = 0 \\ \Rightarrow \lambda x + \mu y &\in M^\perp \end{aligned}$$

Theorem: (Pythagorean formula)

- $x, y \in V$ (inner product space)

Then $x \perp y$ iff $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

Orthogonal systems

- $(V, \langle \cdot, \cdot \rangle)$ inner product space.
- $\{u_n\} \subset V$ is called orthogonal system if $u_n \perp u_m \quad \forall n \neq m$
finite or infinite

It is ON (orthonormal) system if in addition $\|u_n\|=1$.

Example: 1) $\{e_k\} \subset \ell^2 \quad \langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i$

$$e_k = \{0, 0, \dots, 0, 1, 0, \dots\}$$

$\{e_k\}$ is ON-system.
2 k-th position.

2) $C([-π, π])$, $\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) \overline{g(t)} dt$

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{int} : n \in \mathbb{Z} \right\} - \text{ON-system}$$

Definition: Let $\{u_n : n \in \mathbb{N}\}$ be an ON-system in V .

The formal series $\sum_{n=1}^{\infty} \langle x, u_n \rangle u_n$ is called Fourier series of x corresponding to $\{u_n : n \in \mathbb{N}\}$ and $\langle x, u_n \rangle$ are called Fourier coefficients to $\{u_n : n \in \mathbb{N}\}$.

Theorem: (Bessel's equality and inequality)

If $\{u_n\}$ is an ON-system in an inner product space V , then for any $x \in V$

$$\|x - \sum_{k=1}^n \langle x, u_k \rangle u_k\|^2 = \|x\|^2 - \sum_{k=1}^n |\langle x, u_k \rangle|^2$$

and

$$\sum_{k=1}^{\infty} |\langle x, u_k \rangle|^2 \leq \|x\|^2$$

Proof:

$$\begin{aligned} \|x - \sum_{k=1}^n \langle x, u_k \rangle u_k\|^2 &= \left\langle x - \sum_{k=1}^n \langle x, u_k \rangle u_k, x - \sum_{k=1}^n \langle x, u_k \rangle u_k \right\rangle = \\ &= \langle x, x \rangle - \sum_{k=1}^n \overline{\langle x, u_k \rangle} \langle x, u_k \rangle - \sum_{k=1}^n \langle x, u_k \rangle \langle u_k, x \rangle + \\ &\quad + \left\langle \sum_{k=1}^n \langle x, u_k \rangle u_k, \sum_{k=1}^n \langle x, u_k \rangle u_k \right\rangle = \|x\|^2 - \sum_{k=1}^n |\langle x, u_k \rangle|^2 - \\ &\quad - \cancel{\sum_{k=1}^n |\langle x, u_k \rangle|^2} + \cancel{\sum_{k=1}^n |\langle x, u_k \rangle|^2} = \|x\|^2 - \sum_{k=1}^{\infty} |\langle x, u_k \rangle|^2 \end{aligned}$$

This also gives

$$\sum_{k=1}^n |\langle x, u_k \rangle|^2 = \|x\|^2 - \|x - \sum_{k=1}^n \langle x, u_k \rangle u_k\|^2 \leq \|x\|^2. \text{ Holds for any } n.$$

Hence

$$\sum_{k=1}^{\infty} |\langle x, u_k \rangle|^2 \leq \|x\|^2 \quad \square$$

Hilbert spaces

Definition: A Hilbert space is an inner product space which is complete w.r.t the norm defined through the inner product.

Example: • \mathbb{C}^n - inner product space and complete w.r.t Euclidian norm $\Rightarrow \mathbb{C}^n$ - Hilbert space.

• ℓ^2 -Banach space w.r.t. $\|x\|_2 = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2}$

and $\langle x, y \rangle^{1/2} = \|x\|_2$ for

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i \quad \Rightarrow \ell^2 \text{ Hilbert space.}$$

$(C([0,1]), \|\cdot\|_\infty)$ - Banach space but not inner product space.

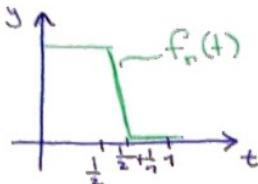
\Rightarrow not Hilbert space.

$(C([0,1]), \langle \cdot, \cdot \rangle)$ - inner product space, corresponding norm

$$\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} dt \quad \|f\|_2 = \langle f, f \rangle^{1/2} = \left(\int_0^1 |f(t)|^2 dt \right)^{1/2}$$

Statement: $(C([0,1]), \langle \cdot, \cdot \rangle)$ is not a Hilbert space as $(C([0,1]), \|\cdot\|_2)$ is not complete.

To prove the statement show that $f_n(t)$



is Cauchy w.r.t $\|\cdot\|_2$
but has no limit
in $C([0,1])$

$C_F = \{(x_1, x_2, \dots) : \text{only finite } x_i \neq 0\}$

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i$$

Show that $(C_F, \langle \cdot, \cdot \rangle)$ is not a Hilbert space.

Convergence in Hilbert spaces: Hilbert space

Definition: A sequence $\{x_n\} \subset H$ is called strongly convergent ($x_n \rightarrow x \in H$) if $\|x_n - x\| \rightarrow 0$ (norm from inner prod)
We say that x_n goes to x weakly ($x_n \rightharpoonup x$) if $\langle x_n, y \rangle \rightarrow \langle x, y \rangle$ for any $y \in H$.

Statement: $x_n \rightarrow x \Rightarrow x_n \rightharpoonup x$

Proof: Assume $x_n \rightarrow x$. Cauchy-Schwartz

$$|\langle x_n, y \rangle - \langle x, y \rangle| = |\langle x_n - x, y \rangle| \leq \underbrace{\langle x_n - x, x_n - x \rangle^{1/2}}_{\|x_n - x\|} \underbrace{\langle y, y \rangle^{1/2}}_{\|y\|}$$

$$\Rightarrow \langle x_n, y \rangle - \langle x, y \rangle \rightarrow 0 \quad \boxed{\text{Q.E.D.}}$$

Remark: The converse is not true in general:

Take $H = \ell^2$. $x_n = e_n = (0, \dots, 0, 1, 0, \dots)$. We have $\langle e_n, y \rangle \rightarrow 0$
Hence $e_n \rightharpoonup 0$ nth position.

But $e_n \not\rightarrow 0$ as $\|e_n - 0\|_2 = \|e_n\|_2 = 1$

Statement: $x_n \rightarrow x, y_n \rightarrow y \Rightarrow \langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$

In particular, $x_n \rightarrow x \Rightarrow \|x_n\| \rightarrow \|x\|$

Proof: $|\langle x_n, y_n \rangle - \langle x, y \rangle| = |\langle x_n, y_n \rangle - \langle x, y_n \rangle + \langle x, y_n \rangle - \langle x, y \rangle| =$
 $= |\langle x_n - x, y_n \rangle + \langle x, y_n - y \rangle| \leq |\langle x_n - x, y_n \rangle| + |\langle x, y_n - y \rangle| \leq$
 $\leq \|x_n - x\| \|y_n\| + \|x\| \|y_n - y\|$

Cauchy-Schwarz

Check that $\{\|y_n\|\}$ is bounded:

$$\|y_n\| \leq \|y_n - y\| + \|y\| \Rightarrow |\langle x_n, y_n \rangle - \langle x, y \rangle| \rightarrow 0.$$

Bounded

Statement: $x_n \xrightarrow{\omega} x$ and $\|x_n\| \rightarrow \|x\| \Rightarrow x_n \rightarrow x$

Proof: $\|x_n - x\|^2 = \langle x_n - x, x_n - x \rangle = \langle x_n, x_n \rangle - \langle x, x_n \rangle - \langle x_n, x \rangle + \langle x, x \rangle =$
 $= \|x_n\|^2 - \overbrace{\langle x_n, x \rangle}^{\|x\|^2} - \langle x_n, x \rangle + \|x\|^2 \rightarrow \|x\|^2 - \|x\|^2 - \|x\|^2 + \|x\|^2 = 0$

$\downarrow \|x\|^2 \quad \downarrow x_n \xrightarrow{\omega} x \quad \downarrow \langle x, x \rangle$

□

We have proved

$x_n \rightarrow x \Rightarrow \{\|x_n\|\}$ is bounded

Theorem:

$$x_n \xrightarrow{\omega} x \Rightarrow \sup_{n \in \mathbb{N}} \|x_n\| < \infty$$

Proof: Let $x_n \xrightarrow{\omega} x$

Consider $f_n: H \rightarrow \mathbb{C}, f_n(\underline{x}) = \langle \underline{x}, x_n \rangle, \underline{x} \in H$.

- f_n is a linear functional for each n
- $\forall n$ f_n is a bounded (\Leftrightarrow continuous) linear functional
as if $y_k \rightarrow y$ then $f_n(y_k) = \langle y_k, x_n \rangle \rightarrow \langle y, x_n \rangle = f_n(y)$
- $f_n(\underline{x}) \rightarrow \langle \underline{x}, x \rangle$
 $\{f_n(\underline{x})\}$ is a convergent sequence in \mathbb{C} and hence bounded.
Hence

$$\exists M_y: |f_n(y)| \leq M_y$$

By Banach-Steinhaus theorem
 $\|f_n\| \leq M$ for some $M > 0$.

We are done if we prove that

$$\|f_n\| = \|x_n\|.$$

$$|f_n(y)| = |\langle y, x_n \rangle| \leq \|y\| \|x_n\|$$

Cauchy-Schwarz

Hence $\|f_n\| \leq \|x_n\| \quad (1)$

On the other hand,

$$\begin{aligned} f_n(x_n) &= \langle x_n, x_n \rangle = \|x_n\|^2 \\ \Rightarrow f_n\left(\frac{x_n}{\|x_n\|}\right) &= \|x_n\| \\ \Rightarrow \|f_n\| &= \sup_{x \in H} \frac{|f_n(x)|}{\|x\|} \geq \frac{|f_n(x_n)|}{\|x_n\|} = \|x_n\| \end{aligned} \quad (2)$$

(1) and (2) gives us $\|f_n\| = \|x_n\|$ □

Orthogonal decomposition in Hilbert spaces

linear algebra: \mathbb{R}^n , $M \subset \mathbb{R}^n$ subspace

$$\forall x \in \mathbb{R}^n \quad x = y + z \text{ where } z \in M, y \in M^\perp$$

This can be done in a unique way.

General Hilbert space case

Proposition: $M \subset H$. Then M^\perp is a closed subspace.

$$(M^\perp)^\perp = M. \quad (\text{Proof left as exercise})$$

Proposition: H -Hilbert space, M -closed subspace of H , $x \in H$.

Then there exists a unique $z \in M$ such that

$$\|x - z\| = \text{dist}(x, M) \stackrel{\text{def}}{=} \inf_{y \in M} \|x - y\|$$

Proposition: Taking $z \in M$ from previous proposition we have

$$x - z \in M^\perp, \text{ i.e.}$$

$$x = z + \underbrace{(x - z)}_{M^\perp}$$

Föreläsning 8

- ① Application of Schauder's fixed point theorem
 - ② Completion of normed spaces.
[L^p -spaces]
 - ③ Continue on Hilbert spaces.
-

① Example: Assume

$k(x,y)$ continuous on $[0,1] \times [0,1]$

$h(y,z)$ continuous on $[0,1] \times \mathbb{R}$ and

$$\sup_{(y,z) \in [0,1] \times \mathbb{R}} |h(y,z)| = B < \infty.$$

Then there exists a solution $f \in C([0,1])$ to

$$f(x) = \int_0^1 k(x,y) h(y, f(y)) dy, \quad x \in [0,1].$$

Method: Set, for $f \in C([0,1])$

$$(*) T(f)(x) = \int_0^1 k(x,y) h(y, f(y)) dy, \quad x \in [0,1]$$

Want to apply (a generalized version of) Schauder's theorem.

• Assume $(E, \| \cdot \|)$ is a Banach space and F closed convex subset of E .

• Moreover, assume $T: F \rightarrow F$ continuous and $T(F)$ relatively compact in $(E, \| \cdot \|)$.

Then T has a fixed point in F .

Step 1: T as in $(*)$.

• Claim: $T(C([0,1])) \subset C([0,1])$

Proof: $k(x,y)$ continuous on $[0,1] \times [0,1]$ and $[0,1] \times [0,1]$ is compact in \mathbb{R}^2 , implies k uniformly continuous on $[0,1] \times [0,1]$.

Fix $\epsilon > 0$. Then

$$\exists \delta = \delta(\epsilon) > 0 : |h(x_1, y_1) - h(x_2, y_2)| < \frac{\epsilon}{B}, \text{ for } |(x_1, y_1) - (x_2, y_2)| < \delta.$$

Fix $f \in C([0,1])$

$$\begin{aligned} |T(f)(x_1) - T(f)(x_2)| &= \left| \int_0^1 (k(x_1, y) - k(x_2, y)) h(y, f(y)) dy \right| \leq \\ &\leq \int_0^1 |k(x_1, y) - k(x_2, y)| |h(y, f(y))| dy < \varepsilon \text{ provided } |x_1 - x_2| < \delta \\ &< \frac{\varepsilon}{B} \text{ if } |x_1 - x_2| < \delta \leq B \end{aligned}$$

Conclusion: $T(f) \in C([0,1])$ for $f \in C([0,1])$

Step 2: Choose F

k continuous function on compact set $[0,1] \times [0,1]$ implies

$$\sup_{(x,y) \in [0,1] \times [0,1]} |k(x,y)| = A < \infty.$$

Hence $|T(f)(x)| \leq AB$, $\forall f \in C([0,1])$.

Set $F = \{f \in C([0,1]) : \|f\|_\infty \leq AB\}$. Clearly, F is closed, convex in $(C([0,1]), \|\cdot\|_\infty)$ which is a Banach space.

Conclusion: $T: F \rightarrow F$, F closed convex in Banach space $(C([0,1]), \|\cdot\|_\infty)$

Step 3:

Claim: $T(F)$ is relatively compact.

To prove this we use Arzela-Ascoli theorem

\boxed{K} be a compact set in \mathbb{R}^n . Let $S \subset C(K)$ (real-valued cont. functions on K).

Then S is relatively compact in $(C(K), \|\cdot\|_\infty)$ if

1) S uniformly bounded, i.e.

$$\sup_{f \in S} \|f\| < \infty$$

2) Equicontinuity of f in S , i.e.

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0, \forall f \in S$$

$$|x_1 - x_2| < \delta, x_1, x_2 \in K \Rightarrow |f(x_1) - f(x_2)| < \varepsilon. \quad \boxed{\quad}$$

In our example: $S = F$, $K = [0,1]$ in \mathbb{R}

• Check that 1) and 2) in AA-theorem are satisfied.

1) F is uniformly bounded since

$$\sup_{f \in F} \|f\| = AB < \infty$$

2) Equicontinuity follows from calculations in step 1.

Conclusion: $T(f)$ is relatively compact.

Step 4: Claim: $T: F \rightarrow F$ continuous.

Step 1 $f \in F$
 $x_n \rightarrow x$ in $[0,1] \Rightarrow T(f)(x_n) \rightarrow T(f)(x)$ in \mathbb{R}

Step 4 $f_n \rightarrow f$ in F
Show $T(f_n) \rightarrow T(f)$ in $C([0,1])$

Note: $h: [0,1] \times [-AB, AB] \rightarrow \mathbb{R}$ is continuous and $[0,1] \times [-AB, AB]$ is compact set in \mathbb{R}^2 .

So $h: [0,1] \times [-AB, AB] \rightarrow \mathbb{R}$ is uniformly continuous.

Fix $\epsilon > 0$. Then

$$\exists \delta(\epsilon) > 0 : |h(y_1, z_1) - h(y_2, z_2)| < \frac{\epsilon}{A} \text{ for } |(y_1, z_1) - (y_2, z_2)| < \delta.$$

For $f_1, f_2 \in F$ with $\|f_1 - f_2\| < \delta$ we have

$$\begin{aligned} |T(f_1)(x) - T(f_2)(x)| &= \left| \int_0^1 k(x,y) (h(y, f_1(y)) - h(y, f_2(y))) dy \right| \leq \\ &\leq \underbrace{\int_0^1 |k(x,y)| dy}_{\leq A} \cdot \underbrace{|h(y, f_1(y)) - h(y, f_2(y))|}_{< \epsilon/A} < \epsilon. \end{aligned}$$

Conclusion: $T: F \rightarrow F$ is continuous.

Step 5: Apply Schauder's fixed point theorem.

Completion of normed spaces

- $(E, \|\cdot\|)$ normed space.

We say that $(\tilde{E}, \|\cdot\|)$ is a completion of $(E, \|\cdot\|)$ if $(\tilde{E}, \|\cdot\|)$ is a normed space such that

- 1) $\exists \Phi: E \rightarrow \tilde{E}$ one-to-one and linear.
- 2) $\|x\| = \|\Phi(x)\|, \forall x \in E$
- 3) $\Phi(E)$ dense in \tilde{E}
- 4) $(\tilde{E}, \|\cdot\|)$ is a Banach space.

Construction: $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}$ Cauchy sequences in $(E, \|\cdot\|)$.

We say that $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}$ are equivalent, denoted $(x_n) \sim (y_n)$, if

$$\|x_n - y_n\| \xrightarrow{n \rightarrow \infty} 0.$$

Set $\tilde{E} = \{[(x_n)]_{\sim} : (x_n)_{n=1}^{\infty} \text{ Cauchy sequence in } (E, \|\cdot\|)\}$

Vector space structure:

$$\begin{cases} \circ [[(x_n)]_{\sim} + [(\tilde{x}_n)]_{\sim}] = [((x_n + \tilde{x}_n)]_{\sim} \\ \circ \lambda [[(x_n)]_{\sim}] = [(\lambda x_n)]_{\sim} \end{cases}$$

Show that these definitions are well-defined, i.e. independent of the choice of representatives.

Norm:

$$\|[(x_n)]_{\sim}\| = \lim_{n \rightarrow \infty} \|x_n\|$$

Note:

$$(x_n) \sim (y_n)$$

$$\Rightarrow \lim_{n \rightarrow \infty} \|x_n\| = \lim_{n \rightarrow \infty} \|y_n\|$$

Since

$$|\|x_n\| - \|y_n\|| \leq \|x_n - y_n\| \xrightarrow{n \rightarrow \infty} 0$$

- Check that the axioms for being a norm are satisfied.

Now we have $(\tilde{E}, \|\cdot\|)$ is a normed space.

Define Φ .

For $x \in E$ set $\Phi(x) = [(x)_{n=1}^{\infty}]_n$ where

$$(x)_{n=1}^{\infty} = (x, x, x, \dots, x, \dots)$$

Claim: 1 & 2 easy to prove.

Claim: 3) $\Phi(E)$ dense in $(\tilde{E}, \|\cdot\|)$.

Fix $[(x_n)]_n \in \tilde{E}$.

Consider $\Phi(x_k)$ where x_k is the element in the k :th pos. in the sequence $(x_1, x_2, \dots, x_n, \dots)$.

$$\|[(x_n)]_n - \Phi(x_k)\| = \lim_{n \rightarrow \infty} \|x_n - x_k\| \xrightarrow{k \rightarrow \infty} 0$$

since $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Claim: 4) $(\tilde{E}, \|\cdot\|)$ is a Banach space.

- Consider a Cauchy sequence $(\tilde{x}_n)_{n=1}^{\infty}$ in $(\tilde{E}, \|\cdot\|)$

To show: $\exists \tilde{x} \in \tilde{E} : \|\tilde{x}_n - \tilde{x}\| \xrightarrow{n \rightarrow \infty} 0$.

By 3) $\Phi(E)$ is dense in \tilde{E} so for $n=1, 2, \dots$ there exists $x_n \in E$ such that

$$\|\tilde{x}_n - \Phi(x_n)\| < \frac{1}{n}, \quad n=1, 2, \dots$$

Set $\tilde{x} = [(x_n)]_n$. Need to show that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence.

$$\begin{aligned} \|x_n - x_m\| &= \|\Phi(x_n) - \Phi(x_m)\| \leq \|\Phi(x_n) - \tilde{x}_n\| + \|\tilde{x}_n - \tilde{x}_m\| + \|\tilde{x}_m - \Phi(x_m)\| + \\ &+ \|\tilde{x}_n - \tilde{x}_m\|. < \frac{1}{n} + \|\tilde{x}_n - \tilde{x}_m\| + \frac{1}{m} \xrightarrow{n, m \rightarrow \infty} 0 \end{aligned}$$

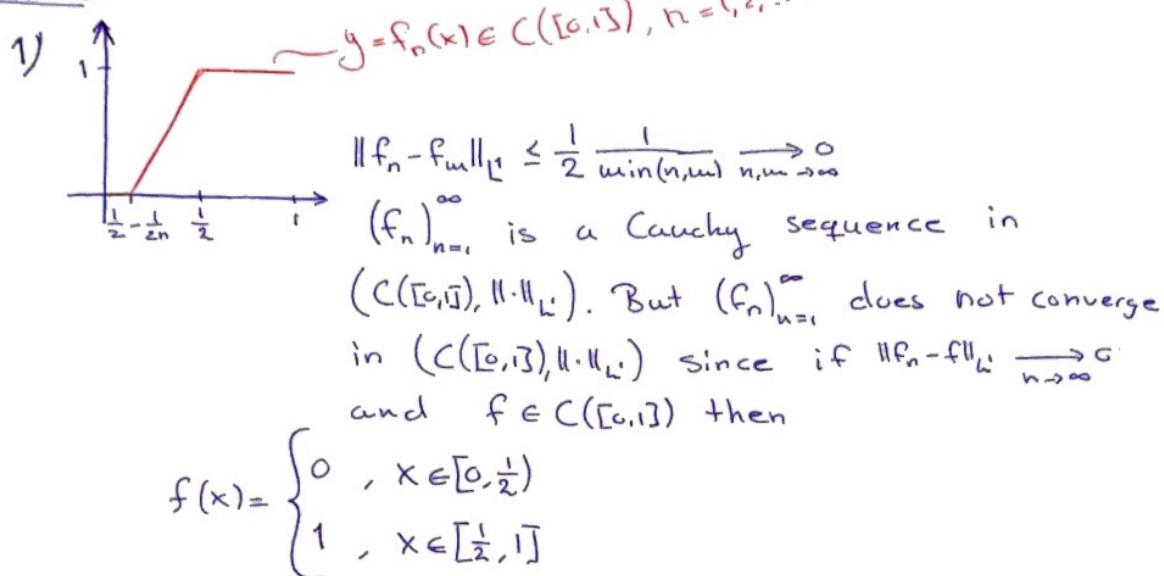
Conclusion: $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in $(E, \|\cdot\|)$

Remains to show: $\|\tilde{x}_n - \tilde{x}\| \xrightarrow{n \rightarrow \infty} 0$

$$\begin{aligned} \|\tilde{x}_n - \tilde{x}\| &\leq \underbrace{\|\tilde{x}_n - \Phi(\tilde{x}_n)\|}_{< \frac{1}{n} \xrightarrow{n \rightarrow \infty} 0} + \underbrace{\|\Phi(\tilde{x}_n) - \tilde{x}\|}_{=\lim_{n \rightarrow \infty} \|x_n - x_m\| \xrightarrow{n \rightarrow \infty} 0} \xrightarrow{n \rightarrow \infty} 0, \quad n \rightarrow \infty \end{aligned}$$

- Consider $C([0,1]) \ni f$
- $(C([0,1]), \|\cdot\|_{\infty})$ is a Banach space.
- $p \geq 1$ $\|f\|_{L^p} = \left(\int_0^1 |f(x)|^p dx \right)^{1/p}$ defines a norm for $C([0,1])$.

Remarks:



Conclusion: $(C([0,1]), \|\cdot\|_{L^1})$ is not a Banach space.

2) Consider

$$f(x) = \begin{cases} 1, & x = \frac{1}{2} \\ 0, & x \in [0,1] \setminus \{\frac{1}{2}\} \end{cases}$$

$$\|f\|_{L^1} = 0 = \|0\|_{L^1}$$

Compare this with the first axiom of a norm function.

Replace $[0,1]$ with \mathbb{R} . For $f: \mathbb{R} \rightarrow \mathbb{R}$ set

$$\text{support}(f) = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$$

Set

$$C_0(\mathbb{R}) = \{f \in C(\mathbb{R}) : \text{support}(f) \text{ compact set in } \mathbb{R}\}$$

Claim: $C_0(\mathbb{R})$ forms a vector space and for every $p \geq 1$

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{1/p} \text{ defines a norm on } C_0(\mathbb{R}).$$

Problem: $(C_0(\mathbb{R}), \|\cdot\|_{L^p})$, $p \geq 1$ are not Banach spaces.

$(L^1(\mathbb{R}), \|\cdot\|_{L^1})$ is a completion of $(C_0(\mathbb{R}), \|\cdot\|_{L^1})$.

(true for all $p \geq 1$)

Note: $A \subset \mathbb{R}$, A bounded

$$f_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Lebesgue measure of $A = \|f_A\|_{L^1} = \mu(A)$

$A \subset \mathbb{R}$, A unbounded

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(A \cap [-n, n])$$

We say that $A \subset \mathbb{R}$ is a δ -set if for all $\epsilon > 0$, there exists ~~closed~~ open intervals $I_n, n=1, 2, \dots$ such that

$$1) A \subset \bigcup_{n=1}^{\infty} I_n$$

$$2) \sum_{n=1}^{\infty} \text{length}(I_n) < \epsilon$$

In particular

$A = \mathbb{Q} = \{r_n; n=1, 2, \dots\}$ is a δ -set.

Back to Hilbert spaces

- $(E, \langle \cdot, \cdot \rangle)$ inner product space
- $\|x\| = \sqrt{\langle \cdot, \cdot \rangle}$, $x \in E$ norm in E .
- $\|\cdot\|$ satisfies the parallelogram-law.

Note: real case

$$\begin{aligned} \|x+y\|^2 - \|x-y\|^2 &= \langle x+y, x+y \rangle - \langle x-y, x-y \rangle = \\ &= 2\langle x, y \rangle + 2\langle y, x \rangle = 4\langle x, y \rangle \end{aligned}$$

$$\Rightarrow \langle x, y \rangle = \frac{1}{4} (\|x+y\|^2 - \|x-y\|^2)$$

Convergence in $(E, \langle \cdot, \cdot \rangle)$

Strong: $x_n \rightarrow x$ if $\|x-x_n\| \xrightarrow{n \rightarrow \infty} 0$

Weak: $x_n \xrightarrow{\omega} x$ if $\langle x_n - x, y \rangle \xrightarrow{n \rightarrow \infty} 0, \forall y \in E$.

But $x_n \xrightarrow{\omega} x$ and $\|x_n\| \rightarrow \|x\| \Rightarrow x_n \rightarrow x$.

Note:

$$x_n \rightarrow x \Rightarrow \|x_n\| \rightarrow \|x\|$$

$x_n \xrightarrow{\omega} \not\Rightarrow \|x_n\| \rightarrow \|x\|$. But $\sup_n \|x_n\| < \infty$.
(theorem)

$A \subset E$

$A^\perp = \{x \in E : \langle x, a \rangle = 0, \forall a \in A\}$ closed subspace of E .

Orthogonal decomposition theorem

- $(E, \langle \cdot, \cdot \rangle)$ Hilbert space
- S closed subspace of E .



$$E = S \oplus S^\perp$$

i.e. $x = y + z$, y, z unique.

Ex: $A \subset E$, E Hilbert space.

$$\Rightarrow \overline{\text{Span } A} = (A^\perp)^\perp$$

Note:

$$A \subset (A^\perp)^\perp$$

subspace
of E

$$\text{Span } A \subset (A^\perp)^\perp$$

closed

$$\overline{\text{Span } A} \subset (A^\perp)^\perp$$

$$A \subset \overline{\text{Span } A}$$

$$\overline{\text{Span } A}^\perp \subset A^\perp$$

$$(A^\perp)^\perp \subset (\overline{\text{Span } A}^\perp)^\perp$$

Hence

$$\overline{\text{Span } A} \subset (A^\perp)^\perp \subset ((\overline{\text{Span } A})^\perp)^\perp$$

ODT

$$E = \overline{\text{Span } A} \oplus \overline{\text{Span } A}^\perp$$

$$E = \boxed{\overline{\text{Span } A}}^\perp \oplus (\overline{\text{Span } A}^\perp)^\perp$$

$$\Rightarrow \overline{\text{Span } A} = (\overline{\text{Span } A}^\perp)^\perp \Rightarrow (A^\perp)^\perp = \overline{\text{Span } A}$$

Proof of ODT

Step 1:

- S closed convex set in Hilbert space E
 $\Rightarrow \forall x \in E, \exists ! \tilde{y} \in S : \|x - y\| \leq \|x - \tilde{y}\|, \forall \tilde{y} \in S.$
i.e. $\|x - y\| = \inf_{\tilde{y} \in S} \|x - \tilde{y}\|$

Fix $x \notin S$.

$$\inf_{\tilde{y} \in S} \|x - \tilde{y}\| = d > 0.$$

Take $(y_n)_{n=1}^{\infty}$ in S such that $\|x - y_n\|_{n=1}^{\infty} \rightarrow d$.

Claim: $(y_n)_{n=1}^{\infty}$ is a Cauchy sequence.

(Use parallelogram-law for $\|\cdot\|$).

: (check book for details)

Step 2:

S as in statement in ODT. Note that S must be convex.

Fix $x \in E$.

Choose $y \in S : \|x - y\| \leq \|x - \tilde{y}\|, \forall \tilde{y} \in S$.

$$\begin{matrix} & y \\ \oplus & \oplus \\ E & S \end{matrix}$$

To show: $x - y \in S^\perp$. Variational argument.

Show: $\langle x - y, v \rangle = 0, \forall v \in S$

$$\|x - y\|^2 \leq \|x - y + \alpha v\|^2, \text{ all scalars } \alpha.$$

$$\begin{aligned} \|x - y\|^2 &\leq \langle x - y + \alpha v, x - y + \alpha v \rangle = \\ &= \|x - y\|^2 + \alpha \langle v, x - y \rangle + \bar{\alpha} \langle x - y, v \rangle + |\alpha|^2 \|v\|^2 \end{aligned}$$

$$\Rightarrow 0 \leq 2\operatorname{Re}(\bar{\alpha} \langle x - y, v \rangle) + |\alpha|^2 \|v\|^2$$

$$\text{Set } \alpha = t \overline{\langle x - y, v \rangle}, \quad t \in \mathbb{R}$$

$$\Rightarrow 0 \leq 2t |\langle x - y, v \rangle|^2 + t^2 |\langle x - y, v \rangle|^2 \|v\|^2$$

Assume $\langle x-y, v \rangle \neq 0$
(otherwise we are done)

We have

$$0 \leq 2t + t^2 \|v\|^2, \quad \forall t \in \mathbb{R}.$$

$$-2t \leq t^2 \|v\|^2 \quad \text{let } t < 0$$

$$2 \leq -t \|v\|^2$$

Let $t \uparrow 0$

$2 \leq 0$. Contradiction.

Conclusion: $\langle x-y, v \rangle = 0 \Rightarrow x-y \in S^\perp$

Föreläsning 9

- ① Bounded linear functionals on Hilbert spaces; Riesz representation theorem.
 - ② ON-basis in H.S. Examples and properties
 - ③ Linear operators on H spaces.
-

- $(H, \langle \cdot, \cdot \rangle)$ - Hilbert space = inner product space which is complete wrt

$$\|x\| = \sqrt{\langle x, x \rangle}$$

- Let M be a closed subspace of H.

$$M^\perp = \{y \in H : \langle x, y \rangle = 0 \quad \forall x \in M\}$$

- Then we know $H = M \oplus M^\perp$, i.e.

$$\forall x \in H, \exists! y \in M, z \in M^\perp : x = y + z.$$

Theorem: (Riesz-Fréchet repr. theorem)

- $(H, \langle \cdot, \cdot \rangle)$ - Hilbert space
- Let f be a bounded linear functional on H.
- Then $\exists! x_f \in H : f(x) = \langle x, x_f \rangle, \forall x \in H$.
- Moreover $\|f\| = \|x_f\|_H$.

Remark: If $f: H \rightarrow \mathbb{C}$ is of the form

$$f(x) = \langle x, y \rangle \quad \forall x \in H \text{ and some } y \in H,$$

then f is bounded and linear.

- f linear follows from elementary calculations.
- f bounded:

$$|f(x)| = |\langle x, y \rangle| \leq \|x\| \|y\| \Rightarrow \|f\| \leq \|y\|$$

Cauchy
-Schwartz

Proof:

Existence:

~~- Consider $N(f) = \text{Ker } f = \{x \in H : f(x) = 0\}$~~

- If f is the zero linear functional, i.e. $f(x) = 0, \forall x \in H$.

Take $x_0 = 0$.

- Assume now that f is not the zero functional, and consider $N(f) = \text{Ker } f = \{x \in H : f(x) = 0\}$.

- $N(f)$ is a closed subspace of H :

$$x_1, x_2 \in N(f), \alpha, \beta \in \mathbb{C} \Rightarrow$$

$$f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2)$$

Hence $\alpha x_1, \beta x_2 \in N(f)$ and $N(f)$ is a subspace.

$N(f)$ is closed, since if $x_n \in N(f)$, $x_n \rightarrow x$ strongly, then $\underbrace{f(x_n)}_{=0} \rightarrow f(x)$ as f is bounded and hence cont,

giving $f(x) = 0 \Rightarrow N(f)$ is closed.

- $N(f)$ is a proper closed subspace ($N(f) \neq H$).

- Consider $N(f)^\perp$, if it is non-zero.

- $\dim N(f)^\perp = 1$:

Let $x_1, x_2 \in N(f)^\perp$

$$f(x_1), f(x_2) \neq 0$$

$$\exists a \in \mathbb{C} : \underbrace{f(x_1) + a f(x_2)}_{=f(x_1 + ax_2)} = 0$$

giving $x_1 + ax_2 \in N(f) \cap N(f)^\perp = \{0\}$.

$$\text{Hence } x_1 + ax_2 = 0$$

- Any two vectors are linearly dependent in $N(f)^\perp$

$$\Rightarrow \dim N(f)^\perp = 1$$

- Take $y' \in N(f)^\perp$, $\|y'\|=1$ and let

$$x_f = \overline{f(y')} y'$$

Vector we need

$$\langle x, x_f \rangle = \begin{cases} 0, & x \in N(f) \\ \underbrace{\langle \lambda y', f(\overline{y'}) y' \rangle}_{\lambda f(y') \langle y', y' \rangle} & x = \lambda y' \\ \lambda f(y') \langle y', y' \rangle = f(\lambda y') = f(x) \end{cases}$$

- Since any element in H is given by $x + \lambda y'$ for $x \in N(f)$
- Using linearity we get

$$f(x + \lambda y') = f(x) + f(\lambda y') = \langle x, x_f \rangle + \langle \lambda y', x_f \rangle = \langle x + \lambda y', x_f \rangle$$

Uniqueness:

- Assume

$$\exists x_1, x_2 : f(x) = \underbrace{\langle x_1, x_f \rangle}_{\Rightarrow \langle x, x_1 - x_2 \rangle = 0, \forall x \in H} = \langle x, x_2 \rangle \quad \forall x \in H$$

Holds in particular for $x = x_1 - x_2$.

$$\langle x_1 - x_2, x_1 - x_2 \rangle = 0 \Rightarrow x_1 - x_2 = 0.$$

Norms equality: (must see $\|f\| = \|x_f\|_H$)

From remark we have

$$f(x) = \langle x, x_f \rangle \Rightarrow \|f\| \leq \|x_f\| \quad (1)$$

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} \geq \frac{|f(x_f)|}{\|x_f\|} = \frac{\|x_f\|^2}{\|x_f\|} = \|x_f\| \quad (2)$$

$$(1) \& (2) \Rightarrow \|f\| = \|x_f\| \quad \square$$

- Bounded linear functionals on $(E, \|\cdot\|)$

= bounded linear maps

$$f : (E, \|\cdot\|) \rightarrow (\mathbb{C}, |\cdot|)$$

Example:

$E = C_F^{\neq} = \{(x_1, x_2, \dots) \text{ only finite number of } x_i \neq 0\} \subset l^2$

- On C_F consider l^2 -inner product

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i, \quad x, y \in C_F$$

- ① C_F is not a Hilbert space as it is not complete wrt

$$\|x\|_{l^2} = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2}$$

- Find a Cauchy sequence which does not converge to an element in C_F .

- Find a proper closed subspace M such that

$$M^\perp = \{0\}.$$

(This would mean in particular that $C_F \neq M \oplus M^\perp$)

Consider $M = \{(x_1, x_2, \dots) \in C_F : \sum_{k=1}^{\infty} x_k \frac{1}{k} = 0\}$

$$x_f = (1, \frac{1}{2}, \frac{1}{3}, \dots) \in l^2$$

$M = \text{Ker } f \cap C_F$ where $f: l^2 \rightarrow \mathbb{C}$, $f(x) = \sum_{k=1}^{\infty} x_k \frac{1}{k}$

$M^\perp = \text{all elements in } C_F \text{ which are in } (\text{Ker } f)^\perp$.

From the proof of Riesz theorem we had $\dim(\text{Ker } f)$ is 1-dim and

$$x_f \in (\text{Ker } f)^\perp$$

Hence $(\text{Ker } f)^\perp = \{\lambda x_f : \lambda \in \mathbb{C}\}$

We have $\underbrace{(\text{Ker } f)^\perp \cap C_F}_{M^\perp} = \{0\}$

- ② $(H; \langle \cdot, \cdot \rangle)$ -Hilbert space

$\{u_i\} \subset H$ $\{u_i\}$ orthogonal system if

$\begin{cases} \text{finite or} \\ \text{infinite} \end{cases}$ $\langle u_i, u_j \rangle = 0, \forall i \neq j.$

ON-system if

$$\langle u_i, u_j \rangle = \delta_{ij}.$$

Proposition: Orthogonal systems of non-zero vectors are linearly independent.
 - Proof left as exercise

- Having linearly independent family of vectors, we can make orthogonal using Gram-Schmidt.

$\{\sigma_i\}_{i=1}^{\infty}$ - l.i. family

$$u_1 = \sigma_1$$

$$u_2 = \sigma_2 - \frac{\langle \sigma_2, \sigma_1 \rangle}{\langle \sigma_1, \sigma_1 \rangle} \sigma_1 \quad \{u_k\}_{k=1}^{\infty} \text{ orth. system}$$

$$u_3 = \sigma_3 - \frac{\langle \sigma_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle \sigma_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 \Rightarrow \text{Span}\{\sigma_1, \dots, \sigma_k\} =$$

:

$$u_k = \sigma_k - \frac{\langle \sigma_k, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \dots - \frac{\langle \sigma_k, u_{k-1} \rangle}{\langle u_{k-1}, u_{k-1} \rangle} u_{k-1}$$

$\forall k$.

Recall: $\{u_i\}$ - ON-system, $x \in H$

$$\sum_{i=1}^{\infty} \langle x, u_i \rangle u_i \leftarrow \text{formal series}$$

Fourier series of x relative ON-system $\{u_i\}$
 $\langle x, u_i \rangle$ - Fourier coefficients.

$$C([-T, T]), \{u_k\} = \left\{ \frac{1}{\sqrt{2\pi}} e^{ikt}, k \in \mathbb{Z} \right\}$$

-ON-system : $(C([-T, T]), \langle \cdot, \cdot \rangle)$

$$\langle f, g \rangle = \int_{-T}^T f(t) \overline{g(t)} dt$$

$$\langle f, u_k \rangle = \hat{f}(k) \leftarrow \text{usual FT-coeff.}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt$$

- Want to see when $\sum_{i=1}^{\infty} \langle x, u_i \rangle u_i$ is convergent to x .

Definition: An ON-system is called an ON-basis for H if its span is dense in H .

We say that ON-system is complete if

every $x \in H$ is $\sum_{i=1}^{\infty} \langle x, u_i \rangle u_i$

Theorem:

• $(H, \langle \cdot, \cdot \rangle)$ Hilbert space, $\{u_k\}$ - ON-system in H .

Following are equivalent

- ① $\{u_n\}$ - complete ON-system $\left(\Leftrightarrow x = \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k\right)$
- ② $\{u_n\}$ is an ON-basis for H .
- ③ (Parseval's identity) $\|x\| = \left(\sum_{k=1}^{\infty} |\langle x, u_k \rangle|^2 \right)^{1/2} \quad \forall x \in H$
- ④ $\langle x, y \rangle = \sum_{k=1}^{\infty} \langle x, u_k \rangle \overline{\langle y, u_k \rangle}, \quad \forall x, y \in H$
- ⑤ $\langle x, u_k \rangle = 0, \forall k \Rightarrow x = 0.$

Proof: ① \Rightarrow ②

We have $x = \sum_{i=1}^{\infty} \langle x, u_i \rangle u_i$, it means $x = \lim \sum_{i=1}^n \langle x, u_i \rangle u_i$
 \Rightarrow any $x \in H$ is in $\overline{\text{Span}\{u_i, i \geq 1\}}$, i.e. $\{u_i\}$ is ON-basis. $\hookrightarrow \text{Span}(u_i, i \geq 1)$

② \Rightarrow ⑤ let $\{u_i\}$ be ON-basis.

Assume $\langle x, u_k \rangle = 0 \quad \forall k$. Then $\langle x, u \rangle = 0, \forall u \in \text{Span}\{u_k, k \geq 1\}$

By the property of strong convergence implies weak convergence (i.e. if $\|y_n - y\| \rightarrow 0 \Rightarrow \langle x, y_n \rangle \rightarrow \langle x, y \rangle$)

We will have $\langle x, u \rangle = 0 \quad \forall u \in \overline{\text{Span}\{u_k, k \geq 1\}} = H$.

In particular $\langle x, u \rangle = 0$ for $u = x$, i.e.

$$\langle x, x \rangle = 0 \Leftrightarrow x = 0.$$

⑤ \Rightarrow ①

Bessel's inequality: $\{u_k\}$ ON-system, then

$$\|x - \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k\|^2 = \|x\|^2 - \sum_{k=1}^{\infty} |\langle x, u_k \rangle|^2$$

Assume ⑤, i.e. $\langle x, u_k \rangle = 0 \quad \forall k \Rightarrow x = 0$.

Must see: $x = \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k, \quad \forall x \in H$.

From Bessel's inequality we have

$$\sum_{k=1}^n |\langle x, u_k \rangle|^2 = \|x\|^2 - \|x - \sum_{k=1}^n \langle x, u_k \rangle u_k\|^2 \leq \|x\|^2, \forall n$$

and hence

$$\sum_{k=1}^{\infty} |\langle x, u_k \rangle|^2 \text{ is convergent } \leq \|x\|^2 \quad (\times)$$

Pythagorean

This implies that

$$\begin{aligned} & \left\| \sum_{k=1}^n \langle x, u_k \rangle u_k - \sum_{k=1}^m \langle x, u_k \rangle u_k \right\|^2 = \left\| \sum_{k=m+1}^n \langle x, u_k \rangle u_k \right\|^2 \\ & \quad (n > m) \\ & = \sum_{k=m+1}^n |\langle x, u_k \rangle|^2 \|u_k\|^2 = \sum_{k=m+1}^n |\langle x, u_k \rangle|^2 \xrightarrow[m \rightarrow \infty]{\text{by (x)}} 0 \end{aligned}$$

Hence the partial sums $s_n = \sum_{k=1}^n \langle x, u_k \rangle u_k$ is a Cauchy seq.
As H is a Hilbert space we know that s_n has a limit in H.

Write $\sum_{i=1}^{\infty} \langle x, u_i \rangle u_i$ for the limit, must see that it is x.

Consider $y = x - \sum_{i=1}^{\infty} \langle x, u_i \rangle u_i$. Then

$$\langle y, u_i \rangle = \langle x, u_i \rangle - \langle x, u_i \rangle = 0 \quad \forall i$$

By ⑤ we have $y = 0$, hence $x = \sum_{i=1}^{\infty} \langle x, u_i \rangle u_i$.

① \Rightarrow ③ From Bessel's equality we have

$$\|x - \sum_{i=1}^n \langle x, u_i \rangle u_i\|^2 = \|x\|^2 - \sum_{i=1}^n |\langle x, u_i \rangle|^2$$

Assuming ①, LHS $\rightarrow 0$ as $n \rightarrow \infty$

On the other hand RHS $\rightarrow \|x\|^2 - \sum_{i=1}^{\infty} |\langle x, u_i \rangle|^2$

which gives Parseval's identity.

③ \Rightarrow ⑤ Trivial.

④ \Rightarrow ③ Trivial. (take $y = x$)

① \Rightarrow ④ We have $x = \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k$

Consider Then

$$\langle x, y \rangle = \sum_{k=1}^{\infty} \langle x, u_k \rangle \langle u_k, y \rangle = \sum_{k=1}^{\infty} \langle x, u_k \rangle \overline{\langle y, u_k \rangle}$$

PF

Example:

$$\cdot L^2([-n,n]) \quad \langle f, g \rangle = \int_{-n}^n f(t) \overline{g(t)} dt$$

$\left\{ e^{int}, n \in \mathbb{Z} \right\}$ ON-system in $L^2([-n,n])$

Statement: The system is an ON-basis, for $L^2([-n,n])$

(In particular, for any $f \in L^2([-n,n])$)

for $\sum_{k \in \mathbb{Z}} f(k) e^{ikt}$ the convergence in L^2 -norm.

$$\Leftrightarrow \|f - \sum_{k=-n}^n f(k) e^{ikt}\|_2 \rightarrow 0.$$

Sketch of proof:

① Stone-Weierstrass theorem

- X compact set
- ~~continuous~~ $C(X, \mathbb{C})$ continuous functions with complex values.
- let $M \subset C(X, \mathbb{C})$
 - subspace that satisfies

① it separates points of X , i.e.

$$\forall x_1, x_2 \in X \quad \exists f \in M : f(x_1) \neq f(x_2)$$

$x_1 \neq x_2$

② M contains ~~the~~ constant function 1.

$$(f(x) = 1 \quad \forall x \in X)$$

③ If ~~M~~ is closed under complex conjugation, i.e.

$$f \in M \rightarrow \bar{f} \in M,$$

and closed under product, i.e.

$$f_1, f_2 \in M \rightarrow f_1 f_2 \in M$$

Then M is dense in $C(X, \mathbb{C})$ w.r.t $\|\cdot\|_\infty$.

- From this it follows that $M = \{\text{all complex polynomials}\}$ are dense in $C([a,b])$.

② $C([a,b])$ is dense in $L^2([a,b])$ w.r.t $\|\cdot\|_2$ -norm.

We shall use ① and ② to show that

$\text{Span}\left\{\frac{1}{\sqrt{2\pi}}e^{int}, n \in \mathbb{Z}\right\}$ is dense in $L^2([-π, π])$.

Let $M = \text{Span}\left\{\frac{1}{\sqrt{2\pi}}e^{int}, n \in \mathbb{Z}\right\} \subset \{f \in C([-π, π], \mathbb{C}) : f(π) = f(-π)\}$



- M separates points.

- M contains the constant funct. 1.

- M closed under complex conj.
closed under products

$$= \{f \in C(X, \mathbb{C})\}$$

Hence by Stone-Weierstrass,

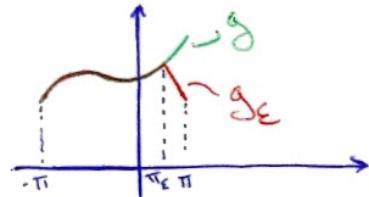
M is dense in $\{f \in C([-π, π], \mathbb{C}) : f(π) = f(-π)\}$.

By ② we have $C([-π, π])$ is dense in $L^2([-π, π])$ wrt $\| \cdot \|_2$.

- From this one can see that even

$\{f \in C([-π, π], \mathbb{C}) : f(π) = f(-π)\}$ is dense in $L^2([-π, π])$

- $\forall \varepsilon > 0, \forall f \in L^2 \exists g \in C([-π, π]) : \|f - g\|_2^2 = \int_{-π}^π |f(t) - g(t)|^2 dt < \varepsilon$



$$g_\varepsilon \in C([-π, π]), \quad g_\varepsilon(-π) = g_\varepsilon(π)$$

$$\|f - g\|_2 \leq \|f - g\|_c + \|g - g_\varepsilon\|_2$$

- We conclude

any $f = L^2$ -limit $g_n, g_n \in C([-π, π]),$
 $g_n(\pi) = g_n(-\pi)$

Each $g_n = \| \cdot \|_\infty$ -norm limit of

element in

$$\text{Span}\left\{\frac{1}{\sqrt{2\pi}}e^{int}, n \in \mathbb{Z}\right\}$$

as $\|g - f\|_2 \leq \|g - f\|_\infty (2\pi)^{1/2}$

We get each g_n can be approximated in L^2 -norm by elements in $\text{Span}\left\{\frac{1}{\sqrt{2\pi}}e^{int}, n \in \mathbb{Z}\right\}$.

Hence $\text{Span}\{\dots\} = L^2([-π, π]).$

$$\begin{aligned} & \int_{-π}^π |g(t) - g_\varepsilon(t)|^2 dt = \\ & = \int_{π-ε}^π |g(t) - g_\varepsilon(t)|^2 dt \leq \\ & \leq \max_{[-π, π]} |g - g_\varepsilon| \cdot \varepsilon \end{aligned}$$

③ linear operators on Hilbert spaces

• $(H_1, \langle \cdot, \cdot \rangle), (H_2, \langle \cdot, \cdot \rangle)$

A bounded linear map $A: H_1 \rightarrow H_2$ is called a bounded linear operator.

Example: $(H_1 = H_2 = L^2([0,1]), K: [0,1] \times [0,1] \rightarrow \mathbb{C})$
Assume K continuous.

$$(Af)(x) = \int_0^1 K(x,y) f(y) dy$$

• A is linear.

• A is bounded

$$\begin{aligned} \|Af\|_2^2 &= \int_0^1 \left| \int_0^1 K(x,y) f(y) dy \right|^2 dx \stackrel{\text{Cauchy-Schwartz}}{\leq} \int_0^1 \left(\int_0^1 |K(x,y)|^2 dy \cdot \int_0^1 |f(y)|^2 dy \right) dx = \\ &= \underbrace{\int_0^1 \left(\int_0^1 |K(x,y)|^2 dy \right) dx}_{\text{finite}} \cdot \underbrace{\int_0^1 |f(y)|^2 dy}_{\|f\|_2^2} \end{aligned}$$

Hence $\|A\| \leq \left(\int_0^1 \int_0^1 |K(x,y)|^2 dx dy \right)^{1/2}$.

Products of operators $H \rightarrow H$

$$A: H \rightarrow H, B: H \rightarrow H$$

Define AB by $(AB)f \stackrel{\text{def}}{=} A(Bf)$

Statement: If A, B bounded, then

$A \cdot B$ also bounded and ~~and~~ $\|A \cdot B\| \leq \|A\| \|B\|$

• In particular,

$\forall n \in \mathbb{N}$ A^n is bounded and

$$\|A^n\| \leq \|A\|^n$$

Föreläsning 10

- $(E, \|\cdot\|)$ Banach space (e.g. Hilbert space)
- $A \in \mathcal{B}(E, E)$ bounded linear operator
- $\|A\| = \sup_{\|x\|=1} \|A(x)\|$
- $A, B \in \mathcal{B}(E, E)$
- $(AB)(x) = A(B(x)), \quad x \in E \quad \cdot \quad \|AB\| \leq \|A\| \|B\|$

In particular $A^n = \underbrace{AA\dots A}_{n \text{ times}}$
 $\|A^n\| = \|A\|^n, \quad n=1, 2, \dots$

Example: $E = L^2([0,1]) \ni f, g$

$$\langle f, g \rangle_{L^2} = \int_0^1 f(x) \overline{g(x)} dx$$

$$\|f\|_{L^2} = \left(\int_0^1 |f(x)|^2 dx \right)^{1/2}$$

$$k(x, y) \in C([0,1] \times [0,1])$$

$$A(f)(x) = \int_0^1 k(x, y) f(y) dy, \quad x \in [0,1] \quad \text{for } f \in L^2([0,1])$$

$$\|A\| \leq \left(\int_0^1 \left(\int_0^1 |k(x, y)|^2 dy \right)^{1/2} dx \right)^{1/2} < \infty$$

Example: ^(xx) $(E, \|\cdot\|)$ normed space

\Rightarrow there are no $A, B \in \mathcal{B}(E, E)$ such that

$$AB - BA = I, \quad I(x) = x, \quad x \in E.$$

Remark: Consider

$$E = C^\infty((0,1)) \ni f$$

$$A = \frac{d}{dx}, \quad B = x$$

$$(AB - BA)(f) = \frac{d}{dx}(xf(x)) - x \frac{d}{dx} f(x) = f(x)$$

Argue by contradiction.

- Assume $A, B \in \mathcal{B}(E, E)$ with $AB - BA = I$.

Hint: $A^nB - BA^n$, $n=1, 2, \dots$ ← consider

$$\begin{aligned} n=2: A^2B - BA^2 &= (A^2B - ABA) + (ABA - BA^2) = \\ &= A(\underbrace{AB - BA}_{=I}) + (\underbrace{AB - BA}_{=I})A = 2A \end{aligned}$$

$$\begin{aligned} n=3: A^3B - BA^3 &= (A^3B - A^2BA) + (A^2BA - BA^3) = \\ &= A^2(\underbrace{AB - BA}_{=I}) + (\underbrace{AB - BA}_{=2A})A = 3A^2 \end{aligned}$$

In general

$$[A^nB - BA^n = nA^{n-1}, n=1, 2, \dots] (*)$$

- Check using an induction argument.

We obtain

$$\begin{aligned} n\|A^{n-1}\| &= \|A^nB - BA^n\| \leq \|A^nB\| + \|BA^n\| \leq \\ &\leq 2\|A^{n-1}\|\|A\|\|B\| \end{aligned}$$

Hence

$$(2\|A\|\|B\| - n)\|A^{n-1}\| \geq 0, n=2, 3, \dots$$

We conclude that

$$\|A^{n-1}\|=0 \text{ for } n \text{ large enough, i.e.}$$

$$A^n = 0 \text{ for } n \text{ large enough.}$$

Repeated use of (*) gives

$$A = 0.$$

This contradicts $AB - BA = I$.

- The implication in (xx) is proven.

Riesz representation theorem:

- $(E, \langle \cdot, \cdot \rangle)$ Hilbert space.

- $f \in \mathcal{B}(E, \mathbb{C})$, i.e. f is a bounded linear functional on E .

$$\Rightarrow \exists! x_f \in E \quad f(x) = \langle x, x_f \rangle, \forall x \in E.$$

Also: $\|f\| = \|x_f\|$ norm of x_f in E .
operator norm of

Generalization of R.R.T.

Lax-Milgram theorem

- $(E, \langle \cdot, \cdot \rangle)$ Hilbert space
 - $\varphi : E \times E \rightarrow \mathbb{C}$ is a bilinear, bounded, coercive functional.
 - $f : E \rightarrow \mathbb{C}$ bounded linear functional on E .
- $\Rightarrow \exists ! \tilde{x}_f \in E : f(x) = \varphi(x, \tilde{x}_f), \forall x \in E.$
-

Here

$$\varphi : E \times E \rightarrow \mathbb{C}$$

• Bilinear $\varphi(\alpha x + \beta y, z) = \alpha \varphi(x, z) + \beta \varphi(y, z), \quad \begin{cases} \forall x, y, z \in E \\ \alpha, \beta \text{ scalars} \end{cases}$

$$\varphi(x, \alpha y + \beta z) = \bar{\alpha} \varphi(x, y) + \bar{\beta} \varphi(x, z)$$

• Bounded

$$\exists M > 0 : |\varphi(x, y)| \leq M \|x\| \|y\|, \forall x, y \in E$$

• Coercive

$$\exists k > 0 : \varphi(x, x) \geq k \|x\|^2, \forall x \in E.$$

Clearly, $\langle \cdot, \cdot \rangle$ on E is a bounded, bilinear, coercive functional on E (with $M=k=1$).

Proof:

Step 1: $\exists ! A \in \mathcal{B}(E, E) : \varphi(x, y) = \langle x, A(y) \rangle, \forall x, y \in E$

Step 2: 1-1 and onto.

Step 3: Apply R.R.T

$$f(x) = \langle x, x_f \rangle = \langle x, A(A^{-1}(x_f)) \rangle = \varphi(x, \tilde{x}_f), \forall x \in E$$

$$\tilde{x}_f = A^{-1}(x_f)$$

Step 1: Fix $y \in E$. Consider

$$E \ni x \mapsto \varphi(x, y) \in \mathbb{C}$$

Claim: $f_y : E \rightarrow \mathbb{C}$ bounded linear functional.

$$f_y(\alpha x + \beta z) = \varphi(\alpha x + \beta z, y) = \alpha \varphi(x, y) + \beta \varphi(z, y) = \alpha f_y(x) + \beta f_y(z)$$

$\Rightarrow f_y$ linear

$$\forall x, z \in E \\ \text{scalars } \alpha, \beta$$

$$|f_y(x)| = |\varphi(x, y)| \leq (M \|y\|) \|x\|, \forall x \in E$$

$\Rightarrow f_y$ bounded.

• RRT implies

$$f_y(x) = \langle x, A(y) \rangle, \forall x \in E \text{ for some } A(y) \in E$$

Now we have $A : E \rightarrow E$.

Claim: $A \in \mathcal{B}(E, E)$

For $x, y, z \in E$, α, β scalars.

$$\begin{aligned} \langle x, A(\alpha y + \beta z) \rangle &= \varphi(x, \alpha y + \beta z) = \bar{\alpha} \varphi(x, y) + \bar{\beta} \varphi(x, z) = \\ &= \bar{\alpha} \langle x, A(y) \rangle + \bar{\beta} \langle x, A(z) \rangle = \langle x, \alpha A(y) + \beta A(z) \rangle \\ &\Rightarrow \langle x, A(\alpha y + \beta z) - \alpha A(y) - \beta A(z) \rangle = 0, \forall x \in E \end{aligned}$$

Implies

$$\|A(\alpha y + \beta z) - \alpha A(y) - \beta A(z)\| = 0$$

$$\Rightarrow A(\alpha y + \beta z) = \alpha A(y) + \beta A(z), \forall y, z \in E, \alpha, \beta \text{ scalars.}$$

So A linear.

A bounded:

$$|\varphi(x, y)| \leq M \|x\| \cdot \|y\|, \forall x, y \in E \quad | \quad |\varphi(x, y)| = |\langle x, A(y) \rangle|$$

Take $x = A(y)$

$$\|A(y)\|^2 \leq M \|A(y)\| \|y\|, \forall y \in E$$

$$\Rightarrow \|A(y)\| \leq M \|y\|, \forall y \in E$$

$$\Rightarrow \|A\| \leq M < \infty. \quad \underline{\text{Step 1 done!}}$$

Step 2:

Note $\Psi(x, y) = \langle x, A(y) \rangle$, $x, y \in E$.

Claim: A is 1-1, i.e. $A(x_1) = A(x_2) \Rightarrow x_1 = x_2$

• Ψ coercive, i.e. $\Psi(x, x) \geq k \|x\|^2$, $\forall x \in E$.

$$\text{So } \|x\|^2 \leq \frac{1}{k} \Psi(x, x) = \frac{1}{k} |\langle x, A(x) \rangle| \leq \frac{1}{k} \|x\| \|A(x)\|, \forall x \in E.$$

Cauchy -
Schwartz

Hence

$$\|x\| \leq \frac{1}{k} \|A(x)\| \quad \forall x \in E$$

If $A(x_1) = A(x_2)$ i.e. $A(x_1 - x_2) = 0 \in E$,

$$\text{then } (+) \|x_1 - x_2\| \leq \frac{1}{k} \|A(x_1 - x_2)\| = 0 \Rightarrow x_1 = x_2.$$

Claim: A is onto, $\underline{R(A)} = E$
 $= \{A(x) : x \in E\}$

First show that

$R(A)$ is a closed subspace in E .

(x) $\circ R(A)$ is a subspace in E , since A linear

• $R(A)$ closed since

$$R(A) \ni y_n \rightarrow y \text{ in } (E, \|\cdot\|) \\ \Rightarrow y \in R(A)$$

Assume.

To show: $y \in R(A)$

For $n = 1, 2, \dots$ there are x_1, x_2, \dots such that

$$y_n = A(x_n), n=1, 2, \dots$$

Claim: $(x_n)_{n=1}^{\infty}$ is a Cauchy seq. in E .

Since

$$\|x_n - x_m\| \leq \{(+)\} \leq \frac{1}{k} \|A(x_n - x_m)\| = \{A \text{ linear}\} = \frac{1}{k} \|A(x_n) - A(x_m)\| = \\ = \frac{1}{k} \|y_n - y_m\| \xrightarrow{n, m \rightarrow \infty} 0 \quad \text{since } y_n \rightarrow y.$$

(x) $R(A)$ linear
 $y_1, y_2 \in R(A)$, i.e.
 $y_1 = A(x_1), y_2 = A(x_2)$
 α_1, α_2 scalars
 $x_1, x_2 \in E$
 $\alpha_1 y_1 + \alpha_2 y_2 = \alpha_1 A(x_1) + \alpha_2 A(x_2)$
 $= \{\text{linear}\} = A(\alpha_1 x_1 + \alpha_2 x_2)$
 $\Rightarrow \alpha_1 x_1 + \alpha_2 x_2 \in R(A)$

Since $(E, \|\cdot\|)$ is a Banach space, $(x_n)_{n=1}^{\infty}$ converges in $(E, \|\cdot\|)$. Call the limit $x \in E$.

Hence

$\boxed{(E, \|\cdot\|) \text{ in } y \leftarrow y_n = A(x_n) = \{x_n \rightarrow x \text{ in } (E, \|\cdot\|) \text{ and } A \text{ bounded linear: } E \rightarrow E \text{ and hence continuous}\} \rightarrow A(x) \text{ in } (E, \|\cdot\|)}$

So $y = A(x)$. We have $y \in R(A)$. Secondly A is onto, i.e $R(A) = E$.

Assume that this is not true, then we can apply ODT which gives

$$E = \underbrace{R(A)}_{\substack{\text{closed} \\ \text{subspace}}} \oplus \underbrace{R(A)^\perp}_{\neq 0}$$

Fix $z \in R(A)^\perp \setminus \{0\}$. Note

$$\varphi(x, y) = \langle x, A(y) \rangle \quad \forall x, y \in E$$

with $x = y = z$, then

$$\begin{aligned} \varphi(z, z) &= \langle z, \underbrace{A(z)}_{\in R(A)^\perp} \rangle = 0 \\ &\in R(A)^\perp \cap R(A) \end{aligned}$$

$$\varphi(z, z) \geq \underbrace{k \|z\|^2}_{\geq 0} \geq 0 \Rightarrow \|z\| = 0 \Rightarrow z = 0$$

Contradiction! Conclusion: $R(A)^\perp = \{0\}$ & $R(A) = E$.

We have $\varphi(x, y) = \langle x, A(y) \rangle \quad \forall x, y \in E$, $A \in \mathcal{B}(E, E)$ 1-1 & onto.

Step 3: See the theorem statement! \square

Definition: (Adjoint operator): $\langle A(x), y \rangle = \langle x, A^*(y) \rangle \quad \forall x, y \in E$

- $(E, \langle \cdot, \cdot \rangle)$ Hilbert space
- $A \in \mathcal{B}(E, E)$
- Fix $y \in E$ and consider
 $E \ni x \xrightarrow{ly} \langle A(x), y \rangle \in \mathbb{C}$

Claim: ly is a bounded linear functional on E .

- linear since A is linear
- Bounded since $|ly(x)| \leq (\|A\| \|y\|) \|x\|, x \in E$

RRT implies

$$l_y(x) = \langle x, A^*(y) \rangle \quad \text{call it}, \quad x \in E$$

$\langle A(x), y \rangle \quad | \text{we have } A^*: E \rightarrow E : \langle A(x), y \rangle = \langle x, A(y) \rangle \quad \forall x, y \in E.$

Proposition:

$$A \in \mathcal{B}(E, E) \Rightarrow A^* \in \mathcal{B}(E, E) \text{ and } \|A^*\| = \|A\|$$

Proof: \square A^* linear ... $\langle x, A^*(\alpha y + \beta z) \rangle = \langle x, \alpha A^*(y) + \beta A^*(z) \rangle \quad \left\{ \begin{array}{l} \forall x, y, z \in E \\ \alpha, \beta \text{ scalars} \end{array} \right.$

\square A^* bounded: $\langle x, A^*(y) \rangle = \langle A(x), y \rangle \quad \forall x, y \in E$

Take $x = A^*(y)$, then

$$\|A^*(y)\|^2 = |\langle A(A^*(y)), y \rangle| \leq \|A(A^*(y))\| \|y\| \leq \|A\| \|A^*(y)\| \|y\| \quad y \in E.$$

(*) $\|A^*(y)\| \leq \|A\| \|y\| \quad \forall y \in E$. Conclusion: $A^* \in \mathcal{B}(E, E)$

(x) implies that

$$\|A^*\| \leq \|A\|$$

But $A^{**} = A$ since $\langle x, A^{**}(y) \rangle = \underbrace{\langle A^*(x), y \rangle}_{\substack{\sim \\ \text{as } A \in \mathcal{B}(E, E)}} = \dots = \langle x, A(y) \rangle \quad \forall x, y \in E.$

So

$$\|A\| = \|A^{**}\| \leq \|A^*\|$$

$$\Rightarrow \|A^*\| = \|A\|$$

Remark: $A, B \in \mathcal{B}(E, E), \alpha \in \mathbb{C}$

- $(A + B)^* = A^* + B^*$
- $(\alpha A)^* = \bar{\alpha} A^*$
- $(AB)^* = B^* A^*$
- $A^{**} = A$
- $I^* = I$

Example:

$$A(f)(x) = \int_0^1 k(x,y) f(y) dy, \quad x \in [0,1] \text{ for } f \in L^2([0,1])$$

$$\begin{aligned} \langle A(f), g \rangle_{L^2} &= \int_0^1 A(f)(x) \overline{g(x)} dx = \int_0^1 \int_0^1 k(x,y) f(y) dy \overline{g(x)} dx = \\ &= \int_0^1 f(y) \overline{\int_0^1 k(x,y) g(x) dx} dy = \langle f, A^*(g) \rangle_{L^2} \end{aligned}$$

This gives

$$A^*(f)(x) = \int_0^1 \overline{k(x,y)} f(y) dy, \quad x \in [0,1]$$

Example: $A \in \mathcal{B}(E, E)$

$$\Rightarrow \mathcal{R}(A)^\perp = N(A^*) = \{x \in E : A^*(x) = 0\}$$

Since

$$x \in \mathcal{R}(A)^\perp \iff \langle x, A(y) \rangle = 0, \quad \forall y \in E$$

$$\iff \langle A^*(x), y \rangle = 0, \quad \forall y \in E$$

$$\iff A^*(x) = 0 \iff x \in N(A^*)$$

$$\Rightarrow N(A^*)^\perp = \overline{\mathcal{R}(A)}$$

Since

$$\begin{aligned} N(A^*)^\perp &= (\mathcal{R}(A)^\perp)^\perp = \overline{\text{Span}(\mathcal{R}(A))} = \left\{ \mathcal{R}(A) \text{ subspace of } E \right\} \\ &\quad \uparrow \\ &\quad C \subset E \\ &\quad C^\perp = \overline{\text{Span } C} \\ &\quad \text{Span } \mathcal{R}(A) = \mathcal{R}(A) \\ &\quad = \overline{\mathcal{R}(A)} \end{aligned}$$

- $A \in \mathcal{B}(E, E)$ is called self-adjoint if

$$\boxed{A^* = A}$$

$A \in \mathcal{B}(E, E)$

$$\Rightarrow \|A\| = \sup_{\substack{\|x\|=1 \\ \|y\|=1}} |\langle A(x), y \rangle|$$

since

$$|\langle A(x), y \rangle| \leq \|A(x)\| \|y\| \leq \|A\| \quad \text{for } \|x\| = \|y\| = 1$$

C.S $\leq \|A\| \|x\|$

If $A(x) = 0$, $\forall x \in E$ then

$$\|A\| = 0 \text{ and also } \sup_{\substack{\|x\|=1 \\ \|y\|=1}} |\langle A(x), y \rangle| = 0$$

For x with $A(x) \neq 0$ then $A\left(\frac{1}{\|A(x)\|}x\right) \neq 0$

For such an x with $\|x\|=1$ we have

$$|\langle A(x), \frac{1}{\|A(x)\|}A(x) \rangle| = \frac{1}{\|A(x)\|} \|A(x)\|^2 = \|A(x)\| \quad (\text{why?})$$

So

$$\|A\| = \sup_{\substack{\|x\|=1 \\ \|y\|=1}} \|A(x)\| \leq \sup_{\substack{\|x\|=1 \\ \|y\|=1}} |\langle A(x), y \rangle| \leq \|A\|$$

Proposition: $A \in \mathfrak{B}(E, E)$ self-adjoint

$$\Rightarrow \|A\| = \sup_{\|x\|=1} |\langle A(x), x \rangle|$$

Proof: Set $M = \sup_{\|x\|=1} |\langle A(x), x \rangle|$

For $\|x\|=1$

$$|\langle A(x), x \rangle| \leq \underbrace{\|A(x)\|}_{\leq \|A\|} \underbrace{\|x\|}_{=1} \leq \|A\|$$

Hence $M \leq \|A\|$

Remains to prove $\|A\| \leq M$.

For $x, z \in E$ consider

$$\begin{aligned} & \underbrace{\langle A(x+z), x+z \rangle}_{A(x)+A(z)} - \underbrace{\langle A(x-z), x-z \rangle}_{A(x)-A(z)} = \\ & \quad \text{self-adjoint} \\ & = 2\langle A(x), z \rangle + 2\langle A(z), x \rangle = 2 \left(\underbrace{\langle A(x), z \rangle}_{-\langle z, A(x) \rangle} + \underbrace{\langle z, A^*(x) \rangle}_{=\langle A(x), z \rangle} \right) = \\ & = 4 \operatorname{Re} \langle A(x), z \rangle \end{aligned}$$

Assume $A(x) \neq 0$. Set $z = \frac{1}{\|A(x)\|} A(x)$

Hence

$$\begin{aligned} \|A(x)\| &= \frac{1}{4} \left(\underbrace{\langle A(x + \frac{1}{\|A(x)\|} A(x)), x + \frac{1}{\|A(x)\|} A(x) \rangle}_{=2} - \right. \\ &\quad \left. - \langle A(x - \frac{1}{\|A(x)\|} A(x)), x - \frac{1}{\|A(x)\|} A(x) \rangle \right) \end{aligned}$$

Note for $y \neq 0$

$$|\langle A(y), y \rangle| = \|y\|^2 |\underbrace{\langle A\left(\frac{1}{\|y\|}y\right), \frac{1}{\|y\|}y \rangle}_{\leq M}| \leq M\|y\|^2$$

We now obtain

$$\begin{aligned} \|A(x)\| &\leq \frac{1}{4} \left(M\|x + \frac{1}{\|A(x)\|} A(x)\|^2 + M\|x - \frac{1}{\|A(x)\|} A(x)\|^2 \right) = \{\text{Parallelogram law}\} = \\ &= \frac{M}{4} 2 \left(\|x\|^2 + \left\| \frac{1}{\|A(x)\|} A(x) \right\|^2 \right) = \frac{M}{2} (\|x\|^2 + 1) \end{aligned}$$

$$\text{So } \|A\| = \sup_{\|x\|=1} \|A(x)\| \leq M$$

We have

$$\|A\| = M \quad \square$$

Föreläsning 11

Recall:

- $(E, \langle \cdot, \cdot \rangle)$ Hilbert space
- $A \in \mathcal{B}(E, E)$
- A^* defined by $\langle A(x), y \rangle = \langle x, A^*(y) \rangle$
- $A^* \in \mathcal{B}(E, E)$ with $\|A\|^* = \|A\|$
- $A, B \in \mathcal{B}(E, E)$
 - $(A + B)^* = A^* + B^*$
 - $(\lambda A)^* = \bar{\lambda} A^*$
 - $(AB)^* = B^* A^*$
 - $A^{**} = A$
 - $I^* = I$

A self-adjoint if
 $A^* = A$

Example: Integral operator

$$E = L^2([0,1])$$

$$\langle f, g \rangle = \int_0^1 f(x) \overline{g(x)} dx$$

$$A(f)(x) = \int_0^1 k(x,y) f(y) dy, \quad x \in [0,1]$$

where $k \in C([0,1] \times [0,1])$

$$\|A\| \leq \|k\|_{L^2([0,1] \times [0,1])}$$

$$A^*(f)(x) = \int_0^1 \overline{k(y,x)} f(y) dy, \quad x \in [0,1]$$

A is self-adjoint if $k(x,y) = \overline{k(y,x)}$, $\forall x,y \in [0,1]$

A self-adjoint

$$\Rightarrow \|A\| = \sup_{\|x\|=1} |\langle A(x), x \rangle|$$

Compact operators on Banach/Hilbert spaces

• $(E, \|\cdot\|)$ Banach space (e.g. Hilbert space)

• $A : E \rightarrow E$ linear

We say that A is compact if for each bounded sequence $(x_n)_{n=1}^{\infty}$ in E , $(A(x_n))_{n=1}^{\infty}$ has a bounded subsequence in E .

Lemma: A compact linear
 $\Rightarrow A$ bounded

since if A not bounded

$$\exists (y_n)_{n=1}^{\infty} \in E : \|A(y_n)\| \geq n \|y_n\|, n=1,2,\dots$$

Set

$$x_n = \frac{1}{\|y_n\|} y_n, \quad n=1,2,\dots$$

Here $\|x_n\|=1, \forall n$ and

$$\|A(x_n)\| = \|A\left(\frac{1}{\|y_n\|} y_n\right)\| = \frac{1}{\|y_n\|} \|A(y_n)\| \geq n, \quad \forall n.$$

$(A(x_n))_{n=1}^{\infty}$ has no converging subsequence since $\|A(x_n)\| \xrightarrow{n \rightarrow \infty} \infty$.

Remark: $A \in \mathcal{B}(E, E)$

$F \subset E$, F bounded $\Rightarrow A(F) = \{A(x) : x \in F\}$ is bounded,
e.g. $F = B(0, R) \Rightarrow A(F) \subset B(0, \|A\|R)$.

$A \in \mathcal{B}(E, E)$ and compact \hookrightarrow closure

$F \subset E$, F bounded $\Rightarrow \overline{A(F)}$ is compact.

Lemma: A, B compact linear operators $E \rightarrow E$, α, β scalars
 $\Rightarrow \alpha A + \beta B$ is compact.

Proof: Fix an arbitrary bounded sequence $(x_n)_{n=1}^{\infty}$ in E .

A compact \Rightarrow there exists a converging subsequence

$(A(x_{n_k}))_{k=1}^{\infty}$ of $(A(x_n))_{n=1}^{\infty}$. Clearly $(\alpha A(x_{n_k}))_{k=1}^{\infty}$ converges in E .

Same argument \Rightarrow for B . Hence

$$(\alpha A(x_{n_k}) + \beta B(x_{n_k}))_{k=1}^{\infty} = ((\alpha A + \beta B)(x_{n_k}))_{k=1}^{\infty} \text{ converges in } E. \quad \square$$

Set $\mathcal{K}(E, E) = \text{set of all compact linear mappings } E \rightarrow E$.
We have

$\mathcal{K}(E, E)$ is a subspace in $(\mathcal{B}(E, E), \| \cdot \|_{E \rightarrow E})$

Proposition: $\mathcal{K}(E, E)$ is a closed subspace in $(\mathcal{B}(E, E), \| \cdot \|_{E \rightarrow E})$

• Before the proof we note:

Assume $(E, \langle \cdot, \cdot \rangle)$ Hilbert space

1) $A \in \mathcal{B}(E, E)$

$$\begin{aligned} x_n \rightarrow x \text{ in } E &\Rightarrow A(x_n) \rightarrow A(x) \text{ in } E \\ x_n \xrightarrow{\omega} x \text{ in } E &\Rightarrow A(x_n) \xrightarrow{\omega} A(x) \text{ in } E \end{aligned}$$

since for $y \in E$

$$\langle A(x_n), y \rangle = \langle x_n, A^*(y) \rangle \longrightarrow \langle x, A^*(y) \rangle = \langle A(x), y \rangle$$

2) $A \in \mathcal{K}(E, E)$

$$x_n \xrightarrow{\omega} x \text{ in } E \Rightarrow A(x_n) \rightarrow A(x) \text{ in } E$$

3) $A \in \mathcal{B}(E, E)$ and finite rank operator

i.e. $\dim \underbrace{\mathcal{R}(A)}_{\substack{\text{subspace} \\ \text{of } E \text{ since} \\ A \text{ linear}}} < \infty$

$$\Rightarrow A \in \mathcal{K}(E, E)$$

since let e_1, \dots, e_N be an ON-basis for $\mathcal{R}(A)$. $N = \dim \mathcal{R}(A)$

$$A(x) = \boxed{\langle A(x), e_1 \rangle e_1 + \dots + \langle A(x), e_N \rangle e_N}$$

• Fix an arbitrary bounded sequence $(x_n)_{n=1}^{\infty}$ in E .

• A bounded $\Rightarrow (A(x_n))_{n=1}^{\infty}$ is bounded sequence.

• $(\langle A(x_n), e_i \rangle)_{n=1}^{\infty}$ bounded sequence in \mathbb{C} .

• Bolzano-Weierstrass then implies that $(\langle A(x_n), e_i \rangle)_{n=1}^{\infty}$ has a converging subsequence $(\langle A(x_{n_k}), e_i \rangle)_{k=1}^{\infty}$.

• Clearly $(\langle A(x_{n_k}), e_i \rangle)_{k=1}^{\infty}$ converges in E .

Hence $A(x) = \langle A(x), e_1 \rangle e_1 + \dots + \langle A(x), e_N \rangle e_N$ is a compact mapping since $\mathcal{K}(E, E)$ is a subspace of $\mathcal{B}(E, E)$.

Proof of proposition:

$\mathcal{K}(E, E) \ni A_n \rightarrow A$ in $(\mathcal{B}(E, E), \| \cdot \|_{E \rightarrow E})$

To show: $A \in \mathcal{K}(E, E)$.

• Fix an arbitrary bounded sequence $(x_n)_{n=1}^{\infty}$ in E .

To show: $(A(x_n))_{n=1}^{\infty}$ has a converging subsequence in E .

• Set $M = \sup_n \|x_n\| < \infty$

• $A_1 \in \mathcal{K}(E, E) \Rightarrow (A_1(x_n))_{n=1}^{\infty}$ has a converging subsequence $(A_1(x_{n,1}))_{n=1}^{\infty}$.

• $A_2 \in \mathcal{K}(E, E) \Rightarrow (A_2(x_{n,1}))_{n=1}^{\infty}$ has a converging subsequence $(A_2(x_{n,2}))_{n=1}^{\infty}$.

• Proceed inductively

• $A_k \in \mathcal{K}(E, E) \Rightarrow (A_k(x_{n,k-1}))_{n=1}^{\infty}$ has a converging subsequence $(A_k(x_{n,k}))_{n=1}^{\infty}$.

Also, $(A_l(x_{n,k}))_{n=1}^{\infty}$ converges in E for $l=1, 2, \dots, k$.

Set $y_n = x_{n,n}$, $n=1, 2, \dots$

Here

$(A_k(y_n))_{n=1}^{\infty}$ converges for $k=1, 2, \dots$

Claim: $(A(y_n))_{n=1}^{\infty}$ converges in E .

Since $(E, \| \cdot \|)$ is a Banach space, it is enough to show that $(A(y_n))_{n=1}^{\infty}$ is a Cauchy sequence in $(E, \| \cdot \|)$.

Fix an arbitrary $\epsilon > 0$.

$$\begin{aligned} \|A(y_n) - A(y_m)\| &= \|A(y_n) - A_k(y_n) + A_k(y_n) - A_k(y_m) + A_k(y_m) - A(y_m)\| \leq \\ &\leq \underbrace{\|A(y_n) - A_k(y_n)\|}_{= \|(A - A_k)(y_n)\|} + \underbrace{\|A_k(y_n) - A_k(y_m)\|}_{\leq \|A - A_k\|_{E \rightarrow E} M} + \underbrace{\|A_k(y_m) - A(y_m)\|}_{\leq \|A - A_k\|_{E \rightarrow E} M} \\ &\leq \|A - A_k\|_{E \rightarrow E} \|y_n\| \leq M \end{aligned}$$

Fix k (large enough) such that

$$\|A_k - A\|_{E \rightarrow E} < \frac{\epsilon}{3M}$$

Then

$$\|A(y_n) - A(y_m)\| < \frac{2}{3}\varepsilon + \|A_k(y_n) - A_k(y_m)\|$$

$(A_k(y_n))_{k=1}^{\infty}$ converges in E

$$\Rightarrow \exists N > 0 : \|A_k(y_n) - A_k(y_m)\| < \frac{\varepsilon}{3}, \forall n, m \geq N.$$

Hence

$$\|A(y_n) - A(y_m)\| < \varepsilon, \forall n, m \geq N$$

and so $(A(y_n))_{n=1}^{\infty}$ is a Cauchy sequence. \square

Proposition:

• $(E, \langle \cdot, \cdot \rangle)$ separable Hilbert space.

• $A \in \mathcal{K}(E, E)$

\Rightarrow there exists finite rank operators $A_n \in \mathcal{K}(E, E)$ such that

$$\|A - A_n\|_{E \rightarrow E} \xrightarrow{n \rightarrow \infty} 0$$

Proof:

Let $(x_k)_{k=1}^{\infty}$ be an ON-basis for E .

For

$$x = \sum_{k=1}^{\infty} \langle x, x_k \rangle x_k, \quad x \in E$$

Set

$$A_n(x) = A \left(\sum_{k=1}^n \langle x, x_k \rangle x_k \right) = \sum_{k=1}^n \langle x, x_k \rangle A(x_k), \quad x \in E$$

$$n = 1, 2, \dots$$

Here $\dim(\mathcal{R}(A_n)) \leq n, n = 1, 2, \dots$

So A_n finite rank operator on E for $n = 1, 2, \dots$

Fix $x \in E$ with $\|x\| = 1$

$$\begin{aligned} \| (A - A_n)(x) \|^2 &= \| A \left(\sum_{k=n+1}^{\infty} \langle x, x_k \rangle x_k \right) \|^2 \leq \\ &\leq \underbrace{\|A\|^2}_{\|A\|^2 = \sum_{k=n+1}^{\infty} |\langle x, x_k \rangle|^2} \sum_{k=n+1}^{\infty} |\langle x, x_k \rangle|^2 \leq \sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2 = \|x\|^2 \\ &\leq \sup_{\substack{\|y\|=1 \\ y \in \{x_1, \dots, x_n\}^\perp}} \|A(y)\|^2 \end{aligned}$$

$$\|A - A_n\|_{E \rightarrow E} \leq \sup_{\substack{\|y\|=1 \\ y \in \{x_1, \dots, x_n\}^\perp}} \|A(y)\|^2$$

Set

$$a_n = \sup_{\substack{\|y\|=1 \\ y \in \{x_1, \dots, x_n\}^\perp}} \|A(y)\|^2 < \infty, \quad n=1, 2, \dots$$

Here

$$a_n \geq a_{n+1} \geq 0, \quad n=1, 2, \dots$$

Clearly $(a_n)_{n=1}^{\infty}$ converges in \mathbb{R}

$$\text{Set } a = \lim_{n \rightarrow \infty} a_n.$$

Remains to prove: $a=0$.

Assume $a > 0$.

There exists $(y_n)_{n=1}^{\infty}$ in E such that

- 1) $\|y_n\|=1$
- 2) $y_n \in \{x_1, \dots, x_n\}^\perp$
- 3) $\|A(y_n)\|^2 \geq \frac{1}{2}a$

Claim: $y_n \xrightarrow{\omega} 0$ in $(E, \langle \cdot, \cdot \rangle)$.

Since

Fix an arbitrary $x \in E$

$$\begin{aligned} |\langle y_n, x \rangle| &= |K y_n, \sum_{k=1}^{\infty} \langle x, x_k \rangle x_k| = |\langle y_n, \sum_{k=n+1}^{\infty} \langle x, x_k \rangle x_k \rangle| \leq \\ &\leq \|y_n\| \left\| \sum_{k=n+1}^{\infty} \langle x, x_k \rangle x_k \right\| \xrightarrow{n \rightarrow \infty} 0 \\ &= \sqrt{\sum_{k=n+1}^{\infty} |\langle x, x_k \rangle|^2} \end{aligned}$$

(Note $\sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2 = \|x\|^2 < \infty$)

$$\left. \begin{array}{l} \circ y_n \xrightarrow{\omega} 0 \text{ in } (E, \langle \cdot, \cdot \rangle) \\ \circ A \in K(E, E) \end{array} \right\} \Rightarrow A(y_n) \rightarrow A(0) = 0$$

Contradiction to 3).

This shows $a=0$.

Proposition:

- $(E, \langle \cdot, \cdot \rangle)$ Hilbert space
- $A \in \mathcal{K}(E, E)$
- $x_n \xrightarrow{\omega} x$ in $(E, \langle \cdot, \cdot \rangle)$
- $\Rightarrow A(x_n) \xrightarrow{\omega} A(x)$ in $(E, \langle \cdot, \cdot \rangle)$

Proof:

$x_n \xrightarrow{\omega} x$ in $(E, \langle \cdot, \cdot \rangle)$
 $\Rightarrow \sup_n \|x_n\| < \infty$ (according to theorem)

$A \in \mathcal{K}(E, E)$
 $\Rightarrow (A(x_n))_{n=1}^{\infty}$ has a converging subsequence $(A(x_{n_k}))_{k=1}^{\infty}$.
 Since $(x_n)_{n=1}^{\infty}$ bounded.

Say $A(x_{n_k}) \rightarrow y$ in E .

$A \in \mathcal{K}(E, E) \subset \mathcal{B}(E, E)$

and

$x_n \xrightarrow{\omega} x$ in $(E, \langle \cdot, \cdot \rangle)$,

then

$A(x_n) \xrightarrow{\omega} A(x)$ in $(E, \langle \cdot, \cdot \rangle)$

We get $y = A(x)$.

We have

$A(x_{n_k}) \rightarrow A(x)$ in E

Assume that $A(x_n) \not\xrightarrow{\omega} A(x)$ in E .

Then there exists $\epsilon > 0$ and a subsequence

$(A(\tilde{x}_n))_{n=1}^{\infty}$ of $(A(x_n))_{n=1}^{\infty}$, such that

$$\|A(\tilde{x}_n) - A(x)\| \geq \epsilon, \forall n.$$

But $\tilde{x}_n \xrightarrow{\omega} x$ in $(E, \langle \cdot, \cdot \rangle)$ and $A \in \mathcal{K}(E, E)$ implies that $(A(\tilde{x}_n))_{n=1}^{\infty}$ has a converging subsequence $(A(\tilde{x}_{n_k}))_{k=1}^{\infty}$, that converges to $A(x)$. Contradiction.

$\Rightarrow A(x_n) \xrightarrow{\omega} A(x)$ in $(E, \langle \cdot, \cdot \rangle)$



Proposition: $\forall A \in \mathcal{K}(E, E)$ and $(E, \langle \cdot, \cdot \rangle)$ Hilbert space.

$$\Rightarrow A^* \in \mathcal{K}(E, E)$$

"Proof": Fix bounded sequence $(x_n)_{n=1}^\infty$ in E

$$\|A^*(x_n) - A^*(x_m)\|^2 = \langle A^*(x_n) - A^*(x_m), A^*(x_n) - A^*(x_m) \rangle =$$

$A^*(x_n - x_m)$

$$= \langle x_n - x_m, A(A^*(x_n) - A^*(x_m)) \rangle, \text{ then use } A \in \mathcal{K}(E, E). \quad (\text{left as exercise}) \quad \square$$

Proposition: $A \in \mathcal{K}(E, E)$, $B \in \mathcal{B}(E, E)$

$$\Rightarrow AB, BA \in \mathcal{K}(E, E). \quad (\text{Proof in book})$$

Example:

- $k \in C([0,1] \times [0,1])$
- $A(f)(x) = \int_0^1 k(x,y) f(y) dy, \quad x \in [0,1]$
- $f \in L^2([0,1])$

We know

- $A \in \mathcal{J}(L^2([0,1]), L^2([0,1]))$
- $\|A\|_{L^2 \rightarrow L^2} \leq \|k\|_{L^2([0,1] \times [0,1])}$

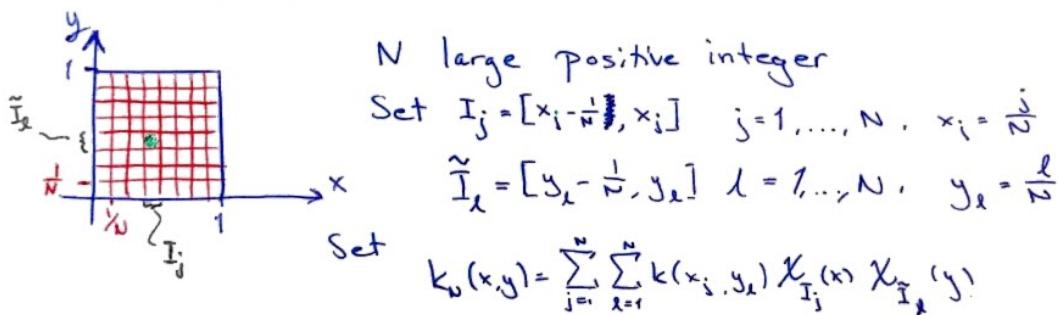
Claim: $A \in \mathcal{K}(L^2([0,1]), L^2([0,1]))$

• Approximate A by finite rank operators.

Note: Set $A = A_k$, $B = A_{k_n}$, k_n nice function on $[0,1] \times [0,1]$

$$A - B = A_k - A_{k_n} = A_{k-k_n} \quad \text{So}$$

$$\|A - B\|_{L^2 \rightarrow L^2} \leq \|k - k_n\|$$



where

$$\chi_{I_j}(x) = \begin{cases} 1 & x \in I_j \\ 0 & \text{otherwise} \end{cases}$$

$$\chi_{I_k}(y) = \dots$$

Since $k \in C([0,1] \times [0,1])$ and $[0,1] \times [0,1]$ compact on \mathbb{R}^2 , then k is uniformly continuous on $[0,1] \times [0,1]$.

Fix $\epsilon > 0$

Claim:

$$\exists N > 0 : \sup_{\substack{(x,y) \in \\ [0,1] \times [0,1]}} |k(x,y) - k_N(x,y)| < \epsilon$$

$$A_{k_N}(f)(x) = \int_0^1 k_N(x,y) f(y) dy = \underbrace{\sum_{j=1}^N \sum_{l=1}^N k(x_j, y_l) \int_0^1 \chi_{I_j}(y) f(y) dy}_{\text{scalar}} \cdot \chi_{I_l}(x)$$

$$\dim(\mathcal{R}(A_{k_N})) = N < \infty$$

Hence

$$A_{k_N} \in \mathcal{K}(L^2([0,1]), L^2([0,1])) \quad \forall N$$

Moreover

$$\|A - A_{k_N}\|_{L^2 \rightarrow L^2} \leq \|k - k_N\|_{L^2([0,1] \times [0,1])} < \epsilon \text{ for } N \text{ large enough.}$$

$\mathcal{K}(E, E)$ closed set in $(\mathcal{B}(E, E), \|\cdot\|_{L^2 \rightarrow L^2})$ So $A \in \mathcal{K}(L^2, L^2)$.

Example: $(E, \langle \cdot, \cdot \rangle)$ Hilbert space

$(x_n)_{n=1}^\infty$ ON-basis

$(\lambda_n)_{n=1}^\infty$ sequence of scalars

Set

$$T(x) = \sum_{n=1}^\infty \lambda_n \langle x, x_n \rangle x_n, \quad x \in E$$

Claim:

1) $T \in \mathcal{B}(E, E) \iff (\lambda_n)_{n=1}^\infty$ bounded sequence in \mathbb{C} .

2) $T \in \mathcal{K}(E, E) \iff \lambda_n \rightarrow 0, \quad n \rightarrow \infty$.

Note: $x \in E$

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \quad (\text{Parseval's formula})$$

1)

$T(x) \in E$

$$\|T(x)\|^2 = \sum_{n=1}^{\infty} |\lambda_n|^2 |\langle x, x_n \rangle|^2$$

If $(\lambda_n)_{n=1}^{\infty}$ is bounded sequence in \mathbb{C} , then

$$\sup_n |\lambda_n| = M < \infty$$

$$\|T(x)\|^2 \leq \sum_{n=1}^{\infty} M^2 |\langle x, x_n \rangle|^2 = M^2 \|x\|^2.$$

If $(\lambda_n)_{n=1}^{\infty}$ is not bounded, then there exists a sequence $(\lambda_{n_k})_{k=1}^{\infty}$ such that $|\lambda_{n_k}| \xrightarrow{k \rightarrow \infty} \infty$.

But

$$\|T(x_{n_k})\| = |\lambda_{n_k}| \|x_{n_k}\| = |\lambda_{n_k}| \xrightarrow{k \rightarrow \infty} \infty$$

$$\sup_{\|x\|=1} \|T(x)\| = \infty \quad . \quad 1) \text{ Done!}$$

2) Assume $\lambda_n \xrightarrow{n \rightarrow \infty} 0$.

Set $T_k(x) = \sum_{n=1}^k \lambda_n \langle x, x_n \rangle x_n, \quad x \in E$.

T_k is a finite rank operator for $k = 1, 2, \dots$

So $T_k \in \mathcal{K}(E, E)$ all k .

$$\begin{aligned} \|T - T_k\|_{E \rightarrow E} &= \sup_{\|x\|=1} \|(T - T_k)(x)\| = \sup_{\|x\|=1} \left\| \sum_{n=k+1}^{\infty} \lambda_n \langle x, x_n \rangle x_n \right\| \leq \\ &\leq \sup_{n=k+1, k+2, \dots} |\lambda_n| \xrightarrow{k \rightarrow \infty} 0. \end{aligned}$$

Assume $\lambda_n \not\rightarrow 0$ as $n \rightarrow \infty$.

Then there exists $\varepsilon > 0$ and a sequence $(\lambda_{n_k})_{k=1}^{\infty}$ such that

$$|\lambda_{n_k}| > \varepsilon.$$

Note:

$$T(x_{n_k}) = \lambda_{n_k} x_{n_k}, \quad k = 1, 2, \dots$$

$$\|T(x_{n_k})\| = |\lambda_{n_k}| \underbrace{\|x_{n_k}\|}_{=1} = |\lambda_{n_k}| \geq \varepsilon, \quad k = 1, 2, \dots$$

$x_{n_k} \xrightarrow{\omega} 0$ since for $y \in E$

$$\langle x_{n_k}, y \rangle = \langle x_{n_k}, \sum_{n=1}^{\infty} \langle y, x_n \rangle x_n \rangle = \overline{\langle y, x_{n_k} \rangle}_{k \rightarrow \infty} \rightarrow 0$$

since $\sum_{n=1}^{\infty} |\langle y, x_n \rangle|^2 = \|y\|^2 < \infty$.

If $T \in \mathcal{K}(E, E)$, then $T(x_{n_k}) \rightarrow T(0) = 0$ but
 $\|T(x_{n_k})\| \geq \epsilon \quad \forall k$, hence $T \notin \mathcal{K}(E, E)$.

(*) Example: $(E, \langle \cdot, \cdot \rangle)$ Hilbert space
 $A \in \mathcal{K}(E, E)$, $I(x) = x, \forall x \in E$.
 $\Rightarrow \boxed{\mathcal{R}(I-A)}$ closed in E .

Remark:

$$\mathcal{R}(I-A)^\perp = N((I-A)^*) = N(I-A^*)$$

$$\overline{\mathcal{R}(I-A)} = \mathcal{R}(I-A)^{\perp\perp} = N(I-A^*)^\perp$$

If $A \in \mathcal{K}(E, E)$, then

$$\overline{\mathcal{R}(I-A)} = \mathcal{R}(I-A)$$

Solve

$$x = A(x) + y$$

$$(I-A)(x) = y$$

: ← Next time

Fredholm alternative

Proof: (x) Assume

$\boxed{\mathcal{R}(I-A) \ni y_n} \rightarrow y \text{ in } (E, \|\cdot\|)$

To show: $y \in \mathcal{R}(I-A)$
i.e. $y = (I-A)(x)$, some $x \in E$

$y_n = (I-A)(x_n)$, some $x_n \in E$.

$$x_n \in E = \boxed{N(I-A)} \oplus \boxed{N(I-A)^\perp}$$

closed
Subspace in E

Step 1 Show $(\hat{x}_n)_{n=1}^{\infty}$ is bounded in E .

Step 2

$$y_n = (I-A)(\hat{x}_n) = I(\hat{x}_n) - A(\hat{x}_n)$$

$$\|x_n\|^2 = \|\tilde{x}_n\|^2 + \|\hat{x}_n\|^2$$

Föreläsning 1

Introductory example

$$\begin{cases} f'' + f = g & \text{on } [0, 1] = I \\ f(0) = 1, \quad f'(0) = 1 \end{cases} \quad (+)$$

① $g=0, \quad f(x) = A\cos(x) + B\sin(x), \quad x \in I, A, B \in \mathbb{R}$

② g arbitrary

- Method of variation of constants

Set $f(x) = A(x)\cos(x) + B(x)\sin(x)$

• Differentiate

$$f'(x) = \underbrace{A'(x)\cos(x) + B'(x)\sin(x)}_{(*)} - A(x)\sin(x) + B(x)\cos(x)$$

Assume (part of method) that $(*) = 0$.

• Differentiate $f'(x)$

$$f''(x) = -A'(x)\sin(x) + B'(x)\cos(x) - \underbrace{A(x)\cos(x) - B(x)\sin(x)}_{= -f(x)}$$

$$g(x) = f''(x) + f(x) = -A'(x)\sin(x) + B'(x)\cos(x)$$

Now

$$\begin{cases} A'(x)\cos(x) + B'(x)\sin(x) = 0 \\ -A'(x)\sin(x) + B'(x)\cos(x) = g(x) \end{cases}, \quad x \in I$$

with $A(0)=1, B(0)=0$.

We get

$$\begin{cases} A'(x) = -g(x)\sin(x) \\ A(0) = 1 \\ B'(x) = g(x)\cos(x) \\ B(0) = 0 \end{cases}$$

This implies

$$A(x) = A(0) + \int_0^x A'(t) dt = 1 - \int_0^x g(t)\sin(t) dt$$

$$B(x) = B(0) + \int_0^x B'(t) dt = \int_0^x g(t)\cos(t) dt$$

\implies

$$\begin{aligned} f(x) &= \cos(x) - \int_0^x g(t)\sin(t)\cos(x) dt + \int_0^x g(t)\cos(t)\sin(x) dt \\ &= \cos(x) + \int_0^x \underbrace{(\cos(t)\sin(x) - \cos(x)\sin(t))}_{=\sin(x-t)} g(t) dt \quad (*) \end{aligned}$$

which satisfies (+).

Special case:

$$g(x) = k(x)f(x), \quad x \in I$$

where k is a known continuous function on I .

Insert $g(x)$ in $(*)$ and we obtain

$$f(x) = \cos(x) + \int_0^x \sin(x-t)k(t)f(t)dt, \quad x \in I \quad (xx)$$

- Observe that f appear in both LHS and RHS.
- (xx) is a reformulation of $(+)$ with $g = kf$.

Set

$$f_1(x) = \cos(x) + \int_0^x \sin(x-t)k(t)f_0(t)dt$$

$$f_2(x) = \cos(x) + \int_0^x \sin(x-t)k(t)f_1(t)dt$$

$$\vdots$$
$$f_m(x) = \cos(x) + \int_0^x \sin(x-t)k(t)f_{m-1}(t)dt, \quad m=1,2,\dots$$

Hope: f_m tends to some continuous function f

on I , denoted $f_m \rightarrow f$. "Tends to" has to be made precise.

$$f_{m+1}(x) = \cos(x) + \int_0^x \sin(x-t)k(t)f_m(t)dt$$

$\downarrow \qquad \parallel \qquad \downarrow$

$$f(x) = \cos(x) + \int_0^x \sin(x-t)k(t)f(t)dt$$

Set

$$\begin{cases} u(x) = \cos(x) \\ Kv(t) = \int_0^x \sin(x-t) k(t)v(t) dt \\ \text{for } v \in C(I) \end{cases}$$

We have

$$(\exists) \begin{cases} f_0 \in C(I) \\ f_{m+1} = u + Kf_m, \quad m=0,1,2,\dots \end{cases}$$

Facts from previous calculus classes.

Definition: $v_m \in C(I), \quad m=1,2,\dots$

We say that $(v_m)_{m=1}^\infty$ converges uniformly on $I = [0,1]$ if

$$\max_{x \in I} |v_n(x) - v_m(x)| \rightarrow 0 \quad \text{as } n,m \rightarrow \infty$$

$$\text{i.e. } \forall \varepsilon > 0 \quad \exists N : n,m \geq N \Rightarrow \max_{x \in I} |v_n(x) - v_m(x)| < \varepsilon$$

Lemma:

Suppose that $(v_m)_{m=1}^\infty$ converges uniformly on I .
Then there exists $v \in C(I)$ such that

$$\max_{x \in I} |v_m(x) - v(x)| \rightarrow 0 \quad \text{as } m \rightarrow \infty$$

Back to (1)

Notations:

- $K(Kv) = K^2v$, $v \in C(I)$
- $K^{n+1}v = K(K^n v)$, $n = 1, 2, \dots$

We have

$$\begin{aligned} f_0 &\in C(I) \\ f_1 &= u + Kf_0 \\ f_2 &= u + K(u + Kf_0) \end{aligned}$$

Note that K has the linear property

$$K(v+w) = \underbrace{K(v)}_{Kv} + \underbrace{K(w)}_{Kw}$$

 for $v, w \in C(I)$

Then

$$\begin{aligned} f_2 &= u + K(u + Kf_0) = u + Ku + K^2f_0 \\ f_3 &= u + Kf_2 = u + Ku + K^2u + K^3f_0 \end{aligned}$$

In general

$$f_n = u + Ku + K^2u + \dots + K^{n-1}u + K^n f_0, \quad n=1, 2, \dots$$

Assume $n > m$, then

$$f_n - f_m = K^m u + \dots + K^{n-1}u + K^n f_0 - K^m f_0$$

Set $\|v\| = \max_{x \in I} |v(x)|$, for $v \in C(I)$

Note

$$\begin{aligned} \|v+w\| &\leq \|v\| + \|w\| \\ \|-v\| &= \|v\| \end{aligned} \quad \text{for } v, w \in C(I)$$

We have

$$\begin{aligned}\|f_n - f_m\| &= \|K^m u + \dots + K^{n-1} u + K^n f_0 - K^m f_0\| \leq \\ &\leq \|K^m u\| + \dots + \|K^{n-1} u\| + \|K^n f_0\| + \underbrace{\|-K^m f_0\|}_{= \|K^m f_0\|}\end{aligned}$$

Assumption: $\sum_{k=1}^{\infty} \|K^k v\| < \infty, \forall v \in C(I) \quad (\times \times \times)$

Under this assumption

$$\|f_n - f_m\| \rightarrow 0 \quad \text{as } n, m \rightarrow \infty$$

Since

- $\sum_{k=1}^{\infty} \|K^k u\| < \infty \quad (u(x) = \cos(x) \in C(I))$
- $\sum_{k=1}^{\infty} \|K^k f_0\| < \infty \quad (f_0 \in C(I))$

Conclusion: $(f_n)_{n=1}^{\infty}$ converges uniformly on I .

By previously mentioned lemma, there exists $f \in C(I)$ such that

$$\max_{x \in I} |f_n(x) - f(x)| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

$$\text{i.e. } \|f_n - f\| \xrightarrow{n \rightarrow \infty} 0$$

Back to "Hope":

" f_n tends to f " denoted $f_n \rightarrow f$ shall be interpreted as $\|f_n - f\| \rightarrow 0, n \rightarrow \infty$.

$$\begin{aligned} f_{n+1}(x) &= u(x) + \left[\underset{\downarrow}{Kf_n(x)} \right], \quad x \in I \\ &\qquad\qquad\qquad \parallel \\ f(x) &= u(x) + \left[\underset{\cdots}{Kf(x)} \right]. \end{aligned}$$

For $x \in I$

$$\begin{aligned} |Kf_n(x) - Kf(x)| &= \left| \int_0^x \sin(x-t) k(t) f_n(t) dt - \int_0^x \sin(x-t) k(t) f(t) dt \right| \leq \\ &\leq \int_0^x |\sin(x-t)| |k(t)| (f_n(t) - f(t)) dt = \\ &= \int_0^x |\sin(x-t)| |k(t)| \underbrace{|f_n(t) - f(t)|}_{\leq \|f_n - f\|} dt \leq \\ &\leq \int_0^x |\sin(x-t)| |k(t)| dt \|f_n - f\| \end{aligned}$$

In particular

$$\begin{aligned} \|Kf_n - Kf\| &\leq \max_{x \in I} \int_0^x \underbrace{|\sin(x-t)|}_{\leq 1} \underbrace{|k(t)|}_{\leq \max_{t \in I} |k(t)| < \infty} dt \|f_n - f\| \leq \\ &\leq \|k\| \|f_n - f\| \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

We have, provided that $(\times \times \times)$ holds, shown

$$\begin{aligned} f_{n+1} &= u + Kf_n \\ \downarrow &\qquad\qquad\qquad \downarrow \\ f &= u + Kf \end{aligned}$$

Let us try to prove $(\times \times \times) \left(\sum_{k=1}^{\infty} \|K^k v\| < \infty, \forall v \in C(I) \right)$

For $v \in C(I)$ arbitrary and $x \in I$

$$\begin{aligned} |Kv(x)| &= \left| \int_0^x \sin(x-t) k(t)v(t) dt \right| \leq \int_0^x \underbrace{|\sin(x-t)|}_{\leq 1} \underbrace{|k(t)|}_{\leq \|k\|} |v(t)| dt \leq \\ &\leq \int_0^x \underbrace{|v(t)|}_{\leq \|v\|} dt \cdot \|k\| \leq \|k\| \|v\| x, \quad x \in I \end{aligned}$$

In particular $\|Kv\| \leq \|k\| \|v\|$

$$|K^2 v| \leq \int_0^x |Kv(t)| dt \cdot \|k\| \leq \int_0^x \|k\| \|v\| t dt \cdot \|k\| = \|k\|^2 \|v\| \frac{x^2}{2}$$

In particular $\|K^2 v\| \leq \|k\|^2 \|v\| \frac{1}{2}$

By induction we get

$$\begin{aligned} |K^n v(x)| &\leq \|k\|^n \|v\| \frac{x^n}{n!}, \quad x \in I \\ \Rightarrow \|K^n v\| &\leq \|k\|^n \|v\| \frac{1}{n!} \end{aligned}$$

So

$$\sum_{l=1}^{\infty} \|K^l v\| \leq \sum_{l=1}^{\infty} \|k\|^l \|v\| \frac{1}{l!} = \|v\| \sum_{l=1}^{\infty} \frac{\|k\|^l}{l!} \leq e^{\|k\|} \|v\| < \infty$$

Conclusion: (xxx) holds true.

So far shown that

$$f = u + Kf$$

has a solution $f \in C(I)$.

Question: Is the solution unique?

Assume $f, \tilde{f} \in C(I)$ such that

$$\begin{cases} f = u + kf \\ \tilde{f} = u + k\tilde{f} \end{cases}$$

Set

$$v = f - \tilde{f} \in C(I)$$

$$v = (u + kf) - (u + k\tilde{f}) = kf - k\tilde{f} = k(f - \tilde{f}) = Kv$$

We have

$$v = Kv \Rightarrow Kv = K(Kv) = K^2v$$

$$\Rightarrow v = Kv = \dots = K^n v, \quad n = 1, 2, \dots$$

We know

$$\sum_{n=1}^{\infty} \|K^n v\| < \infty, \quad \forall v \in C(I)$$

Apply this to $\hat{v} = v$.

$$\text{Then } \sum_{n=1}^{\infty} \underbrace{\|K^n v\|}_{\substack{= \|v\| \\ \text{for all } n}} < \infty \Rightarrow \|v\| = 0$$

i.e. $v(x) = 0, \forall x \in I$.

We have

$$f(x) = \tilde{f}(x), \quad \forall x \in I$$

Answer to question: Yes!

We have more or less proved following theorem.

Theorem 1:

Set $I = [0, 1]$ and suppose $u \in C(I)$ and $k \in C(I \times I)$

Consider

$$f(x) = u(x) + \int_0^x k(x, t) f(t) dt, \quad x \in I \quad (1).$$

Then (1) has a unique solution $f \in C(I)$.

By using the same technology we can prove

Theorem 2:

Set $I = [0, 1]$ and $u \in C(I)$ and $k \in C(I \times I)$
suppose

and

$$\max_{(x,t) \in I \times I} |k(x, t)| < 1.$$

Consider

$$f(x) = u(x) + \int_0^x k(x, t) f(t) dt, \quad x \in I \quad (2)$$

Then (2) has a unique solution $f \in C(I)$.

Remark:

(1) is called a Volterra integral equation

(2) is called a Fredholm integral equation

Different notions used in the example:

Vector space: $C(I)$ with the operations

addition: $(v+w)(x) = v(x) + w(x)$, $v, w \in C(I)$

Multiplication by scalars: $(\lambda v)(x) = \lambda v(x)$, $v \in C(I), \lambda \in \mathbb{R}$

$x \in I$

Note that $v+w, \lambda v \in C(I)$.

Norm on $C(I)$

$$\|v\| = \max_{x \in I} |v(x)|$$

With norm given, we can talk about convergence and continuity.

Cauchy sequence in $C(I)$

In our example $\|f_n - f_m\| \rightarrow 0$, $n, m \rightarrow \infty$.

We say that the sequence $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Banach space $C(I)$ with the max-norm $\|\cdot\|$.

Lemma says every Cauchy sequence converges.

i.e. $\|v_n - v_m\| \rightarrow 0$, $n, m \rightarrow \infty$

$$\Rightarrow \exists v \in C(I) : \|v_n - v\| \xrightarrow{n \rightarrow \infty} 0.$$

K linear mapping: $C(I) \rightarrow C(I)$

$$K(v+w) = Kv + Kw \quad , \quad v, w \in C(I), \lambda \in \mathbb{R}$$

$$K(\lambda v) = \lambda Kv$$

K bounded linear

$$\|Kv\| \leq M\|v\|, \quad \forall v \in C(I)$$

when M independent of v .

Definition: $\|K\| = \inf \{M > 0 : \|Kv\| \leq M\|v\|, \forall v \in C(I)\}$

Fixed point results

Our example:

$$f = u + Kf \equiv T(f)$$

$$f_0 \in C(I)$$

Formed the sequence of iterates $(f_n)_{n=0}^{\infty}$

$$f_n = T(f_{n-1}), \quad n=1,2,\dots$$

If $\|T(v) - T(w)\| \leq c\|v-w\|$, for all $v,w \in C(I)$
and for some $c < 1$.

Then there is a unique $v \in C(I)$ such that

$$v = T(v).$$

(This is Banach's fixed point theorem)

Green's function

Our example

$$L = \left(\frac{d}{dx}\right)^2 + 1 \quad \text{differential operator}$$

Boundary conditions $v(0) = v'(0) = 0$.

$$\text{Then } f(x) = \underbrace{\int_0^x g(x,t) h(t) dt}_{\text{Green's function}}$$

Vector spaces

Definition:

We say that E is a real vector space if it is a nonempty set with the operations

- addition: $E \times E \rightarrow E$ $(x, y) \mapsto x + y$
- M. B. S : $\mathbb{R} \times E \rightarrow E$ $(\lambda, x) \mapsto \lambda x$

satisfying the axioms:

- 1) $x + y = y + x, \quad \forall x, y \in E$
- 2) $(x + y) + z = x + (y + z), \quad \forall x, y, z \in E$
- 3) For all $x, y \in E$ there exists $z \in E$ such that $x + z = y$.
- 4) $\alpha(\beta x) = (\alpha\beta)x, \quad \forall \alpha, \beta \in \mathbb{R}, \quad \forall x \in E$
- 5) $\alpha(x+y) = \alpha x + \alpha y, \quad \forall \alpha \in \mathbb{R}, \quad \forall x, y \in E$
- 6) $(\alpha + \beta)x = \alpha x + \beta x, \quad \forall \alpha, \beta \in \mathbb{R}, \quad \forall x \in E$
- 7) $1x = x, \quad \forall x \in E$

Remark: E is a complex vector space if all \mathbb{R} above are replaced by \mathbb{C} .

Remark: $\exists! 0 \in E : x + 0 = x, \quad \forall x \in E$

Fix $x \in E$. By 3) there exists 0_x such that $x + 0_x = x$.

Fix $y \in E$. Want to show that $y + 0_x = y$. By 3) there exists $z \in E$ such that $x + z = y$

$$y + 0_x = (x + z) + 0_x = (x + 0_x) + z = x + z = y.$$

Assume $x + \mathbb{O}_1 = x$, $x + \mathbb{O}_2 = x$, $\forall x \in E$.

We want to show that $\mathbb{O}_1 = \mathbb{O}_2$ (implying the uniqueness of the zero vector).

$$\mathbb{O}_1 = \mathbb{O}_1 + \mathbb{O}_2 = \mathbb{O}_2 + \mathbb{O}_1 = \mathbb{O}_2.$$

2) $\forall x \in E \exists! (-x) \in E : x + (-x) = \mathbb{O}$.

3) $\mathbb{O}x = \mathbb{O}$, $\forall x \in E$

$$(-1)x = -x, \quad \forall x \in E.$$

Examples of real vector spaces

1) \mathbb{R} , with standard addition and M.B.S.

2) \mathbb{R}^n , $n = 1, 2, \dots$

$$\text{addition: } (x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

$$\text{M.B.S: } \lambda(x_1, x_2, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$$

3) $\mathbb{R}^\infty = \{(x_1, x_2, \dots, x_n, \dots) : x_n \in \mathbb{R}, n = 1, 2, \dots\}$

4) $1 \leq p < \infty$

$$\ell^p = \left\{ (x_1, \dots, x_n, \dots) \in \mathbb{R}^\infty : \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} < \infty \right\}$$

with the same addition and M.B.S as \mathbb{R}^∞

To check:

$$1) x, y \in \ell^p \Rightarrow x+y \in \ell^p$$

$$2) x \in \ell^p \Rightarrow \lambda x \in \ell^p$$

1) Assume $x = (x_1, \dots, x_n, \dots)$, $y = (y_1, \dots, y_n, \dots)$

$$\begin{cases} x \in l^p & \sum_{n=1}^{\infty} |x_n|^p < \infty \\ y \in l^p & \sum_{n=1}^{\infty} |y_n|^p < \infty \end{cases}$$

$$x+y = (x_1+y_1, \dots, x_n+y_n, \dots) \stackrel{?}{\in} l^p$$

We note that

$$|x_n+y_n| \leq |x_n| + |y_n| \leq 2 \max(|x_n|, |y_n|)$$

$$\Rightarrow |x_n+y_n|^p \leq 2^p \max(|x_n|^p, |y_n|^p) \leq 2^p (|x_n|^p + |y_n|^p)$$

$$\begin{aligned} \Rightarrow \sum_{n=1}^{\infty} |x_n+y_n|^p &\leq \sum_{n=1}^{\infty} 2^p (|x_n|^p + |y_n|^p) = \\ &= \underbrace{2^p \sum_{n=1}^{\infty} |x_n|^p}_{< \infty} + \underbrace{2^p \sum_{n=1}^{\infty} |y_n|^p}_{< \infty} < \infty. \end{aligned}$$

$$2) \quad \sum_{n=1}^{\infty} |\lambda x_n|^p \leq \sum_{n=1}^{\infty} |\lambda|^p |x_n|^p = |\lambda|^p \sum_{n=1}^{\infty} |x_n|^p < \infty.$$

$\Rightarrow l^p$ vector space.

5) Function spaces, say real valued functions on I.

addition: $(f+g)(x) = f(x) + g(x)$

M.B.S : $(\lambda f)(x) = \lambda f(x)$

for functions f and g.

6) $C(I)$, addition and M.B.S as in 5)

• f, g continuous on I $\Rightarrow f+g$ continuous on I

• f cont, $\lambda \in \mathbb{R} \Rightarrow \lambda f$ cont on I.

7) $P(I)$, polynomials on I.

8) $P_k(I)$, polynomials of degree at most k on I.

Föreläsning 2

Hölder's inequality:

Theorem:

Assume $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Let (x_1, \dots, x_n, \dots) and (y_1, \dots, y_n, \dots) be sequences of complex numbers.

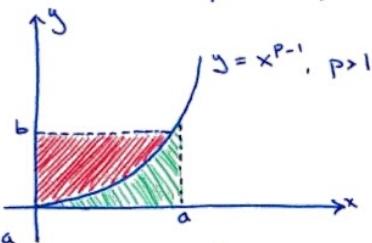
$$\Rightarrow \sum_{n=1}^{\infty} |x_n y_n| \leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |y_n|^q \right)^{\frac{1}{q}}$$

Remark: Here the LHS can be $=\infty$, but then the RHS is also $=\infty$.

Proof:

Step 1: $ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \forall a, b > 0$.

Proof:



$$\text{Area}_1 = \int_0^a x^{p-1} dx = \frac{a^p}{p}$$

$$\text{Note: } y = x^{p-1} \text{ gives } x = y^{\frac{1}{p-1}} = \left\{ P = \frac{1}{1-\frac{1}{q}} \right\} = y^{\frac{1}{\frac{1}{q}-1}} = y^{\frac{q}{q-1}}$$

$$= y^{\frac{q}{q-1}-1} = y^{q-1}$$

$$\text{Area}_2 = \int_0^b y^{q-1} dy = \frac{b^q}{q}$$

$$\text{We get } ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Step 2: It is enough to consider the case $LHS > 0$ and $RHS < \infty$ (the other cases are trivial).

There exists integer N such that

$$0 < \sum_{n=1}^N |x_n|^p, \quad \sum_{n=1}^N |y_n|^q < \infty.$$

Set

$$\begin{cases} a = \frac{|x_k|}{\left(\sum_{n=1}^N |x_n|^p\right)^{1/p}}, & k = 1, 2, \dots, N \\ b = \frac{|y_k|}{\left(\sum_{n=1}^N |y_n|^q\right)^{1/q}}, & k = 1, 2, \dots, N \end{cases}$$

Insert into $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$:

$$\frac{|x_k y_k|}{\left(\sum_{n=1}^N |x_n|^p\right)^{1/p} \left(\sum_{n=1}^N |y_n|^q\right)^{1/q}} \leq \frac{|x_k|^p}{p \sum_{n=1}^N |x_n|^p} + \frac{|y_k|^q}{q \sum_{n=1}^N |y_n|^q}, \quad k = 1, 2, \dots, N$$

• Sum over k from 1 to N .

$$\Rightarrow \sum_{k=1}^N |x_k y_k| \leq \left(\sum_{n=1}^N |x_n|^p \right)^{1/p} \left(\sum_{n=1}^N |y_n|^q \right)^{1/q}.$$

Let $N \rightarrow \infty$, first in RHS and then in LHS. □

Minkowski's inequality.

Theorem: Assume $1 \leq p < \infty$ and $x, y \in l^p$

$$\rightarrow \|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

$$\left(\|x\|_{l^p} = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \right)$$

Proof: $p=1$

$$\begin{aligned} \|x + y\|_{l^1} &= \|(x_1, \dots, x_n, \dots) + (y_1, \dots, y_n, \dots)\|_{l^1} = \\ &= \|(x_1 + y_1, \dots, x_n + y_n, \dots)\| = \sum_{n=1}^{\infty} |x_n + y_n| \leq \sum_{n=1}^{\infty} (|x_n| + |y_n|) = \\ &= \sum_{n=1}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = \|x\|_{l^1} + \|y\|_{l^1}. \end{aligned}$$

$1 < p < \infty$

$$\begin{aligned} \|x + y\|_p^p &= \sum_{n=1}^{\infty} |x_n + y_n|^p = \sum_{n=1}^{\infty} \underbrace{|x_n + y_n|}_{\leq |x_n| + |y_n|} |x_n + y_n|^{p-1} \leq \\ &\leq \sum_{n=1}^{\infty} |x_n| |x_n + y_n|^{p-1} + \sum_{n=1}^{\infty} |y_n| |x_n + y_n|^{p-1}; \end{aligned}$$

$$\begin{aligned} \sum_{n=1}^{\infty} |x_n| |x_n + y_n|^{p-1} &\leq \left\{ \text{Holder's inequality} \right\} \leq \underbrace{\left(\sum_{n=1}^{\infty} |x_n|^p \right)}^{p} \cdot \underbrace{\left(\sum_{n=1}^{\infty} |x_n + y_n|^q \right)}_{= \|x\|_{l^p}}^{q/p} \\ &= \left\{ (p-1)q = (p-1) \frac{1}{1 - \frac{1}{p}} = p \right\} = \|x\|_{l^p} \left(\sum_{n=1}^{\infty} |x_n + y_n|^p \right)^{1/q} = \|x\|_{l^p} \end{aligned}$$

We have

$$\|x + y\|_p^p \leq (\|x\|_{l^p} + \|y\|_{l^p}) \|x + y\|_{l^p}^{p/q}$$

If $\|x + y\|_p \neq 0$, then

$$\|x + y\|_p^{p - \frac{p}{q}} \leq \|x\|_{l^p} + \|y\|_{l^p} \quad \text{with} \quad p - \frac{p}{q} = p \left(1 - \frac{1}{q} \right) = p \cdot \frac{1}{p} = 1. \quad \square$$

Remark: $f \in C([0,1])$

$$\|f\|_{L^p} = \left(\int_0^1 |f(t)|^p dt \right)^{1/p}, \quad 1 \leq p < \infty.$$

Claim:

$$(x) \|fg\|_{L^1} = \int_0^1 |f(t)g(t)| dt \leq \|f\|_{L^p} \cdot \|g\|_{L^{p'}} , \quad \frac{1}{p} + \frac{1}{p'} = 1$$

$$(xx) \|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

(x): Corresponding Hölder's

(xx): Corresponding Minkowski's

This can be proven with the same technique as we used with ℓ^p with

$$\sum_{n=1}^{\infty} \text{ replaced by } \int_0^1 dt$$

- E is a real/complex vector space

$$x_1, \dots, x_n \in E$$

$$\lambda_1, \dots, \lambda_n \in \mathbb{F} \quad (\mathbb{F} \text{ arbitrary scalar field})$$

- We say that

$$\lambda_1 x_1 + \dots + \lambda_n x_n$$

is a linear combination of x_1, \dots, x_n .

- We say x_1, \dots, x_n are linearly independent if

$$\alpha_1 x_1 + \dots + \alpha_n x_n = 0$$

$$\Rightarrow \alpha_1 = \dots = \alpha_n = 0.$$

- $A \subset E$. A is linearly independent if every linear combination of vectors in A is linearly independent.

Example:

$$E = \mathcal{P}([0,1])$$

$$A = \{P_k : P_k(x) = x^k, x \in [0,1], k=0,1,2,\dots\}$$

- A is linearly independent.

• Let's consider

$$\alpha_0 P_0 + \dots + \alpha_n P_n = 0$$

$$\text{i.e. } \alpha_0 P_0(x) + \dots + \alpha_n P_n(x) = 0(x), x \in [0,1]$$

$$\text{i.e. } \alpha_0 + \alpha_1 x + \dots + \alpha_n x^n = 0, x \in [0,1]$$

$$x=0 \Rightarrow \alpha_0 = 0$$

$$\alpha_1 x + \dots + \alpha_n x^n = 0$$

- Differentiate $\Rightarrow \alpha_1 = 0$, repeat and get that

$$\alpha_0 = \alpha_1 = \dots = \alpha_n = 0.$$

□

- $B \subset E$

- $\text{Span } B = \text{set of all linear combination of elements in } B = \left\{ \sum_{k=1}^n \lambda_k x_k : x_k \in B, \lambda_k \in F, k=1,2,\dots,n \right\}$

where n is a positive integer}

Remark: $\sum_{k=1}^n \lambda_k x_k \in E$

$\sum_{k=1}^{\infty} \lambda_k x_k$ has no meaning

$C \subset E$ is called a basis for E if

- 1) C is linearly independent
- 2) $\text{Span } C = E$

Example:

$$E = \mathbb{P}(\{0,1\})$$

$$A = \{P_k : k=0,1,\dots\}$$

Claim: A is a basis for E .

Example:

$$E = \ell^2$$

$$A = \{x_k : k=1,2,\dots\}$$

$$x_k = (0, 0, \dots, 0, \underset{\substack{\uparrow \\ \text{Position } k}}{1}, 0, \dots)$$

Claim: A is linearly independent, since

$$\underbrace{\alpha_1 x_1}_{(\alpha_1, 0, \dots)} + \dots + \underbrace{\alpha_n x_n}_{(0, 0, \dots, 0, \alpha_n, \dots)} = 0 \Rightarrow (\alpha_1, \dots, \alpha_n) = (0, 0, \dots, 0)$$

$$\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0.$$

Question: is A a basis for ℓ^2 ?

We note: if $x \in \text{Span } A$, then

$$x = (x_1, \dots, x_n, 0, 0, 0, \dots) \text{ for some positive } n.$$

i.e. x has finitely many non-zero positions.

Consider: $x = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots)$

$$\|x\|_{\ell^2} = \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} < \infty$$

$$\text{So } x \in \ell^2 \setminus \text{Span } A$$

Remark: Every vector space has basis (if we are allowed to use axioms of choice/Zorn's lemma).

Assume x_1, \dots, x_n is a basis for E . Then every basis for E must contain n different elements.
 $n = \dim E$ is well-defined.

NORMS

- E vector space
- We say that a mapping $\|\cdot\|: E \rightarrow [0, \infty)$ is a norm on E if
 - 1) $\|x\| = 0 \Rightarrow x = 0$
 - 2) $\|\lambda x\| = |\lambda| \|x\|, \forall x \in E, \forall \lambda \in F$
 - 3) $\|x+y\| \leq \|x\| + \|y\|, \forall x, y \in E$

Remark: $\|0\| = \|0 \cdot 0\| = \underbrace{\|0\|}_{=0} \|0\| = 0$

Example: $1 \leq p < \infty$

$\|x\|_p$ is a norm on ℓ^p . Need to check 1, 2 and 3.

$$1) \|0\|_{\ell^p} = \left(\sum_{n=1}^{\infty} |0_n|^p \right)^{1/p} \Rightarrow x_n = 0, n=1,2,\dots \Rightarrow x = (x_1, \dots) = (0, \dots) = 0$$

$$2) \|\lambda x\|_{\ell^p} = \left(\sum_{n=1}^{\infty} |\lambda x_n|^p \right)^{1/p} = \left(|\lambda|^p \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} = |\lambda| \|x\|_{\ell^p}$$

$$3) \|x+y\|_{\ell^p} \stackrel{\text{Minkowski's}}{\leq} \|x\|_{\ell^p} + \|y\|_{\ell^p}$$

Example: $E = C([0,1]) \ni f$

$$\|f\| = \max_{t \in [0,1]} |f(t)| \in [0, \infty)$$

Check 1, 2 and 3.

$$1) \|f\| = 0 \Rightarrow |f(t)| = 0, \forall t \in [0,1] \Rightarrow f = \emptyset.$$

$$2) \|\lambda f\| = \max_{t \in [0,1]} |\underbrace{(\lambda f)(t)}_{\lambda f(t)}| = |\lambda| \max_{t \in [0,1]} |f(t)| = |\lambda| \|f\|$$

$$3) \|f+g\| = \max_{t \in [0,1]} |\underbrace{(f+g)(t)}_{f(t)+g(t)}| \leq \max_{t \in [0,1]} (|f(t)| + |g(t)|) \leq \\ \leq \max_{t \in [0,1]} |f(t)| + \max_{t \in [0,1]} |g(t)| = \|f\| + \|g\|$$

Example: $E = C([0,1]) \ni f$

$$\|f\|_L = \int_0^1 |f(t)| dt \text{ defines a norm on } E.$$

Check 1, 2, 3.

$$3) \|f+g\|_L = \int_0^1 |f+g| dt \leq \int_0^1 |f| dt + \int_0^1 |g| dt = \|f\|_L + \|g\|_L$$

$$2) \|\lambda f\|_L = \int_0^1 |\underbrace{(\lambda f)(t)}_{\lambda f(t)}| dt = |\lambda| \int_0^1 |f(t)| dt = |\lambda| \|f\|_L$$

$$1) 0 = \|f\|_L - \int_0^1 |f(t)| dt \Rightarrow f(t) = 0, \forall t \in [0,1]$$

Since f is continuous.

i.e. $f = \emptyset$.

Equivalence of norms:

- E vector space with norms $\|\cdot\|$ and $\|\cdot\|_*$.
- We say that $\|\cdot\|, \|\cdot\|_*$ are equivalent if there exists $\alpha, \beta > 0$ such that

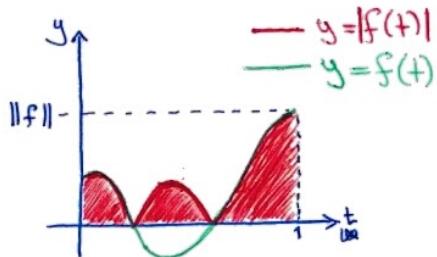
$$\alpha \|x\|_* \leq \|x\| \leq \beta \|x\|_*, \quad \forall x \in E.$$

Example:

$$E = C([0,1])$$

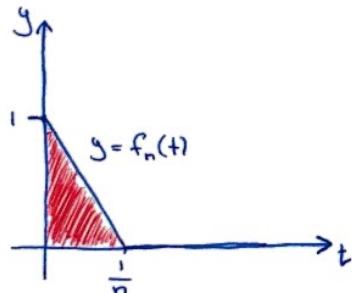
$$\|f\| = \max_{t \in [0,1]} |f(t)|$$

$$\|f\|_* = \|f\|_{L^1} = \text{area}$$



Question: Are these two norms equivalent?

$$\|f\|_* = \int_0^1 |f(t)| dt \leq \|f\|$$



$$\|f_n\| = 1$$

$$\|f_n\| = \frac{1}{2n} \quad (\text{area})$$

$$\text{We want } \|f_n\|_* \geq \alpha \|f_n\|$$

$$\Rightarrow \frac{\|f_n\|_*}{\|f_n\|} \geq \alpha > 0$$

$$\text{But } \frac{\|f_n\|_*}{\|f_n\|} = \frac{1}{2n} \xrightarrow{n \rightarrow \infty} 0$$

Answer: No! Not equivalent.

Theorem:

E vector space with finite dimension ($\dim E < \infty$)

\Rightarrow All norms on E are equivalent.

(Actually \Leftrightarrow but need Zorn's lemma to prove)

Proof:

Assume that $n = \dim E$ (n positive integer) and let x_1, x_2, \dots, x_n be a basis for E .

For every $x \in E$

$$x = \alpha_1(x)x_1 + \dots + \alpha_n(x)x_n \text{ where } \alpha_1(x), \dots, \alpha_n(x) \text{ unique}$$

Set

$$\|x\|_* = |\alpha_1(x)| + \dots + |\alpha_n(x)|, \quad x \in E.$$

Claim: $\|x\|_*$ defines a norm on E (to prove this, simply check norm conditions 1, 2, 3 as usual).

Fix an arbitrary norm $\|\cdot\|$ on E .

Claim: $\|\cdot\|, \|\cdot\|_*$ are equivalent.

Note: for $x \in E$

$$\begin{aligned} \|x\| &= \|\alpha_1(x)x_1 + \dots + \alpha_n(x)x_n\| \leq |\alpha_1(x)|\|x_1\| + \dots + |\alpha_n(x)|\|x_n\| \leq \\ &\leq \max_{k=1,2,\dots,n} \|x_k\| \underbrace{(|\alpha_1(x)| + \dots + |\alpha_n(x)|)}_{=\|x\|_*}, \end{aligned}$$

$$\text{Set } \beta = \max_{k=1,2,\dots,n} \|x_k\|.$$

$$\text{Then } \|x\| \leq \beta \|x\|_*, \quad \forall x \in E.$$

Remains to prove there exists $\alpha > 0$ such that

$$\alpha \|x\|_* \leq \|x\|, \quad \forall x \in E.$$

(Following not part of proof)

- Let E be a vector space with norm $\|\cdot\|$ and $(v_n)_{n=1}^{\infty}$ a sequence in E . We say that $(v_n)_{n=1}^{\infty}$ converges in $(E, \|\cdot\|)$ if

$$\exists v \in E : \|v_n - v\| \xrightarrow{n \rightarrow \infty} 0$$

Notation: $v_n \rightarrow v$ in $(E, \|\cdot\|)$

Note: if $\|\cdot\|, \|\cdot\|_*$ are equivalent, then

$$v_n \rightarrow v \text{ in } (E, \|\cdot\|) \iff v_n \rightarrow v \text{ in } (E, \|\cdot\|_*)$$

(Back to proof):

- We argue by contradiction.
- Assume there is no $\alpha > 0$ such that $\alpha \|x\|_* \leq \|x\|, \forall x \in E$.
- For $k = 1, 2, \dots$ there are $y_k \in E$ such that

$$\frac{1}{k} \|y_k\|_* > \|y_k\| \quad (\times \times)$$

- We have

$$y_k = \alpha_1^{(k)} x_1 + \dots + \alpha_n^{(k)} x_n, \quad \alpha_1^{(k)}, \dots, \alpha_n^{(k)} \text{ unique scalars, } k = 1, 2, \dots$$

$(\times \times)$:

$$k \|y_k\| < |\alpha_1^{(k)}| + \dots + |\alpha_n^{(k)}|$$

We have

$$k \|y_k\| < 1, \quad k=1, 2, \dots$$

which implies that $y_k \rightarrow 0$ in $(E, \|\cdot\|)$.

If

$$\alpha_1^{(k)} \rightarrow \bar{\alpha}_1, \quad k \rightarrow \infty$$

$$\alpha_2^{(k)} \rightarrow \bar{\alpha}_2, \quad k \rightarrow \infty$$

⋮

$$\alpha_n^{(k)} \rightarrow \bar{\alpha}_n, \quad k \rightarrow \infty$$

then set $\bar{y} = \bar{\alpha}_1 x_1 + \dots + \bar{\alpha}_n x_n$.

$$\begin{aligned} \|y_k - \bar{y}\| &= \|(\alpha_1^{(k)} - \bar{\alpha}_1)x_1 + \dots + (\alpha_n^{(k)} - \bar{\alpha}_n)x_n\| \leq \\ &\leq |\alpha_1^{(k)} - \bar{\alpha}_1| \underbrace{\|x_1\|}_{\substack{\longrightarrow 0 \\ k \rightarrow \infty}} + \dots + |\alpha_n^{(k)} - \bar{\alpha}_n| \underbrace{\|x_n\|}_{\substack{\longrightarrow 0 \\ k \rightarrow \infty}} \xrightarrow{k \rightarrow \infty} 0 \end{aligned}$$

$$\|\bar{y}\| = \|\bar{y} + y_k - y_k\| \leq \underbrace{\|\bar{y} - y_k\|}_{\substack{\longrightarrow 0 \\ k \rightarrow \infty}} + \underbrace{\|y_k\|}_{\substack{\longrightarrow 0 \\ k \rightarrow \infty}}$$

$$\text{So } \|\bar{y}\| = 0 \Rightarrow \bar{y} = 0$$

But

$$|\bar{\alpha}_1| + |\bar{\alpha}_2| + \dots + |\bar{\alpha}_n| = 1$$

and this contradicts the fact that

x_1, \dots, x_n is a basis!

$$\left(\boxed{\alpha_{1,1}^{(k)}} \alpha_2^{(k)}, \dots, \alpha_n^{(k)} \right) \quad k=1,2,\dots$$

where

$$|\alpha_{1,1}^{(k)}| + \dots + |\alpha_n^{(k)}| = 1$$

$$(x): \quad |\alpha_{1,1}^{(k)}| \leq 1, \quad k=1,2,\dots$$

By Bolzano-Weierstrass theorem there exists a convergent subsequence $(\alpha_{1,1}^{(k)})_{k=1}^{\infty}$ of $(\alpha_{1,1}^{(k)})_{k=1}^{\infty}$

$$\text{Set } \bar{\alpha}_1 = \lim_{k \rightarrow \infty} \alpha_{1,1}^{(k)}$$

Consider

$$(\alpha_{1,1}^{(k)}, \alpha_{2,1}^{(k)}, \dots, \alpha_{n,1}^{(k)}) , \quad k=1,2,\dots$$

We have that

$$|\alpha_{2,1}^{(k)}| \leq 1, \quad k=1,2,\dots$$

$$\text{Set } \bar{\alpha}_2 = \lim_{k \rightarrow \infty} \alpha_{2,2}^{(k)}, \quad \text{etc.}$$

etc.

Föreläsning 13

- $(E, \langle \cdot, \cdot \rangle)$ complex Hilbert space

- $A \in \mathcal{B}(E, E)$

- Consider the equation

$$x = A(x) + y, \quad y \in E.$$

$$(I - A)(x) = y$$

- Consider this problem, for $\lambda \in \mathbb{C}$

$$(A - \lambda I)(x) = y$$

- Set

$$\rho(A) = \{\lambda \in \mathbb{C} : (A - \lambda I)^{-1} \in \mathcal{B}(E, E)\}$$

- $\rho(A)$ is called the resolvent set for A .

- Set

$$\sigma(A) = \mathbb{C} \setminus \rho(A)$$

- $\sigma(A)$ is called the spectrum for A

- Clearly, a necessary condition for $(A - \lambda I)^{-1} \in \mathcal{B}(E, E)$ is that $A - \lambda I : E \rightarrow E$ is a bijection.

- Linearity for $(A - \lambda I)^{-1}$ follows from the linearity of $A - \lambda I$.

Banach's inverse mapping theorem

- $(E, \|\cdot\|)$ Banach space

- $A \in \mathcal{B}(E, E)$

- $A - \lambda I : E \rightarrow E$ bijection

$$\Rightarrow (A - \lambda I)^{-1} \in \mathcal{B}(E, E)$$

Proof based on the open mapping theorem
Proof is omitted.

Assume $\lambda \in \sigma(A)$

Then $A - \lambda I : E \rightarrow E$ is not a bijection.

- If $A - \lambda I : E \rightarrow E$ is not injective, then there exists $0 \neq x \in E$ such that $(A - \lambda I)(x) = 0$, i.e. λ is an eigenvalue of A . Set

$$\sigma_p(A) = \{\lambda \in \mathbb{C} : \lambda \text{ eigenvalue of } A\}$$

- If $A - \lambda I$ is injective, densely defined but not bounded, then $\lambda \in \sigma(A)$.

The set of such λ 's is called the continuous spectrum of A , denoted $\sigma_c(A)$.

- If $A - \lambda I$ is not surjective, then the set of such λ 's is called the residual spectrum, denoted $\sigma_r(A)$.

Lemma:

- $(E, \|\cdot\|)$ Banach space
- $A \in \mathcal{B}(E, E)$ with $\|A\| < 1$

$$\Rightarrow (I - A)^{-1} \in \mathcal{B}(E, E) \quad \text{Called a Neumann Series}$$

$$(I - A)^{-1} = I + \sum_{n=1}^{\infty} A^n \quad \leftarrow$$

Proof: Observe

$$\|A^n\| = \underbrace{\|(AA \dots A\|}_{n \text{ times}} \leq \|A\|^n, \quad n = 1, 2, \dots$$

and

$$\sum_{n=1}^{\infty} \|A^n\| < \infty.$$

Since E is a Banach space we have

$$\sum_{n=1}^{\infty} A^n \text{ converges in } \mathcal{B}(E, E)$$

since E Banach space implies $\mathcal{B}(E, E)$ Banach space.

Now

$$(I - A)\left(I + \sum_{n=1}^N A^n\right) = I - A^{N+1} \longrightarrow I \quad \text{in } \mathcal{B}(E, E)$$

$$\left(I + \sum_{n=1}^N A^n\right)(I - A) = I - A^{N+1} \longrightarrow I \quad \text{in } \mathcal{B}(E, E)$$

We get

$$\left(I + \sum_{n=1}^{\infty} A^n\right)(I - A) = I = (I - A)\left(I + \sum_{n=1}^{\infty} A^n\right)$$

We have $(I - A)^{-1}$ exists and $= I + \sum_{n=1}^{\infty} A^n$ ◻

Lemma:

- $(E, \|\cdot\|)$ Banach space
- $A \in \mathcal{B}(E, E)$

\Rightarrow

- 1) $\sigma(A) \neq \emptyset$
- 2) $\sigma(A)$ closed set in \mathbb{C}
- 3) $\sigma(A) \subset \overline{B(0, \|A\|)}$

Proof:

1) Omitted

2) Enough to proof that $\rho(A)$ is open set in \mathbb{C} .

Fix $\lambda_0 \in \rho(A)$. So $(A - \lambda_0 I)^{-1} \in \mathcal{B}(E, E)$.

Note

$$\begin{aligned} A - \lambda I &= A - \lambda_0 I - (\lambda - \lambda_0)I = \\ &= \underbrace{(A - \lambda_0 I)}_{\substack{\text{invertible} \\ \text{since } \lambda_0 \in \rho(A)}} \underbrace{(I - (\lambda - \lambda_0)(A - \lambda_0 I)^{-1})}_{\substack{\text{invertible if} \\ \|(I - (\lambda - \lambda_0)(A - \lambda_0 I)^{-1})\| < 1 \\ \text{by previous lemma,} \\ \text{i.e. } |\lambda - \lambda_0| < \frac{1}{\|(A - \lambda_0 I)^{-1}\|}}} \end{aligned}$$

Clearly

$A - \lambda I$ is invertible if

$$|\lambda - \lambda_0| < \frac{1}{\|(A - \lambda_0 I)^{-1}\|}$$

3) It is enough to show that $\lambda \in \rho(A)$ if

$$|\lambda| > \|A\|.$$

$$\text{Note } A - \lambda I = -\lambda \left(I - \frac{1}{\lambda} A \right)$$

Here $\left\| -\frac{1}{\lambda} A \right\| = \frac{1}{|\lambda|} \|A\| < 1$ so $I - \frac{1}{\lambda} A$ is invertible by previous lemma.

So $\lambda \in \rho(A)$. \square

Now assume $(E, \langle \cdot, \cdot \rangle)$ complex Hilbert space.

$A \in \mathcal{K}(E, E)$ (we don't assume A self-adjoint).

Then

1) $\lambda \in \sigma(A) \setminus \{0\} \Rightarrow \lambda$ is an eigenvalue of A .

2) $\lambda \in \sigma(A) \setminus \{0\} \Rightarrow \dim \{x \in E : Ax = \lambda x\} < \infty$

3) 0 is the only cluster point for $\sigma(A)$.

4) $0 \in \sigma(A)$

since if $0 \notin \sigma(A)$ then $A^{-1} \in \mathcal{B}(E, E)$ and

$$\underbrace{AA^{-1}}_{\in \mathcal{K}(E, E)} = I$$

$\underbrace{\in \mathcal{B}(E, E)}$

But $I \notin \mathcal{K}(E, E)$ since E ∞ -dim. Just take an ON-sequence $(x_n)_{n=1}^{\infty}$ in E .

Then $x_n \rightarrow 0$ in E , but $\|x_n\| = 1$ for all n and if $I \in \mathcal{K}(E, E)$ then

$$x_n = I(x_n) \longrightarrow I(0) = 0 \text{ in } E.$$

which implies $\|x_n\| \xrightarrow{n \rightarrow \infty} 0$

□

Moreover (by Hilbert-Schmidt theorem) $(E, \langle \cdot, \cdot \rangle)$ complex Hilbert separable ∞ -dim.

$A \in \mathcal{K}(E, E)$ and self-adjoint

$\Rightarrow (u_n)_{n=1}^{\infty}$ ON-basis for E where

$$A(u_n) = \lambda_n u_n, \quad n=1, 2, \dots$$

(λ_n eigenvalue of A . with normalized eigenvector u_n)
with

$$\lim_{n \rightarrow \infty} \lambda_n = 0$$

For $x \in E$

$$x = \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n$$

and

$$A(x) = \sum_{n=1}^{\infty} \langle x, \lambda_n u_n \rangle u_n$$

Fredholm alternative

• E, A as above.

Then

(1) $x = A(x) + y$ is solvable for all $y \in E$.

iff

(2) $x = A(x)$ has no non-trivial solution $x \in E$.

Exactly one of the statements hold

[1] as (1) above

[2] (2) has a non-trivial solution $x \in E$.]

In general:

(1) is solvable for $y \in E$ iff $y \in \{x \in E : A(x) = x\}$.

If so, if x is a solution (1), then also $x + \tilde{x}$ is a solution to (1) where

$$\tilde{x} = \{x \in E : A(x) = x\}.$$

Proof: Look at (1).

Let $(u_n)_{n=1}^{\infty}$ be the ON-basis from the previous theorem.

$$x = \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n$$

$$y = \sum_{n=1}^{\infty} \langle y, u_n \rangle u_n$$

$$A(x) = \sum_{n=1}^{\infty} \lambda_n \langle x, u_n \rangle u_n$$

(1) takes the form

$$\sum_{n=1}^{\infty} (\langle x, u_n \rangle - \lambda_n \langle x, u_n \rangle - \langle y, u_n \rangle) u_n = 0$$

This implies

$$(1 - \lambda_n) \langle x, u_n \rangle - \langle y, u_n \rangle = 0, \quad n = 1, 2, \dots$$

If $\lambda_n \neq 1$ then

$$\langle x, u_n \rangle = \frac{\langle y, u_n \rangle}{1 - \lambda_n}$$

If $\lambda_n = 1$ then y must be orthogonal to every u_n corresponding to eigenvalue 1.

$$\sum_{n=1}^{\infty} \frac{\langle y, u_n \rangle}{1 - \lambda_n} u_n \in E,$$

since

$$\left(\frac{\langle y, u_n \rangle}{1 - \lambda_n} \right)_{\substack{n=1 \\ \lambda_n \neq 1}}^{\infty} \in l^2,$$

since

$$\sup_{\substack{n \\ \lambda_n \neq 1}} \left| \frac{1}{1 - \lambda_n} \right| < \infty$$

since

$$\lim_{n \rightarrow \infty} \lambda_n = 0$$

and

$$(\langle y, u_n \rangle)_{n=1}^{\infty} \in l^2$$

□

Boundary value problems for ODE's

Consider

$$(*) \quad \begin{cases} Lu = f \in C([0,1]) \\ R_j u = 0 \quad , \quad j = 1, 2, \dots, n \end{cases} \quad \begin{matrix} DE \\ BC \text{ (homogeneous!) } \end{matrix}$$

where

$$Lu = u^{(n)} + C_{n-1}(x)u^{(n-1)} + \dots + C_1(x)u' + C_0(x)u, \quad u \in C^n([0,1])$$

$$C_0(x), C_1(x), \dots, C_{n-1}(x) \in C([0,1])$$

$$R_j u = \sum_{k=0}^{n-1} (\alpha_{jk} u^{(k)}(0) + \beta_{jk} u^{(k)}(1)) \quad j = 1, 2, \dots, n$$

$$\alpha_{jk}, \beta_{jk} \in \mathbb{C} \quad j = 1, \dots, n, \quad k = 0, \dots, n-1$$

Reformulate (*)

$$u(x) = \int_0^x g(x,y) f(y) dy \in C^n([0,1]) \text{ and satisfies}$$

Green's function BC: $R_j u = 0, j = 1, 2, \dots, n$.

for L and $R_j, j = 1, \dots, n$.

Consider the problem

$$(\ast\ast) \quad \begin{cases} Lu = f(x, u), & x \in [0, 1] \\ R_j u = 0, & j=1, 2, \dots, n. \end{cases}$$

The reformulation above gives

$$u(x) = \int_0^x g(x, y) f(y, u(y)) dy, \quad x \in [0, 1].$$

To find a solution, set

$$T(u)(x) = \int_0^x g(x, y) f(y, u(y)) dy, \quad x \in [0, 1]$$

$$T: C([0, 1]) \rightarrow C([0, 1]).$$

A fixed point to T gives a solution to $(\ast\ast)$.

Note that if $u \in C([0, 1])$ then $T(u) \in C^n([0, 1])$ and satisfies $R_j u = 0$, $j=1, \dots, n$.

Given h and R_j , $j=1, 2, \dots, n$, find the corresponding Green's function.

Example:

$$\begin{cases} Lu = u'' - u & \text{on } [0, 1] \\ R_1 u = u(0) = 0 \\ R_2 u = u(1) = 0 \end{cases}$$

Solve $Lu = 0 = u'' - u$ (1)

$u_1(x) = e^x, u_2(x) = e^{-x}$

$(r^2 - 1 = 0 \Rightarrow r_{1,2} = \pm 1)$

$\Rightarrow u(x) = A e^x + B e^{-x}$

Theorem 1 $Lu = f \in C([0, 1])$

where $Lu = u^{(n)} + c_{n-1}(x)u^{(n-1)} + \dots + c_1(x)u' + c_0(x)u$

$c_k(x) \in C([0, 1]), k=0, \dots, n-1$

$\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{C}^n, x_0 \in [0, 1]$

$\Rightarrow \exists! u \in C([0, 1]) \text{ satisfying } Lu = f$
 and $(u, u', \dots, u^{(n-1)}) \Big|_{x_0} = \xi$.

"Proof": Reformulate the problem as a system of 1st order differential equations.

$$\begin{cases} Lu = f \\ (u, \dots, u^{(n-1)}) \Big|_{x_0} = \bar{u} \end{cases} \Leftrightarrow \begin{cases} \tilde{u}' = \tilde{f} \\ \tilde{u}(x_0) = \bar{u} \end{cases}$$

$$\Leftrightarrow \tilde{u}(x) = \bar{u} + \int_{x_0}^x \tilde{f}(s) ds$$

\tilde{f} contains \tilde{u} implicitly.

The statement in the proof follows from an application of Banach's fixed point theorem. (see course home page for proof of Picard's theorem).

Set $N(L) = \{u \in C^n([0,1]) : Lu = 0\}$.

Corollary: $\dim N(L) = n$.

Set $C_R^n([0,1]) = \{u \in C^n([0,1]) : R_j u = 0, j=1,2,\dots,n\}$

$$\text{and } L_0 = L \Big|_{C_R^n([0,1])}$$

Let $u_1, \dots, u_n \in N(L)$.

Theorem 2: The following statements are equivalent.

Let $u_1, \dots, u_n \in N(L)$

(1) $w(x) \neq 0$ all $x \in [0,1]$

(2) $w(x) \neq 0$ for some $x \in [0,1]$

(3) u_1, u_2, \dots, u_n is a basis for $N(L)$

where

$$w(x) = \begin{bmatrix} u_1(x) & \dots & u_n(x) \\ u_1'(x) & \dots & u_n'(x) \\ \vdots & & \vdots \\ u_1^{(n-1)}(x) & \dots & u_n^{(n-1)}(x) \end{bmatrix}, \quad x \in [0,1]$$

Theorem 3: With notation from above, the following statements are equivalent

(1) $L_0 : C_R^n([0,1]) \longrightarrow C([0,1])$ is a bijection

(2) $\det(R_j u_k)_{1 \leq j, k \leq n} \neq 0$.

Ex: (■) from before.

$$\left. \begin{array}{l} u_1(x) = e^x \quad u_2(x) = e^{-x} \\ R_1 u_1 = u_1(0) = e^0 = 1 \\ R_2 u_2 = u_2(0) = e^0 = 1 \\ R_1 u_2 = u_2(1) = e^{-1} \\ R_2 u_1 = u_1(1) = e \end{array} \right| \det(R_j u_k) = \begin{vmatrix} 1 & 1 \\ e & e^{-1} \end{vmatrix} = \frac{1}{e} - e \neq 0$$

Theorem 4: Assume u_1, \dots, u_n basis for $N(L)$ and $\det(R_j u_k) \neq 0$. Set $G = L_0^{-1}$

$\Rightarrow \exists!$ continuous $g \in C([0,1] \times [0,1])$

such that

$G(f) = \int_0^1 g(xy) f(y) dy$ is a solution to

$$\begin{cases} Lu = f \\ R_j u = 0, j=1 \end{cases}$$

Here

$$g(x,y) = \underbrace{\left(\sum_{k=1}^n a_k(y) u_k(x) \right)}_{\equiv e(x,y)} \underbrace{\Theta(x-y)}_{\text{Heaviside}} + \sum_{k=1}^n b_k(y) u_k(x)$$

$$\begin{cases} e_x^{(k)}(y,y) = 0, k=0,1,\dots,n-2 \\ e_x^{(n-1)}(y,y) = 1 \quad (Lu = c_0 u^{(n)} + c_{n-1} u^{(n-1)} + \dots + c_0 u) \end{cases}$$

and

$$R_j(g(\cdot, y)) = 0, \quad 0 < y < 1, \quad j=1,2,\dots,n.$$

Note:

$$\begin{aligned} \int_0^1 g(x,y) f(y) dy &= \int_0^1 e(x,y) \Theta(x-y) f(y) dy + \int_0^1 \sum_{k=1}^n b_k(y) u_k(x) f(y) dy = \\ &= \underbrace{\int_0^x \sum_{k=1}^n a_k(y) u_k(x) f(y) dy}_{L[\cdot \cdot] = f} + \underbrace{\sum_{k=1}^n \int_0^1 b_k(y) f(y) dy}_{L[\cdot] = 0} \underbrace{[u_k(x)]}_{\text{basis for } N(L)} \end{aligned}$$

Calculate $g(x, y) : n=2$

Set $e(x, y) = a_1(y)u_1(x) + a_2(y)u_2(x)$

$$\begin{cases} e(y, y) = a_1(y)e^y + a_2(y)e^{-y} = 0 \end{cases} \dots \quad (1)$$

$$\begin{cases} e'_x(y, y) = a_1(y)e^y - a_2(y)e^{-y} = 1 \end{cases} \dots \quad (2)$$

$$\begin{cases} a_1(y) = \frac{1}{2}e^{-y} \\ a_2(y) = -\frac{1}{2}e^y \end{cases}$$

$$e(x, y) = \frac{1}{2}e^{-y}e^x - \frac{1}{2}e^y e^{-x} = \frac{1}{2}(e^{x-y} - e^{y-x}), \quad (x, y) \in [0, 1] \times [0, 1]$$

Set

$$g(x, y) = e(x, y) \Theta(x-y) + b_1(y)u_1(x) + b_2(y)u_2(x)$$

For $0 < y < 1$

R, $g(\cdot, y) = 0$, i.e. $g(0, y) = 0$ for $y \in (0, 1)$

i.e. $b_1(y)u_1(0) + b_2(y)u_2(0) = 0$ for $y \in (0, 1)$

$$\text{So } \boxed{b_1(y) + b_2(y) = 0} \text{ for } y \in (0, 1). \quad (1)$$

$$R_2 g(\cdot, y) = 0 \quad y \in (0, 1)$$

i.e. $g(1, y) = 0$ for $y \in (0, 1)$

i.e.

$$e(1, y) + b_1(y)u_1(1) + b_2(y)u_2(1) = 0 \quad \text{for } y \in (0, 1)$$

$$\boxed{\frac{1}{2}(e^{1-y} - e^{y-1}) + b_1(y)e + b_2(y)e^{-1} = 0} \quad \text{for } y \in (0, 1) \quad (2)$$

(1), (2) gives

$$\begin{cases} b_1(y) = -b_2(y) & y \in (0, 1) \\ b_2(y)(e^{-1} - e) = \frac{1}{2}(e^{y-1} - e^{1-y}) & y \in (0, 1) \end{cases}$$

$$\text{So } \begin{cases} b_2(y) = \frac{\frac{1}{2}(e^{y-1} - e^{1-y})}{e^{-1} - e} = \frac{1}{2} \frac{e^{2-y} - e^y}{e^2 - 1} \\ b_1(y) = \frac{1}{2} \frac{e^y - e^{2-y}}{e^2 - 1} \end{cases}$$

We obtain

$$g(x,y) = \frac{1}{2}(e^{x-y} - e^{y-x})\theta(x-y) + \frac{1}{2} \frac{e^{x+y} - e^{x+2-y}}{e^2 - 1} + \\ + \frac{1}{2} \frac{e^{x-y} - e^{y-x}}{e^2 - 1}$$

Question: $g(x,y) = g(y,x)$ all $x,y \in [0,1]$?

In general, we say that $h_0 = h \Big|_{C_R^n([0,1])}$ is symmetric if

$$\langle h_0(u), v \rangle_{L^2} = \langle u, h(v) \rangle_{L^2} \quad \forall u, v \in C_R^n([0,1])$$

Example (cont.)

$$h(u) = u'' - u \quad \text{DE}$$

$$u(0) = u(1) = 0 \quad \text{BC}$$

$$u, v \in C^2_R([0,1])$$

$$\begin{aligned} \langle h_0(u), v \rangle_{L^2} &= \int_0^1 h_0(u) \bar{v} dx = \int_0^1 u'' \bar{v} dx = - \int_0^1 \underbrace{u' \bar{v}'}_{-u\bar{v}} dx + \underbrace{[u' \bar{v}]'}_{=0} = \\ &= - \int_0^1 (u' \bar{v}' + u \bar{v}') dx = \int_0^1 u (\bar{v}'' - \bar{v}) dx = \int_0^1 \bar{u} h_0 v dx = \langle u, h_0(v) \rangle_{L^2} \end{aligned}$$

Föreläsning 14

BVP

$$\begin{cases} Lu = u^{(n)} + c_{n-1}(x)u^{(n-1)} + \dots + c_1(x)u' + c_0(x)u = f \in C([0,1]), \quad x \in [0,1] \\ R_j u = \sum_{k=0}^{n-1} (\alpha_{kj}u^{(k)}(0) + \beta_{kj}u^{(k)}(1)) = 0, \quad j = 1, \dots, n \end{cases}$$

$$\alpha_{kj}, \beta_{kj} \in \mathbb{C} \quad k = 0, \dots, n-1, \quad j = 1, \dots, n$$

- $N(L) = \{u \in C^n([0,1]) : Lu = 0\}$, u_1, u_2, \dots, u_n basis for $N(L)$
- $L_0 = L \Big|_{C_R^n([0,1])}$
- $C_R^n([0,1]) = \{u \in C^n([0,1]) : R_j u = 0, \quad j = 1, \dots, n\}$
- $L_0 u \in C([0,1])$ for $u \in C_R^n([0,1])$

Theorem: Assume

$$\det(R_j u_k) \neq 0$$

Then

- ① $L_0 : C_R^n([0,1]) \rightarrow C([0,1])$ is a bijection
- ② Set $G = L_0^{-1}$. Then there exists a continuous function $g(x,y)$ in $[0,1] \times [0,1]$ such that

$$G(f) = \int_0^1 g(x,y) f(y) dy, \quad x \in [0,1]$$

and $f \in C([0,1])$ and g , called the Green's function for L and R_j , $j = 1, \dots, n$ can be given by

$$g(x,y) = e(x,y)\Theta(x-y) + \sum_{k=1}^n b_k(y)u_k(x)$$

$$\text{when } e(x,y) = \sum_{k=1}^n a_k(y)u_k(x)$$

where

$$\left. \left(\frac{\partial}{\partial x} \right)^l e(x,y) \right|_{x=y} = 0, \quad l = 0, 1, \dots, n-2$$

$$\left. \left(\frac{\partial}{\partial x} \right)^{n-1} e(x,y) \right|_{x=y} = 1 \quad \text{for } y \in [0,1]$$

and

$$R_j(g(\cdot, y)) = 0, \quad 0 < y < 1, \quad j=1, 2, \dots, n.$$

Remark: Consider the problem

$$\begin{cases} Lu = f(x, u), \quad x \in [0, 1] \\ R_j u = c_j, \quad j=1, 2, \dots, n. \end{cases} \quad \text{not zero!}$$

Pick any $\tilde{u} \in C^n([0, 1])$ such that

$$R_j \tilde{u} = c_j, \quad j=1, \dots, n.$$

Set $u = \tilde{u} + v$.

Note that $R_j v = R_j(u - \tilde{u}) = R_j u - R_j \tilde{u} = 0, \quad j=1, 2, \dots, n.$

$$h(\tilde{u} + v) = f(x, \tilde{u} + v)$$

gives

$$hv = f(x, \tilde{u} + v) - h\tilde{u} = \hat{f}(x, v)$$

Solve

$$\begin{cases} Lv = \hat{f}(x, v) \\ R_j v = 0, \quad j=1, 2, \dots, n \end{cases}$$

Moreover

$$\text{Set } T(v)(x) = \int_0^x g(x, y) \hat{f}(y, v(y)) dy, \quad x \in [0, 1]$$

Apply a fixed point theorem.

$$T: C^2([0, 1]) \rightarrow C^2([0, 1])$$

• Calculate with $\|u\| = \max_{x \in [0, 1]} |u(x)|$

• $(C^2([0, 1]), \|\cdot\|)$ is not a Banach space

Instead

$$T: C([0, 1]) \rightarrow C([0, 1])$$

$(C([0, 1]), \|\cdot\|)$ is a Banach space.

If $u \in C([0, 1])$ is a fixed point, then

$$u(x) = \int_0^x g(x, y) f(y, u(y)) dy, \quad x \in [0, 1].$$

We actually have $u \in C^2([0, 1]).$

Call h_0 symmetric if

$$\langle h_0 u, v \rangle_{L^2} = \langle u, h_0 v \rangle_{L^2} \quad \forall u, v \in C_c([0,1])$$

where

$$\langle f, g \rangle_{L^2} = \int_0^1 f(x) \cdot \overline{g(x)} dx$$

Theorem:

$$L_0 : C_c([0,1]) \rightarrow C([0,1]) \text{ bijection}$$

Then ① \Leftrightarrow ② \Leftrightarrow ③

- ① h_0 symmetric
- ② \tilde{G} self-adjoint
- ③ $g(x,y) = \overline{g(y,x)}$ all $(x,y) \in [0,1] \times [0,1]$

Here

$$G(f) = \int_0^1 g(x,y) f(y) dy, \quad f \in C([0,1])$$

$$\tilde{G}(f) = \int_0^1 g(x,y) f(y) dy, \quad f \in L^2([0,1]) \quad L^2\text{-norm}$$

$$\tilde{G} \in \mathcal{B}(L^2([0,1]), L^2([0,1])) \quad \text{and} \quad \overline{C([0,1])^{L^2([0,1])}} = L^2([0,1]).$$

So

$$\textcircled{1} \quad \langle h_0(G(f)), G(h) \rangle = \langle G(f), h_0(G(h)) \rangle, \quad \forall f, h \in C([0,1]).$$

Hence

$$\langle f, G(h) \rangle = \langle G(f), h \rangle \quad \forall f, h \in C([0,1]).$$

Similar arguments

$$\Rightarrow \langle f, \tilde{G}(h) \rangle = \langle \tilde{G}(f), h \rangle \quad \forall f, h \in L^2([0,1]) \quad (*)$$

Hence \tilde{G} is self-adjoint in $\mathcal{B}(L^2([0,1]), L^2([0,1]))$.

Given ②. From $(*)$ we get

$$\begin{aligned} \int_0^1 f(x) \int_0^1 g(x,y) h(y) dy dx &= \int_0^1 \int_0^1 g(x,y) f(y) dy \overline{h(x)} dx = \\ &= \int_0^1 f(y) \int_0^1 g(x,y) \overline{h(x)} dx dy = \int_0^1 f(x) \int_0^1 \overline{g(y,x)} h(y) dy dx = \end{aligned}$$

$$\Rightarrow \int_0^1 f(x) \int_0^1 (g(x,y) - \overline{g(y,x)}) h(y) dy dx = 0 \quad f, g, h \text{ continuous}$$

This implies

$$\int_0^1 (g(x,y) - \overline{g(y,x)}) h(y) dy = 0, \quad \forall x \in [0,1]$$

This implies

$$g(x,y) = \overline{g(y,x)} \quad \forall x, y \in [0,1]$$

Theorem: Assume L_0 symmetric and bijection.

Then

- ① 0 is not an eigenvalue of L_0
- 0 is not an eigenvalue of \tilde{G}
- ② f is an eigenfunction for L_0 with the eigenvalue μ if and only if f is an eigenfunction for \tilde{G} with the eigenvalue $\frac{1}{\mu}$.

Since

$$\text{① } N(L_0) = \{0\} \text{ so 0 is not an eigenvalue of } L_0.$$

If f is an eigenfunction for \tilde{G} with eigenvalue 0, then for $u \in C_R^n([0,1])$

$$\langle f, u \rangle = \langle f, G(L_0(u)) \rangle = \langle f, \tilde{G}(L_0(u)) \rangle = \langle \tilde{G}(f), L_0 u \rangle = \langle 0, L_0 u \rangle = 0$$

$$\Rightarrow \langle f, u \rangle = 0, \quad \forall u \in C_R^n([0,1])$$

Claim: $C_R^n([0,1])$ dense in $L^2([0,1])$

If so $f = 0$.

$$\text{② Assume } L_0(f) = \mu f, \quad f \in C_R^n([0,1]) \setminus \{0\}$$

$$0 \neq f = G(L_0(f)) = G(\mu f) = \tilde{G}(\mu f) = \mu \tilde{G}(f)$$

$$\text{So } \tilde{G}(f) = \frac{1}{\mu} f.$$

Assume $\tilde{G}(f) = \frac{1}{\mu} f$, $f \in L^2([0,1])$

We have

$\tilde{G}(f)(x) = \frac{1}{\mu} f(x)$ for all $x \in [0,1]$ except
for x in a zero-set.

Consider

$$\underbrace{\begin{array}{c} \tilde{\mu} \tilde{G}(f)(x) \\ \text{Continuous} \\ \text{function} \end{array}}_{\in C([0,1])}$$

Set $h(x) = \tilde{\mu} \tilde{G}(f)(x)$, $x \in [0,1]$

$$h(x) = \mu \tilde{G}(h)(x) = \underbrace{\mu \tilde{G}(h)}_{\in C_B([0,1])}(x)$$

$$L_0 h = L_0 (\mu \tilde{G}(h)) = \mu L_0 \tilde{G}(h) = \mu h$$

So

$$L_0(h) = \mu h \quad \square$$

Theorem: Assume L_0 symmetric and bijection.

Let $(\mu_n)_{n=1}^\infty$ be eigenvalues of L_0 counted with multiplicity,
and $(e_n)_{n=1}^\infty$ corresponding ON-sequence.

Then

① $(e_n)_{n=1}^\infty$ is an ON-basis for $L^2([0,1])$

② The solution of $\begin{cases} Lu = f \in C([0,1]) \\ R_j u = 0, \quad j=1, 2, \dots, n \end{cases}$

is given by

$$u = \sum_{n=1}^{\infty} \frac{1}{\mu_n} \langle f, e_n \rangle e_n, \quad \text{in } L^2([0,1])$$

(Note $L e_n = \mu_n e_n$)

Method of continuity — Assume

Assume there exists $C > 0$
such that

$$1) \|x\| \leq C \|A_t(x)\| \quad \forall x \in E, t \in [0,1]$$

$$2) \|A_t(x) - A_s(x)\| \leq C |t-s| \|x\|, \quad \forall x \in E \text{ and } s, t \in [0,1]$$

$$3) A_0 \text{ invertible} \implies A_1 \text{ invertible}$$

• $(E, \|\cdot\|)$ Banach space

• $A_t \in \mathcal{B}(E, E)$, $t \in [0,1]$

Proof: Assume A_+ is invertible

$$A_s = \underbrace{A_+}_{\text{invertible}} \underbrace{(I + A_+^{-1}(A_s - A_+))}_{\substack{\text{invertible if} \\ \|A_+^{-1}(A_s - A_+)\| < 1}} \quad \text{by the "Neumann series lemma".}$$

But

$$\|A_+^{-1}(A_s - A_+)\| \leq \underbrace{\|A_+^{-1}\|}_{\leq C} \cdot \underbrace{\|A_s - A_+\|}_{\leq C|t-s|}$$

So if $|t-s| < \frac{1}{C^2}$ then A_s is invertible.

Pick a sequence t_k , $k=1, \dots, N$ such that

$$\max_{k=1, \dots, N-1} |t_{k+1} - t| < \frac{1}{C^2}$$

where $0 = t_1 < t_2 < \dots < t_N = 1$

A_0 invertible $\Rightarrow A_{t_1}$ invertible $\Rightarrow A_{t_2}$ invertible $\Rightarrow \dots \Rightarrow A_1$ invertible.

Orthogonal projections

- $(E, \langle \cdot, \cdot \rangle)$ Hilbert space
- S closed subspace $E = S \oplus S^\perp$

For every $x \in E$ there are unique $y \in S$, $z \in S^\perp$ such that

$$x = y + z.$$

Define $P_S : E \rightarrow E$

$$P_S(x) = y, \quad \forall x \in E$$

We note

$$\bullet P_S \text{ linear: } \begin{aligned} x_1 &= y_1 + z_1, \\ x_2 &= y_2 + z_2 \end{aligned} \quad y_1, y_2 \in S, z_1, z_2 \in S^\perp$$

For scalars α_1, α_2 , we have

$$\alpha_1 x_1 + \alpha_2 x_2 = (\underbrace{\alpha_1 y_1 + \alpha_2 y_2}_{\in S}) + (\underbrace{\alpha_1 z_1 + \alpha_2 z_2}_{\in S^\perp})$$

$$P_S(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 y_1 + \alpha_2 y_2 = \alpha_1 P_S(x_1) + \alpha_2 P_S(x_2)$$

$$P_S(x_1) = y_1 \quad \& \quad P_S(x_2) = y_2$$

• P_s bounded:

$$\|P_s(x)\|^2 = \|y\|^2 \leq \|y\|^2 + \|z\|^2 = \|x\|^2$$

So $\|P_s\| \leq 1$, but $\|P_s(y)\| = \|y\| \Rightarrow \|P_s\| = 1$

• P_s self-adjoint

$$\langle P_s(x_1), x_2 \rangle = \langle y_1, x_2 \rangle = \langle y_1, y_2 \rangle = \langle x_1, y_2 \rangle = \langle x_1, P(x_2) \rangle, \forall x_1, x_2 \in E$$

• $P_s^2 = P_s$

$$P_s^2(x) = P_s(P_s(x)) = P_s(y) = y = P_s(x)$$

Proposition: Assume $P \in \mathcal{B}(E, E)$. Then

$P^2 = P$ and P self-adjoint \Rightarrow there exists a closed subspace S in E such that $P = P_s$.

What is S ?

Set $S = \{x \in E : P(x) = x\}$.

Claim: S is a subspace in E .

$$\begin{aligned} & x_1, x_2 \in S, \alpha_1, \alpha_2 \text{ scalars} \xrightarrow{?} \alpha_1 x_1 + \alpha_2 x_2 \in S \\ & P(x_1) = x_1, \quad P(x_2) = x_2 \\ & P(\alpha_1 x_1 + \alpha_2 x_2) = \{P \text{ linear}\} = \alpha_1 P(x_1) + \alpha_2 P(x_2) = \alpha_1 x_1 + \alpha_2 x_2 \end{aligned}$$

Claim: S is closed $\xrightarrow{P(x) \in E \text{ since } P \text{ cont.}}$

Assume $S \ni x_n \xrightarrow{\text{"P(x)} \rightarrow \text{"}} x$ in E . So $x \in S$

By ODT we have $E = S \oplus S^\perp$. Remains to show: $P = P_s$

Fix $x \in E$

$$x = \underbrace{P(x)}_{\in S} + \underbrace{x - P(x)}_{\in S^\perp} \quad \leftarrow \text{Show this}$$

① $P(x) \in S$ since

$$P(P(x)) = P^2(x) = P(x) \text{ by assumption.}$$

② $x - P(x) \in S^\perp$ since for $y \in S$

$$\begin{aligned} \langle y, x - P(x) \rangle &= \langle P(y), x - P(x) \rangle = \langle y, P^*(x - P(x)) \rangle = \\ &= \langle y, P(x - P(x)) \rangle = \langle y, P(x) - P^2(x) \rangle = \langle y, P(x) - P(x) \rangle = 0 \quad \blacksquare \end{aligned}$$

Ex:

- $(E, \langle \cdot, \cdot \rangle)$ Hilbert space
- $(e_n)_{n=1}^{\infty}$ ON-basis
- $(f_n)_{n=1}^{\infty}$ ON-sequence

Assume

$$\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < \infty \implies (f_n)_{n=1}^{\infty} \text{ ON-basis.}$$

Proof:

Step 1: Assume $\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < 1$.

Assume $(f_n)_{n=1}^{\infty}$ is not an ON-basis.

Then there exists $0 \neq x \in E$ such that

$$\langle x, f_k \rangle = 0, \quad k=1,2,\dots$$

But $(e_n)_{n=1}^{\infty}$ is an ON-basis

$$x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$$

Parseval's formula

$$\begin{aligned} 0 < \|x\|^2 &= \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 = \sum_{n=1}^{\infty} |\langle x, e_n - f_n \rangle|^2 \leq \\ &\leq \sum_{n=1}^{\infty} \|x\|^2 \|e_n - f_n\|^2 = \|x\|^2 \underbrace{\sum_{n=1}^{\infty} \|e_n - f_n\|^2}_{\geq 1} < \|x\|^2 \end{aligned}$$

Contradiction.

Conclusion: $(f_n)_{n=1}^{\infty}$ is an ON-basis.

Step 2: Assume $\sum_{n=1}^{\infty} \|e_n - f_n\| < \infty$

Assume $(f_n)_{n=1}^{\infty}$ is not an ON-basis.

Then there exists $0 \neq x \in E$ such that

$$\langle x, f_k \rangle = 0, \quad k=1,2,\dots$$

Since $\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < \infty$

$$\exists N > 0 : \sum_{n=N+1}^{\infty} \|e_n - f_n\|^2 < 1$$

Note that $\text{Span}\{x, f_1, f_2, \dots, f_N\}$ has dimension $N+1$.

Claim: There exists a non-trivial solution to
 $\alpha_0 x + \alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_N f_N \perp e_k \quad k=1,2,\dots,N.$

N equations, $N+1$ unknowns. Homogeneous linear system. There exists

$$y = \alpha_0 x + \alpha_1 f_1 + \dots + \alpha_N f_N$$

where not all α_k 's are 0 such that

$$y \perp e_k \quad \text{for } k=1,2,\dots,N.$$

Note $y \neq 0$.

$(e_n)_{n=1}^{\infty}$ is ON-basis for E .

$$y = \sum_{n=1}^{\infty} \langle y, e_n \rangle e_n = \sum_{n=N+1}^{\infty} \langle y, e_n \rangle e_n$$

Parseval's formula

$$0 < \|y\|^2 = \sum_{n=N+1}^{\infty} |\langle y, e_n \rangle|^2 = \sum_{n=N+1}^{\infty} |\langle y, e_n - f_n \rangle|^2 \leq \|y\|^2 \underbrace{\sum_{n=N+1}^{\infty} \|e_n - f_n\|^2}_{< 1} < \|y\|^2$$

Contradiction!

Conclusion: $(f_n)_{n=1}^{\infty}$ is an ON-basis.