Mathematics Chalmers & GU

TMA372/MMG800: Partial Differential Equations, 2020–08–27, 8:30-12:30

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An open book exam.
Each problem gives max 6p. Valid bonus points will be added to the scores.
Breakings for **Chalmers; 3**: 15-21p, **4**: 22-28p, **5**: 29p-, and for **GU; G**: 15-26p, **VG**: 27p-

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1. Let $u(x)$ be a function such that $u(1) = 3$ and $u(2) = -1$ and

$$\int_1^2 x u'(x) v'(x)\, dx = 0, \quad \forall v: \ v(1) = v(2) = 0.$$

a) Which differential equation and including boundary data solves u?
b) Formulate a suitable finite element method for the problem
c) Give a suitable a priori error estimate for this problem.

2. Determine if the assumptions of the Lax-Milgram theorem are satified for

$$a(v, w) = \int_I v'w'\, dx + v(0)w(0), \quad I = (0, 1), \ L(v) = \int_I fv\, dx, \quad f \in L_2(I), \ V = H^1(I).$$

3. Determine the stifness matrix and load vector in $cG(1)$ finite element method applied to Poisson equation

$$-\Delta u = 2 \quad \text{in } \Omega = \{(x, y) : 0 < x < 2, \ 0 < y < 1\},$$

with a combination of, homogeneous, Neumann boundary conditions at $\Gamma_2 := \{(2, y) : 0 < y < 1\}$ and Dirichlet boundary condition at $\Gamma_1 := \partial\Omega \setminus \Gamma_2$, on a mesh with stepsize 2/3 in the x-direction and 1/3 in y-direction.

4. Derive an *a posteriori* error estimate for the cG(1) solution of the problem

$$-u'' + 2u' + u = f, \ \text{ in } \ I = (0.1), \qquad u(0) = u(1) = 0,$$

in the energy norm $||v||_E^2 = (v, v) = \int_I (v'^2 + v^2)\, dx, \ \ (f \in L_2(I))$.

5. Let Ω be a convex polygonal domain and u_h, the continuous piecewise linear, finite element solution of the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \quad u = 0 & \text{on } \Gamma. \end{cases}$$

Show that there is a constant C independent of u and h such that

$$||u - u_h||_{L_2(\Omega)} \le Ch^2 |u|_{H^2(\Omega)}.$$

Hint: Assume that, for the inhomogeneous equation $-\Delta u = f$, with $f \in H^s(\Omega)$,

$$||u - u_h||_{H^{s+2}(\Omega)} \le C||f||_{H^s(\Omega)}.$$

Use also the interpolation estimate:

$$||u - \pi_h u||_{L_2(\Omega)} \le Ch^2 |u|_{H^2(\Omega)}.$$

6. Let δ denote Dirac delta function and $i = \sqrt{-1}$. Find a solution for the $3D$-problem

$$\ddot{u}(x, t) - \Delta u(x, t) = e^{it}\delta(x), \quad x \in \mathbb{R}^3.$$

Hint: Set $u = e^{it}v$, with $v(x) = w(x)/r$ wgere $r = |x|$. One can show that $rv = w \to \frac{1}{4\pi}$ as $r \to 0$.

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TMA372/MMG800: Partial Differential Equations, 2020–08–27, 8:30-12:30.
Solutions.

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1. Solution: Partial integration with boundary data $v(1) = v(2) = 0$ gives

$$(1) \qquad 0 = \int_1^2 xu'v'\,dv = -\int_1^2 (xu')'x\,dx,$$

which, since $v = v(x)$ is arbitrary, yields

$$\Big(x\,(u'(x)\Big)' = 0, \qquad u(1) = 3, \quad u(2) = -1.$$

Now consider the partition $\mathcal{T}_h : 1 = x_0 < x_1 < \ldots < x_{M+1} = 2$, subintervals $I_k = (x_{k-1}, x_k)$, and the subspace

$$V_h := \{v = v(x) : v \text{ is continuous, and } v|_{I_k} \text{ is linear } \forall k\},$$

and

$$V_h^0 := \{v \in V_h : v(1) = v(2) = 0\}.$$

<u>**FEM**</u>: Vi seek $u_h \in V_h$ such that $u_h(1) = 3$, $u_h(2) = -1$ and

$$(2) \qquad \int_1^2 xu_h'v'\,dv = 0, \qquad \forall v \in V_h^0.$$

From the subtraction (1)-(2) one gets

$$\int_1^2 x(u' - u_h')\,v'\,dx = 0, \qquad \forall v \in V_h^0.$$

Then with the L_2-norm: $||\cdot||$, on $(1,2)$ we have

$$\begin{aligned}||\sqrt{x}(u' - u_h')||^2 &= \int_1^2 x(u' - u_h')\,(u' - u_h')\,dx \\ &\int_1^2 x(u' - u_h')\,(u' - u_h' - v)\,dx \\ &\le ||\sqrt{x}(u' - u_h')||||\sqrt{x}(u' - u_h' - v)||.\end{aligned}$$

Now a suitable choice of v, interploating $u - u_h$ yields

$$||\sqrt{x}(u' - u_h')|| \le ||\sqrt{x}(u' - u_h' - v)|| \le C_i||\sqrt{x}hu''||.$$

2. Solution: For the formulation of the Lax-Milgram theorem see the book, Chapter 2.
As for the given case: $I = (0,1),\ f \in L_2(I),\ V = H^1(I)$ and

$$a(v,w) = \int_I (v'w')\,dx + v(0)w(0), \quad L(v) = \int_I fv\,dx,$$

it is trivial to show that $a(\cdot,\cdot)$ is bilinear and $b(\cdot)$ is linear. We have that

$$(3) \qquad a(v,v) = \int_I (v')^2\,dx + v(0)^2 \ge \frac{1}{2}\int_I (v')^2\,dx + \frac{1}{2}v(0)^2 + \frac{1}{2}\int_I (v')^2\,dx.$$

Further

$$v(x) = v(0) + \int_0^x v'(y)\,dy, \quad \forall x \in I$$

implies

$$v^2(x) \le 2\Big(v(0)^2 + (\int_0^x v'(y)\,dy)^2\Big) \le \{C-S\} \le 2v(0)^2 + 2\int_0^1 v'(y)^2\,dy,$$

so that

$$\frac{1}{2}v(0)^2 + \frac{1}{2}\int_0^1 v'(y)^2\,dy \geq \frac{1}{4}v^2(x), \quad \forall x \in I.$$

Integrating over x we get

$$\frac{1}{2}v(0)^2 + \frac{1}{2}\int_0^1 v'(y)^2\,dy \geq \frac{1}{4}\int_I v^2(x)\,dx. \tag{4}$$

Now combining (3) and (4) we get

$$\begin{aligned} a(v,v) &\geq \frac{1}{4}\int_I v^2(x)\,dx + \frac{1}{2}\int_I (v')^2(x)\,dx \\ &\geq \frac{1}{4}\Big(\int_I v^2(x)\,dx + \int_I (v')^2(x)\,dx\Big) = \frac{1}{4}||v||_V^2, \end{aligned}$$

so that we can take $\kappa_1 = 1/4$. Further

$$\begin{aligned} |a(v,w)| &\leq \Big|\int_I v'w'\,dx\Big| + |v(0)w(0)| \leq \{C-S\} \leq ||v'||_{L_2(I)}||w'||_{L_2(I)} + |v(0)||w(0)| \\ &\leq ||v||_V||w||_V + |v(0)||w(0)| \end{aligned}$$

Now we have that

$$v(0) = -\int_0^x v'(y)\,dy + v(x), \quad \forall x \in I, \tag{5}$$

and by the Mean-value theorem for the integrals: $\exists \xi \in I$ so that $v(\xi) = \int_0^1 v(y)\,dy$. Choose $x = \xi$ in (5) then

$$\begin{aligned} |v(0)| &= \Big| -\int_0^\xi v'(y)\,dy + \int_0^1 v(y)\,dy\Big| \\ &\leq \int_0^1 |v'|\,dy + \int_0^1 |v|\,dy \leq \{C-S\} \leq ||v'||_{L_2(I)} + ||v||_{L_2(I)} \leq 2||v||_V, \end{aligned}$$

implies that

$$|v(0)||w(0)| \leq 4||v||_V||w||_V,$$

and consequently

$$|a(u,w)| \leq ||v||_V||w||_V + 4||v||_V||w||_V = 5||v||_V||w||_V,$$

so that we can take $\kappa_2 = 5$. Finally

$$|L(v)| = \Big|\int_I fv\,dx\Big| \leq ||f||_{L_2(I)}||v||_{L_2(I)} \leq ||f||_{L_2(I)}||v||_V,$$

taking $\kappa_3 = ||f||_{L_2(I)}$ all the conditions in the Lax-Milgram theorem are fulfilled.

3. Solution: We use the notation $(x,y) = (x_1,x_2)$ and hence $\Gamma_1 := \partial\Omega \setminus \Gamma_2$ where $\Gamma_2 := \{(2,x_2) : 0 \leq x_2 \leq 1\}$. Define

$$V = \{v : v \in H^1(\Omega),\ v = 0 \text{ on } \Gamma_1\}.$$

Multiply the equation by $v \in V$ and integrate over Ω; using Green's formula

$$\int_\Omega \nabla u \cdot \nabla v - \int_\Gamma \frac{\partial u}{\partial n}v = \int_\Omega \nabla u \cdot \nabla v = 2\int_\Omega v,$$

where we have used $\Gamma = \Gamma_1 \cup \Gamma_2$ and the fact that $v = 0$ on Γ_1 and $\frac{\partial u}{\partial n} = 0$ on Γ_2.

Variational formulation: Find $u \in V$ such that

$$\int_\Omega \nabla u \cdot \nabla v = 2\int_\Omega v, \qquad \forall v \in V.$$

FEM: cG(1):

Find $U \in V_h$ such that

$$\int_\Omega \nabla U \cdot \nabla v = 2\int_\Omega v, \qquad \forall v \in V_h \subset V, \tag{6}$$

where

$$V_h = \{v : v \text{ is piecewise linear and continuous in } \Omega,\ v = 0 \text{ on } \Gamma_1, \text{ on the given mesh }\}.$$

A set of bases functions for the finite dimensional space V_h can be written as $\{\varphi_i\}_{i=1}^6$, where

$$\begin{cases} \varphi_i \in V_h, & i = 1,2,3,4,5,6 \\ \varphi_i(N_j) = \delta_{ij}, & i,j = 1,2,3,4,5,6 \end{cases}$$

Then the equation (6) is equivalent to: Find $U \in V_h$ such that

$$\int_\Omega \nabla U \cdot \nabla \varphi_i = 2\int_\Omega \varphi_i, \qquad i = 1,2,3,4,5,6. \tag{7}$$

Set $U = \sum_{j=1}^6 \xi_j \varphi_j$. Invoking in the relation (3) above we get

$$\sum_{j=1}^6 \xi_j \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i = 2\int_\Omega \varphi_i, \qquad i = 1,2,3,4,5,6.$$

Now let $a_{ij} = \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i$ and $b_i = \int_\Omega \varphi_i$, then we have that

$$A\xi = b, \quad A \text{ is the stiffness matrix } \; b \text{ is the load vector.}$$

To compute compute a_{ij} and b_i we note that area of the standard element T, with base $= 2/3$ and hight $= 1/3$, is

$$|T| = 1/2 \cdot 2/3 \cdot 1/3 = 1/9$$

and the bases functions, and their gradients, for the standard element, with base $= 2/3$ and hight $= 1/3$, are

$$\begin{cases} \phi_1(x,y) &= 1 - 3(\frac{x}{2} + y) \\ \phi_2(x,y) &= \frac{3}{2}x \\ \phi_3(x,y) &= 3y \end{cases} \Longrightarrow \begin{cases} \nabla\phi_1(x,y) &= -3\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \\ \nabla\phi_2(x,y) &= 3\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \\ \nabla\phi_3(x,y) &= 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{cases}$$

Thus

$$b_i = \int_\Omega \varphi_i = \begin{cases} 6 \cdot \frac{1}{3} \cdot |T| \cdot 1 = 2/9, & i = 1,2,3,4 \\ 3 \cdot \frac{1}{3} \cdot |T| \cdot 1 = 1/9, & i = 5,6. \end{cases}$$

and the standard stiffness matrix elements are

$$\begin{aligned} s_{11} &= (\nabla\phi_1, \nabla\phi_1) = \textstyle\int_T \nabla\phi_1 \cdot \nabla\phi_1 = \frac{1}{9} \cdot 9(\frac{1}{4} + 1) = \frac{5}{4} \\ s_{12} &= (\nabla\phi_1, \nabla\phi_2) = \textstyle\int_T \nabla\phi_1 \cdot \nabla\phi_2 = \frac{1}{9} \cdot (-9)\frac{1}{4} = -\frac{1}{4} \\ s_{13} &= (\nabla\phi_1, \nabla\phi_3) = \textstyle\int_T \nabla\phi_1 \cdot \nabla\phi_3 = \frac{1}{9} \cdot (-9) \cdot 1 = -1 \\ s_{22} &= (\nabla\phi_2, \nabla\phi_2) = \textstyle\int_T \nabla\phi_2 \cdot \nabla\phi_2 = \frac{1}{9} \cdot 9 \cdot \frac{1}{4} = \frac{1}{4} \\ s_{23} &= (\nabla\phi_2, \nabla\phi_3) = \textstyle\int_T \nabla\phi_2 \cdot \nabla\phi_3 = 0 \\ s_{33} &= (\nabla\phi_3, \nabla\phi_3) = \textstyle\int_T \nabla\phi_3 \cdot \nabla\phi_3 = \frac{1}{9} \cdot 9 \cdot 1 = 1. \end{aligned}$$

and hence the local element-stiffness matrix, taking the symmetry into account, is:

$$S = \begin{pmatrix} 5/4 & -1/4 & -1 \\ -1/4 & 1/4 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

To compute elements a_{ij} for the global stiffeness matrix A we have that

$$a_{ii} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_i = \begin{cases} 2 \cdot (\frac{5}{4} + \frac{1}{4} + 1) = 5, & i = 1,2,3,5 \\ \frac{5}{4} + \frac{1}{4} + 1 = 5/2, & i = 5,6 \end{cases}$$

Further

$$\begin{cases} a_{12} = a_{34} = 2s_{13} = -2 \\ a_{13} = a_{24} = a_{35} = a_{46} = 2s_{12} = -\frac{1}{2} \\ a_{14} = a_{36} = 2s_{12} = -\frac{1}{2} \\ a_{15} = a_{16} = a_{23} = a_{25} = a_{26} = a_{45} = 0 \\ a_{56} = s_{13} = -1 \end{cases}$$

Thus we have

$$A = \begin{pmatrix} 5 & -2 & -1/2 & -1/2 & 0 & 0 \\ -2 & 5 & 0 & -1/2 & 0 & 0 \\ -1/2 & 0 & 5 & -2 & -1/2 & -1/2 \\ -1/2 & -1/2 & -2 & 5 & 0 & -1/2 \\ 0 & 0 & -1/2 & 0 & 5/2 & -1 \\ 0 & 0 & -1/2 & -1/2 & -1 & 5/2 \end{pmatrix} \qquad b = \frac{1}{9}\begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 1 \\ 1 \end{pmatrix}.$$

4. Solution: The Variational formulation: Let $V^0 := H_0^1(0,1)$, Multiply the equation by $v \in V^0$, integrate by parts over $(0,1)$ and use the boundary conditions to obtain

$$\text{(8)} \qquad \text{Find } u \in V^0: \quad \int_0^1 u'v'\,dx + 2\int_0^1 u'v\,dx + \int_0^1 uv\,dx = \int_0^1 fv\,dx, \quad \forall v \in V^0.$$

cG(1): Let $V_n^0 := \{w \in V^0 : w \text{ is cont., p.l. on a partition of } I, w(0) = w(1) = 0\}$

$$\text{(9)} \qquad \text{Find } U \in V_h^0: \quad \int_0^1 U'v'\,dx + 2\int_0^1 U'v\,dx + \int_0^1 Uv\,dx = \int_0^1 fv\,dx, \quad \forall v \in V_h^0.$$

From (1)-(2), we find The Galerkin orthogonality:

$$\text{(10)} \qquad \int_0^1 \Big((u-U)'v' + 2(u-U)'v + (u-U)v\Big)\,dx = 0, \quad \forall v \in V_h^0.$$

We define the inner product $(\cdot,\cdot)_E$ associated to the energy norm to be

$$(v,w)_E = \int_0^1 (v'w' + vw)\,dx, \qquad \forall v, w \in V^0.$$

Note that

$$\text{(11)} \qquad 2\int_0^1 e'e\,dx = [e^2]_0^1 = 0$$

Thus using (11) we have

$$\text{(12)} \qquad ||e||_E^2 = \int_0^1 (e'e' + ee)\,dx = \int_0^1 (e'e' + 2e'e + ee)\,dx.$$

We split the second factor e as $e = u - U = u - v + v - U$, with $v \in V_h$ and write

$$\begin{aligned} ||e||_E^2 &= \int_0^1 \Big(e'(u-U)' + 2e'(u-U) + e(u-U)\Big)\,dx = \Big\{v \in V_h^0\Big\} \\ &= \int_0^1 \Big(e'(u-v)' + 2e'(u-v) + e(u-v)\Big)\,dx \\ &+ \int_0^1 \Big(e'(v-U)' + 2e'(v-U) + e(v-U)\Big)\,dx \\ &= \int_0^1 \Big(e'(u-v)' + 2e'(u-v) + e(u-v)\Big)\,dx, \end{aligned}$$

where, in the last step, we have used the Galerkin orthogonality to eliminate terms involving U. Now we can write

$$\begin{aligned} ||e||_E^2 &= \int_0^1 \Big(e'(u-v)' + 2e'(u-v) + e(u-v)\Big)\,dx \\ &\leq 2||e'||\cdot||u-v||_E + ||e||\cdot||u-v|| \\ &\leq 2||e||_E \cdot ||u-v||_E \end{aligned}$$

and derive the a priori error estimate:

$$||e||_E \leq ||u-v||_E(1+\alpha), \quad \forall v \in V_h.$$

To obtain a posteriori error estimates the idea is to eliminate u-terms, by using the differential equation, and replacing their contributions by the data f. Then this f combined with the remaining U-terms would yield to the residual error:

A posteriori error estimate:

$$\begin{aligned} ||e||_E^2 &= \int_0^1 (e'e' + ee)\,dx = \int_0^1 (e'e' + 2e'e + ee)\,dx \\ &= \int_0^1 (u'e' + 2u'e + ue)\,dx - \int_0^1 (U'e' + 2U'e + Ue)\,dx. \end{aligned} \tag{13}$$

Now using the variational formulation (8) we have that

$$\int_0^1 (u'e' + 2u'e + ue)\,dx = \int_0^1 fe\,dx.$$

Inserting in (13) and using (9) with $v = \Pi_k e$ we get

$$\begin{aligned} ||e||_E^2 &= \int_0^1 fe\,dx - \int_0^1 (U'e' + 2U'e + Ue)\,dx \\ &+ \int_0^1 (U'\Pi_h e' + 2U'\Pi_h e + U\Pi_h e)\,dx - \int_0^1 f\Pi_h e\,dx. \end{aligned} \tag{14}$$

Thus

$$\begin{aligned} ||e||_E^2 &= \int_0^1 f(e-\Pi_h e)\,dx - \int_0^1 \Big(U'(e-\Pi_h e)' + 2U'(e-\Pi_h e) + U(e-\Pi_h e)\Big)\,dx \\ &= \int_0^1 f(e-\Pi_h e)\,dx - \int_0^1 (2U'+U)(e-\Pi_h e)\,dx - \sum_{j=1}^{M+1}\int_{I_j} U'(e-\Pi_h e)'\,dx \\ &= \{\text{partial integration}\} \\ &= \int_0^1 f(e-\Pi_h e)\,dx - \int_0^1 (2U'+U)(e-\Pi_h e)\,dx + \sum_{j=1}^{M+1}\int_{I_j} U''(e-\Pi_h e)\,dx \\ &= \int_0^1 (f + U'' - 2U' - U)(e-\Pi_h e)\,dx = \int_0^1 R(U)(e-\Pi_h e)\,dx \\ &= \int_0^1 hR(U)h^{-1}(e-\Pi_h e)\,dx \leq ||hR(U)||_{L_2}||h^{-1}(e-\Pi_h e)||_{L_2} \\ &\leq C_i||hR(U)||_{L_2}\cdot||e'||_{L_2} \leq ||hR(U)||_{L_2}\cdot||e||_E. \end{aligned}$$

This gives the a posteriori error estimate:

$$||e||_E \leq C_i||hR(U)||_{L_2},$$

with $R(U) = f + U'' - 2U' - U = f - 2U' - U$ on (x_{i-1}, x_i), $i = 1, \dots, M+1$.

5. Solution: The variational formulation to this problems reads, find $u \in V$ such that

$$a(u, v) = (f, v), \qquad \forall v \in V, \tag{15}$$

where

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v, \qquad (f, v) = \int_\Omega f v \, dx,$$

and

$$V := \{v : \; v \text{ is continuous on } \; \Omega \; \text{ and } \; v = 0 \text{ on } \; \Gamma\}$$

Considering a triangulation $\mathcal{T}_h$ with elements K so that $\Omega = \cup_{K \in \mathcal{T}_h} K$ and the mesh size h as the maximum diameter of triangles $K \in \mathcal{T}_h$ we define the finite element space

$$V_h := \{v : \; v \text{ is continuous on } \; \Omega \, v|_K \; \text{ is linear for } \; K \in \mathcal{T}_h \; \text{and} \; v = 0 \text{ on } \; \Gamma\}.$$

With the standard continuous piecewise linear bases function φ_i, the finite element representation for $v \in V_h$ is

$$v(x) = \sum_{j=1}^{M} \xi_j \varphi_j(x), \quad \xi_j = v(N_j), \quad x \in \Omega \cup \Gamma, \quad N_j : \; j - \text{th node} \; .$$

We can now formulate the finite element method for our problem as: Find $u_h \in V_h$ such that

$$a(u_h, v) = (f, v) \qquad \forall v \in V_h. \tag{16}$$

Subtracting (15) and (16) we have that

$$a(e, v) = 0 \qquad \forall v \in V_h, \tag{17}$$

where $e = u - u_h$. Now let ψ be the solution of the following auxiliary dual problem:

$$-\Delta \psi = e, \quad \text{in } \; \Omega, \qquad \psi = 0 \quad \text{on } \; \Gamma. \tag{18}$$

Now using the first hint (with $s = 0$) we get

$$||\psi||_{H^2(\Omega)} \leq C||e||_{L_2(\Omega)}, \tag{19}$$

where the constant C does not depend on e. Using Green's formula and the fact that $e = 0$ on Γ yields

$$(e, e) = -(e, \Delta \psi) = a(e, \psi) = a(e, \psi - \pi_h \psi),$$

where the last equality follows from (17) since $\pi_h \psi \in V_h$ so that $a(e, \pi_h \psi) = 0$. Applying the interpolation estimate (hint 2) and using (19) we get

$$\begin{aligned} ||e||^2_{L_2(\Omega)} &\leq ||e||_{H^1(\Omega)} ||\psi - \pi_h \psi||_{H^1(\Omega)} \leq C||e||_{H^1(\Omega)} h |\psi|_{H^2(\Omega)} \\ &\leq Ch||e||_{H^1(\Omega)} ||e||_{L_2(\Omega)} \end{aligned} \tag{20}$$

Now dividing by $||e||_{L_2(\Omega)}$ and using the first oder estimate

$$||e||_{H^1(\Omega)} \leq Ch|u|_{H^2(\Omega)}, \tag{21}$$

we finally get the desires estimate

$$||e||_{L_2(\Omega)} \leq Ch||e||_{H^1(\Omega)} \leq Ch^2 |u|_{H^2(\Omega)}.$$

6. Solution: Inserting the ansatz in equation yields

$$-e^{it}v - e^{it}\Delta v = e^{it}\delta, \qquad \text{i.e.} \quad -\Delta v - v = \delta.$$

Now letting $v = w/r$ we end up with the equation

$$-w'' - w = 0,$$

with the solution

$$w(r) = a\cos(r) + b\sin(r), \qquad r > 0.$$

FÃ¶r the solution the equality $a = \frac{1}{4\pi}$ should be valid (just compare with the solution $v = \frac{1}{4\pi}\frac{1}{r}$ of the equation $-\Delta v = \delta$), while b may be choosen arbitrary, e.g. $b = 0$ (note that $\frac{\sin r}{r}$ solves the homogeneous equation $-\Delta v - v = 0$). Thus we have found the solution

$$v = \frac{1}{4r}\frac{\cos(r)}{r},$$

and hence the corresponding

$$u = e^{it}\frac{1}{4r}\frac{\cos(r)}{r}.$$