Mathematics Chalmers & GU

TMA372/MMG800: Partial Differential Equations, 2020–06–09, 14:00-18:00

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An open book exam.
Each problem gives max 6p. Valid bonus poits will be added to the scores.
Breakings for **Chalmers; 3**: 15-21p, **4**: 22-28p, **5**: 29p-, and for **GU; G**: 15-26p, **VG**: 27p-

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1. Consier the following Dirichlet boundary value problem

$$-u'' = f, \quad x \in I = (0,1), \qquad u(0) = u(1) = 0.$$

Let u_h be a finite element approximation of the solution u and show the a posteriori error estimte

$$\|(u - u_h)'\|^2 \le C \sum_{j=1}^{N} r_j^2(u_h),$$

where r_j is the L_2-norm of the elementwise residual in a partition $0 = x_0 < x_1 < x_2 < \ldots < x_N = 1$ of $I = \cup_j I_j;\ I_j = (x_{j-1}, x_j),\ j = 1, 2, \ldots N$, given by $r_j(u_h) = h_j \|f + u_h''\|_{L_2(I_j)}$.

2. A model problem for the traffic flow of cars with speed u and density ρ can be written as

$$\dot{\rho} + (u\rho') = 0. \tag{1}$$

Assuming $u = c - \varepsilon(\rho'/\rho)$ (**), yields convection-diffusion equation $\dot{\rho} + c\rho' - \varepsilon\rho'' = 0$. Give a full motivation for this choice (**) of u.

3. Determine the two point boundary value problem having FEM linear system of equations, viz.

$$S\xi = \mathbf{a} + \mathbf{b} + \mathbf{c},$$

where S is $(m+1) \times (m+1)$ matrix with EVEN m and with $s_{ii} = 2/h;\ i = 1, \ldots m$, and $s_{m+1,m+1} = 1/h$, $s_{i,i+1} = s_{i+1,i} = -1/h$, $i = 1, \ldots m$. Further $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are $(m+1) \times 1$ vectors:

$$a_i = 0, \quad i = 1, 2, \ldots m, \qquad a_{m+1} = 1$$

$$b_i = 7h, \quad i = 1, \ldots, \frac{m}{2} + 1, \qquad b_i = 0, \quad i = \frac{m}{2} + 2, \ldots, m+1$$

$$c_1 = -\frac{5}{h}, \qquad c_i = 0, \quad i = 2, \ldots m+1.$$

4. Formulate and prove the Lax-Milgram theorem in full details for the $2d$-problem

$$-\Delta u + \alpha u = f, \quad x \in \Omega, \qquad n \cdot \nabla u = 0, \quad x \in \partial\Omega.$$

5. a) Consider the Schrödinger equation

$$i\,\dot{u} - \Delta u = 0, \quad in\ \Omega, \qquad u = 0, \quad on\ \partial\Omega,$$

where $i = \sqrt{-1}$ and $u = u_1 + u_2$. Show that the total probability: $\|u\|_{L_2(\Omega)}$ is time independent.
b) Consider the corresponding eigenvalue problem:

$$-\Delta u = \lambda u, \quad in\ \Omega, \qquad u = 0, \quad on\ \partial\Omega.$$

Show that for the eigenpair (λ, u), $\lambda > 0$. Give the relation between $\|u\|$ and $\|\nabla u\|$.
c) Give the smallest constant C in $\|u\| \le C\|\nabla u\|$ in terms of the smallest eigenvalue λ_1.

6. Consider the problem

$$-\varepsilon u'' + xu' + u = f \quad \text{in } I = (0,1), \qquad u(0) = u'(1) = 0,$$

where ε in a positive constant, and $f \in L_2(I)$. Prove the following L_2-stability:

$$\|\varepsilon u''\| \le \|f\|.$$

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