Mathematics Chalmers & GU

**TMA372/MMG800: Partial Differential Equations, 2018–06–07, 14:00-18:00**

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*Calculators, formula notes and other subject related material are not allowed.*
Each problem gives max 6p. Valid bonus poits will be added to the scores.
Breakings: **3**: 15-21p, **4**: 22-28p och **5**: 29p- GU: **G**: 15-25p, **VG**: 26p-
For solutions and information about gradings see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1718/coursediary

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**1.** Consider the Dirichlet boundary value problem

$$-\nabla\cdot(a(x)\nabla u)=f(x),\quad x\in\Omega\subset\mathbb{R}^2,\qquad u=0,\ \text{for } x\in\partial\Omega.$$

Assume that $c_0$ and $c_1$ are constants such that $c_0\le a(x)\le c_1,\ \forall x\in\Omega$ and let $U=\sum_{j=1}^N\alpha_j w_j(x)$ be a Galerkin approximation of $u$ in a finite dimensional subspace $M$ of $H_0^1(\Omega)$. Prove the a priori error estimate below and specify $C$ as best you can

$$||u-U||_{H_0^1(\Omega)}\le C\inf_{\chi\in M}||u-\chi||_{H_0^1(\Omega)}.$$

**2.** Consider the heat equation

$$\begin{cases}\dot u-u''=0, & 0<x<1,\quad t>0,\\ u(0,t)=u'(1,t)=0, & t>0,\\ u(x,0)=u_0(x), & 0<x<1.\end{cases}$$

a) Show, with $||u||=\Big(\int_0^1 u(x)^2\,dx\Big)^{1/2}$, that $||u||$ and $||u'||$ are not increasing in time.

b) Show that $||u'||\to 0$, as $t\to\infty$. Give a physical interpretation for a) and b).

**3.** Prove *a posteriori* error estimate, in the energy norm $||v||_E^2=||v'||^2+a||v||^2$, for the $cG(1)$ approximation of the boundary value problem

$$-u''(x)+u'(x)+au(x)=f(x),\quad 0<x<1,\quad u(0)=u(1)=0,\quad a\ge 0.$$

**4.** a) Formulate a $cG(1)$ finite element method for the following system

$$\begin{cases}u(x)+v''(x)=f(x), & v(0)=v(1)=0,\quad 0<x<1,\\ u''(x)-v(x)=0, & u(0)=u(1)=0,\end{cases}$$

and show how the approximate solution $(U,V)$ can be computed from the load vector $F$, using mass- and stiffness matrises.

b) Derive stability estimates for $u$ and $v$, in terms of $f$, (e.g., through multiplying the first equation by $v$ and the second by $u$).

**5.** Consider the following *Schrödinger* equation

$$\dot u+i\Delta u=0,\quad \text{in }\Omega,\qquad u=0,\quad \text{on }\partial\Omega,$$

where $i=\sqrt{-1}$ and $u=u_1+iu_2$. a) Show that the *toatal probability* $\int_\Omega |u|^2$ is time independent. Hint: Multiply the equation by $\bar u=u_1-iu_2$, integrate over $\Omega$ and consider the real part.

b) Consider the corresponding *eigenvalue problem*, of finding $(\lambda,u\neq 0)$, such that

$$-\Delta u=\lambda u\quad \text{in }\Omega,\qquad u=0,\quad \text{on }\partial\Omega.$$

Show that $\lambda>0$. Give the relation between $||u||$ and $||\nabla u||$ for the corresponding eigenfunction $u$.

c) What is the optimal constant $C$ (expressed in terms of smallest eigenvalue $\lambda_1$), for which the inequality $||u||\le C||\nabla u||$ can fullfil for all functions $u$, such that $u=0$ on $\partial\Omega$?

**6.** Formulate and prove the Lax-Milgram theorem (Lecture Notes/Compendium version).

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**TMA372/MMG800: Partial Differential Equations, 2018–06–07, 14:00-18:00. Solutions.**

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**1.** Recall the continuous and approximate weak formulations:

$$(a\nabla u, \nabla v) = (f, v), \qquad \forall v \in H_0^1(\Omega), \tag{1}$$

and

$$(a\nabla U, \nabla v) = (f, v), \qquad \forall v \in M, \tag{2}$$

respectively, so that

$$(a\nabla (u-U), \nabla v) = 0, \qquad \forall v \in M. \tag{3}$$

We may write

$$u - U = u - \chi + \chi - U,$$

where $\chi$ is an arbitrary element of $M$, it follows that

$$\begin{aligned}(a\nabla(u-U), \nabla(u-U)) =& (a\nabla(u-U), \nabla(u-\chi))\\ &\le ||a\nabla(u-U)|| \cdot ||u-\chi||_{H_0^1(\Omega)}\\ &\le c_1||u-U||_{H_0^1(\Omega)}||u-\chi||_{H_0^1(\Omega)},\end{aligned} \tag{4}$$

on using (3), Schwarz's inequality and the boundedness of $a$. Also, from the boundedness condition on $a$, we have that

$$(a\nabla(u-U), \nabla(u-U)) \ge c_0||u-U||^2_{H_0^1(\Omega)}. \tag{5}$$

Combining (4) and (5) gives

$$||u-U||_{H_0^1(\Omega)} \le \frac{c_1}{c_0}||u-\chi||_{H_0^1(\Omega)}.$$

Since $\chi$ is an arbitrary element of $M$, we obtain the result.

**2.** a) Multiply the equation $\dot u = u''$ by $u$ and integrate over $x \in (0,1)$:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2 &= \int_0^1 \dot u u\,dx = \int_0^1 u'' u\,dx = \{\text{part. int.}\}\\ &= u'u|_0^1 - \int_0^1 u'u'\,dx = -\|u'\|^2 \le 0,\end{aligned}$$

i.e., $\|u\|^2$ and hence $\|u\|$ is decreasing in $t$.
Now multiply the equation $\dot u = u''$ by $-u''$ and integrate over $x \in (0,1)$:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u'\|^2 &= \int_0^1 \dot u' u'\,dx = \dot u u'|_0^1 - \int_0^1 \dot u u''\,dx\\ &= -\int_0^1 u''u''\,dx = -\|u''\|^2 \le 0,\end{aligned}$$

i.e., $\|u'\|^2$ and hence $\|u'\|$ is decreasing in $t$.
b) According the first relation above $\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|u'\|^2 = 0$. Integrating over $t$ yields:

$$\frac{1}{2}\|u\|^2(t) + \int_0^t \|u'\|^2\,d\tau = \frac{1}{2}\|u_0\|^2.$$

Thus, it follows that $\int_0^\infty \|u'\|^2\,dt$ must converge, which is possible only if the decreasing function $\|u'\|^2$ tends to 0 as $t \to \infty$, i.e., $\|u'\| \to 0$ as $t \to \infty$.

c) In the absence of a heat source, the temperature and heat flux are decreasing (non-increasing) in time, especially the heat flux tends to 0 as $t \to \infty$.

**3.** (a) The Variational formulation:

(Multiply the equation by $v \in V$, integrate by parts over $(0,1)$ and use the boundary conditions.)

$$\text{(6)} \qquad \text{Find } u \in V: \ \int_0^1 u'v'\,dx + \int_0^1 u'v\,dx + \int_0^1 auv\,dx = \int_0^1 fv\,dx, \quad \forall v \in V.$$

cG(1):

$$\text{(7)} \qquad \text{Find } U \in V_h: \ \int_0^1 U'v'\,dx + \int_0^1 U'v\,dx + \int_0^1 aUv\,dx = \int_0^1 fv\,dx, \quad \forall v \in V_h,$$

where

$$V_h := \{v : v \text{ is continuous piecewise linear in } (0,1),\ v(0) = v(1) = 0\}.$$

From (1)-(2), we find

The Galerkin orthogonality:

$$\text{(8)} \qquad \int_0^1 \Big((u-U)'v' + (u-U)'v + a(u-U)v\Big)\,dx = 0, \quad \forall v \in V_h.$$

We define the inner product $(\cdot,\cdot)_E$ associated to the energy norm to be

$$(v,w)_E = \int_0^1 (v'w' + avw)\,dx, \qquad \forall v, w \in V.$$

A posteriori error estimate: We have that

$$\begin{aligned} ||e||_E^2 &= \int_0^1 (e'e' + aee)\,dx = \int_0^1 (e'e' + e'e + aee)\,dx \\ &= \int_0^1 (u'e' + u'e + aue)\,dx - \int_0^1 (U'e' + U'e + aUe)\,dx. \end{aligned}$$

Thus using (1) we get

$$\text{(9)} \qquad ||e||_E^2 = \int_0^1 fe\,dx - \int_0^1 (U'e' + U'e + aUe)\,dx,$$

which by (2) can be written as

$$\begin{aligned} ||e||_E^2 = \int_0^1 fe\,dx &- \int_0^1 (U'e' + U'e + aUe)\,dx \\ &+ \int_0^1 \Big(U'(\Pi_h e)' + U'\Pi_h e + aU\Pi_h e\Big)\,dx - \int_0^1 f\Pi_h e\,dx. \end{aligned}$$

Observe that the last line above is identically 0. Adding up we have

$$\begin{aligned}
||e||_E^2 &= \int_0^1 f(e-\Pi_h e)\,dx - \int_0^1 \Big(U'(e-\Pi_h e)' + U'(e-\Pi_h e) + aU(e-\Pi_h e)\Big)\,dx \\
&= \int_0^1 f(e-\Pi_h e)\,dx - \int_0^1 (U'+aU)(e-\Pi_h e)\,dx - \sum_{j=1}^{M+1}\int_{I_j} U'(e-\Pi_h e)'\,dx \\
&=\{\text{partial integration}\} \\
&= \int_0^1 f(e-\Pi_h e)\,dx - \int_0^1 (U'+aU)(e-\Pi_h e)\,dx + \sum_{j=1}^{M+1}\int_{I_j} U''(e-\Pi_h e)\,dx \\
&= \int_0^1 (f+U''-U'-aU)(e-\Pi_h e)\,dx = \int_0^1 R(U)(e-\Pi_h e)\,dx \\
&= \int_0^1 hR(U)h^{-1}(e-\Pi_h e)\,dx \le ||hR(U)||_{L_2}||h^{-1}(e-\Pi_h e)||_{L_2} \\
&\le C_i||hR(U)||_{L_2}\cdot||e'||_{L_2} \le C_i||hR(U)||_{L_2}\cdot||e||_E.
\end{aligned}$$

This gives the a posteriori error estimate:

$$||e||_E \le C_i||hR(U)||_{L_2},$$

with $R(U) = f + U'' - U' - aU = f - U' - aU$, on $(x_{i-1}, x_i)$, $i = 1, \dots, M+1$.

**4.** a) Multiplying the equations in the system by the test functions $\varphi$ and $\psi$, with $\varphi = \psi = 0$ for $x = 0$ and $x = 1$, and integrating by parts gives that

$$\left\{\begin{array}{l} \int_0^1(\varphi u - \varphi' v') = \int_0^1 \varphi f, \\ \int_0^1(-\psi' u' - \psi v) = 0. \end{array}\right. \tag{10}$$

Partitioning of $[0,1]$ into subintervals (elements) $I_j = [x_{j-1}, x_j]$, $x_j = j/(m+1)$, the linear approximations $U(x) = \sum_{j=1}^m U_j\varphi_j(x)$ and $V(x) = \sum_{j=1}^m V_j\varphi_j(x)$, with $\varphi_j(x)$ s being the usual piecewise linear basis functions, the $cG(1)$ approximation of the above system (4) can be formulated as: Find the nodal values $U_j$ and $V_j$ such that

$$\left\{\begin{array}{l} \int_0^1(\varphi_i U - \varphi_i' V') = \int_0^1 \varphi f, \quad i = 1, \dots, m, \\ \int_0^1(-\varphi_i' U' - \varphi_i V) = 0, \quad i = 1, \dots, m. \end{array}\right. \tag{11}$$

This gives $2m$ equations with the $2m$ unkown nodal values $U = [U_1, \dots, U_m]^T$ and $V = [V_1, \dots, V_m]^T$, which can be written in the matrix form as

$$\left\{\begin{array}{l} MU - SV = F, \\ -SU - MV = 0, \end{array}\right. \tag{12}$$

where $M$ and $S$ are the usual, 3-diagonal, mass- and stiffness matrises, respectively: $M$ has $2h/3$ diagonal elements and $h/6$ super and subdiagonal elements. Corrseponding elements for $S$ are $2/h$ diagonal elements and $-1/h$ sub and superdiagonal elements. All other elemts are zeros. $F$ is the load vector with elements $\int_0^1 \varphi_i f$.

From the second equation above we get that $V = -M^{-1}SU$, which inserting in the first equation gives $U = (M + SM^{-1}S)^{-1}F$.

b) Multiply the first equation by $u$ and the second equation by $-v$, add the two resulting equations and integrate over $[0,1]$. By partial integration we have then

$$\int_0^1 u^2 + v^2 = \int_0^1 uf,$$

Using Cauchy-Schwartz inequality we get

$$||u||^2 + ||v||^2 = \int_0^1 uf \le ||u||||f|| \le \frac{1}{2}||u||^2 + \frac{1}{2}||f||^2$$

This gives that $||u|| \le ||f||$, and consequently even $||v|| \le ||f||$.

We could alternatively multiply the first equation by $-v$ and the second by $-u$, add the two resulting equations and integrate over $[0, 1]$. By partial integration we have this time

$$\int_0^1 (u')^2 + (v')^2 = \int_0^1 (-v)f,$$

Using, first Poincare', and then the Cauchy-Schwartz inequality we get

$$||u'||^2 + ||v'||^2 \le ||v||||f|| \le ||v'||||f|| \le \frac{1}{2}||v'||^2 + \frac{1}{2}||f||^2,$$

so that we have now $||v'|| \le ||f||$, and consequently even $||u'|| \le ||f||$. We could obviously continue in this manner and get basic stability estimates for the, e.g., moment $v = u''$, through

$$||u''|| = ||v|| \le ||f||,$$

and for $v''$:

$$||v''|| = ||f - u|| \le ||f|| + ||u|| \le 2||f||.$$

**5.** a) We multiply the shrödinger equation by $\bar{u}$ and integrate over $\Omega$ to obtain

$$\int_\Omega \bar{u}\dot{u} + i\int_\Omega \bar{u}\nabla u = \int_\Omega (u_1\dot{u}_1 + u_2\dot{u}_2) + i\int_\Omega (u_1\dot{u}_2 - u_2\dot{u}_1 - \nabla\bar{u}\cdot\nabla u) = 0.$$

Now both real and imaginary part of the above expression is 0. Thus, considering the real part, we have

$$\int_\Omega (u_1\dot{u}_1 + u_2\dot{u}_2) = \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega (u_1^2 + u_2^2) = 0,$$

therefore $\int_\Omega |u|^2$ is independent of the time.

b) Multiplying the eigenvalue equation $-\Delta u = \lambda u$ by $u$, integrating over $\Omega$, and using partial integration we get

$$\lambda\int_\Omega u^2 = \int_\Omega u(-\Delta u) = \int_\Omega |\nabla u|^2,$$

which gives $\lambda \ge 0$ (and also $\lambda > 0$, for $u \ne 0$). Further $||u|| = \frac{1}{\sqrt{\lambda}}||\nabla u||$. This indicates that the constant in the estimate $||u|| \le C||\nabla u||$, satisfying for all functions $u$ with $u = 0$ on $\Gamma := \partial\Omega$, can not be smaller than $\frac{1}{\sqrt{\lambda_1}}$, with $\lambda_1 > 0$ being the smallest eigenvalue. As a matter of fact we have the inequality $||u|| \le \frac{1}{\sqrt{\lambda_1}}||\nabla u||$, for all $u$ with $u = 0$ on $\Gamma$. This is due to the fact that we can represent $u$ in terms of orthogonal eigenfunctions both "with and without gradient", i.e. $\int_\Omega u_i u_j = \int_\Omega \nabla u_i \cdot \nabla u_j = 0$, for $i \ne j$.
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**6.** See the Lecture Notes.

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