Mathematics Chalmers & GU

TMA372/MMG800: Partial Differential Equations, 2016–08–24, 8:30-12:30

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Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 6p. Valid bonus poits will be added to the scores.
Breakings: **3**: 15-21p, **4**: 22-28p och **5**: 29p- For GU students **G:** 15-26p, **VG:** 27p-
For solutions see the couse diary: http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1516/

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1. Let $\pi_1 f(x)$ be the linear interpolant of a, twice continuously differentiable, function f on an interval $I = (a, b)$. Prove the following *optimal* interpolation error estimate fotr the first derivative:

$$||f' - (\pi_1 f)'||_{L_\infty(a,b)} \le \frac{1}{2}(b-a)||f''||_{L_\infty(a,b)}$$

Note: It is importat to derive the error estimate with the interploation constant 1/2.

2. Let $b \ge 0$, consider the problem

$$\ddot{u} + b\dot{u} - u'' = 0 \quad 0 < x < 1 \qquad u(0,t) = u(1,t) = 0.$$

a) Describe the phenomena that these equations can model. What is the meaning of each term?
b) Prove that there is a natural energy associted with the solution u, which is preserved in time (as t increases) for $b = 0$, and decreases as time increases for $b > 0$.

3. Consider the problem

(1) $\qquad -\Delta u(x) + u(x) = f(x) \quad \text{for} \quad x \in \Omega \subset \mathbb{R}^d \qquad \mathbf{n} \cdot \nabla u(x) = g(x) \quad x \in \Gamma := \partial\Omega,$

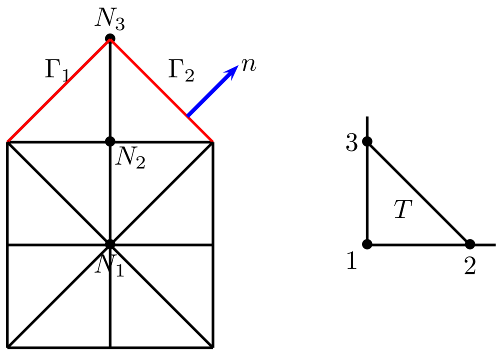
where $\mathbf{n}$ is the outward unit normal to Γ and f and g are given functions
Use the inequality $||u||_{L_2(\Gamma)} \le C_\Omega(||\nabla u||_{L_2(\Omega)} + ||u||_{L_2(\Omega)})$ and show the stabilty estimate:

$$||\nabla u||^2_{L_2(\Omega)} + ||u||^2_{L_2(\Omega)} \le C(||f||^2_{L_2(\Omega)} + ||g||^2_{L_2(\Gamma)}).$$

4. Compute the stiffness and mass matrices as well as load vector for the cG(1) approximation for

$$-\varepsilon\Delta u + u = 3, \quad x \in \Omega; \qquad u = 0, \quad x \in \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2), \quad \nabla u \cdot n = 0, \quad x \in \Gamma_1 \cup \Gamma_2,$$

where $\varepsilon > 0$ and n is the outward unit normal to $\partial\Omega$, (obs! 3 nodes N_1, N_2 and N_3, see Fig.)



5. Formulate the Lax-Milgram theorem. Verify the assumptions of the Lax-Milgram Theorem and determine the constants of the assumptions in the case: $I = (0,1)$, $f \in L_2(I)$, $V = H^1(I)$ and

$$a(v,w) = \int_I (uw + v'w')\,dx + v(0)w(0), \quad L(v) = \int_I fv\,dx. \qquad ||w||^2_V = ||w||^2_{L_2(I)} + ||w'||^2_{L_2(I)}.$$

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TMA372/MMG800: Partial Differential Equations, 2016–08–24, 8:30-12:30. Solutions.

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1. We have that

$$\pi_1 f(x) = \lambda_a(x) f(a) + \lambda_b(x) f(b) = f(a)\frac{b-x}{b-a} + f(b)\frac{x-a}{b-a} \Longrightarrow$$

$$(\pi_1 f(x))' = \frac{1}{b-a}\Big(f(b) - f(a)\Big). \tag{2}$$

By Taylor expansions of $f(b)$ and $f(a)$ about x: We have that $\exists, \eta_b \in (x, b)$ and $\eta_a \in (a, x)$:

$$f(b) = f(x) + (b-x)f'(x) + \frac{1}{2}(b-x)^2 f''(\eta_b) \tag{3}$$

$$f(a) = f(x) + (a-x)f'(x) + \frac{1}{2}(a-x)^2 f''(\eta_a) \tag{4}$$

so that

$$f(b) - f(a) = (b-a)f'(x) + \frac{1}{2}(b-x)^2 f''(\eta_b) - \frac{1}{2}(a-x)^2 f''(\eta_a).$$

Inserting in (2) yields

$$(\pi_1 f(x))' = f'(x) + \frac{1}{2}\frac{1}{b-a}\Big((b-x)^2 f''(\eta_b) - (a-x)^2 f''(\eta_a)\Big)$$

$$|(\pi_1 f(x))' - f'(x)| \le \frac{1}{2(b-a)} \max_{x\in[a,b]} |f''(x)| \Big((b-x)^2 + (a-x)^2\Big).$$

Let now $g(x) = (b-x)^2 + (a-x)^2$, then g is convex, symmetric and hence has its maximum at the endpoints of the interval $I = [a, b]$ $g_{\max} = g(a) = g(b) = (b-a)^2$. Thus

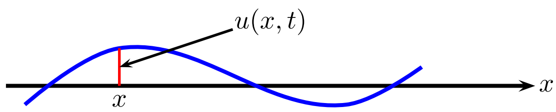
$$|(\pi_1 f(x))' - f'(x)| \le \frac{1}{2} \max_{x\in[a,b]} |f''(x)|.$$

which is the desired result.

Not that $g'(x) = 0$ yields to the minimipoint $x_{\min} = (a+b)/2$ and $g(x_{\min}) = (b-a)^2/2$.

2. a) $\ddot{u} + b\dot{u} - u'' = 0$ models, e.g. the transversal oscillation of a wire fixed at its two endpoints, where $u(x, t)$ is the displacement coordinates.

Te terms are corresponding to *inertia, friction (from the surrounding media)* and the resultant powers of all *tension and stress*.



b) Multiplying the equation by $\dot{u}$ and integrating in spatial variable over $[0,1]$ tields

$$\frac{1}{2}\frac{d}{dt}\Big(\|\dot{u}\|^2 + \|u'\|^2\Big) + b\|\dot{u}\|^2 = 0.$$

Hence considering the Energy as

$$E(u) = \frac{1}{2}\|\dot{u}\|^2 + \frac{1}{2}\|u'\|^2,$$

we have that

$$\frac{d}{dt}E(u) = -b\|\dot{u}\|^2.$$

Thus

$$\frac{d}{dt}E(u) = 0, \quad \text{if } b = 0, \qquad \text{and} \quad \frac{d}{dt}E(u) \leq 0 \quad \text{if } b > 0.$$

3. Multiplying the equation by u and using partial integration (Green's formula) yields

$$\int_\Omega (\nabla u \cdot \nabla u + u\,u)\,dx - \int_\Gamma \mathbf{n}\cdot\nabla u\,u\,d\sigma = \int_\Omega f\,u\,dx,$$

i.e.

$$\|\nabla u\|^2 + \|u\|^2 = \int_\Omega f\,u\,dx + \int_\Gamma g\,u\,d\sigma \leq \|f\|\|u\| + ||g||_\Gamma C_\Omega(\|\nabla u\| + \|u\|),$$

where $\|\cdot\| = ||\cdot||_{L_2(\Omega)}$ and we used the inequality $\|u\| \leq C_\Omega(\|\nabla u\| + \|u\|)$. Now using the inequality $ab \leq a^2 + \frac{1}{4}b^2$ we get

$$\|\nabla u\|^2 + \|u\|^2 \leq \|f\|^2 + \frac{1}{4}\|u\|^2 + C||g||_\Gamma^2 + \frac{1}{4}\|\nabla u\|^2 + \frac{1}{4}\|u\|^2$$

which gives the desired inequality.

4. Let V be the linear function space defined by

$$V_h := \{v : v \ \text{ is continuous in } \Omega, \ \ v = 0, \ \text{ on } \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)\}.$$

Multiplying the differential equation by $v \in V$ and integrating over Ω we get that

$$-(\Delta u, v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Now using Green's formula we have that

$$\begin{aligned} -(\Delta u, \nabla v) &= (\nabla u, \nabla v) - \int_{\partial\Omega} (n\cdot\nabla u) v\,ds \\ &= (\nabla u, \nabla v) - \int_{\partial\Omega\setminus(\Gamma_1\cup\Gamma_2)} (n\cdot\nabla u) v\,ds - \int_{\Gamma_1\cup\Gamma_2} (n\cdot\nabla u) v\,ds \\ &= (\nabla u, \nabla v), \qquad \forall v \in V. \end{aligned}$$

Thus the variational formulation is:

$$(\nabla u, \nabla v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Let V_h be the usual finite element space consisting of continuous piecewise linear functions satisfying the boundary condition $v = 0$ on $\partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)$: The $cG(1)$ method is: Find $U \in V_h$ such that

$$(\nabla U, \nabla v) + (U, v) = (f, v) \qquad \forall v \in V_h$$

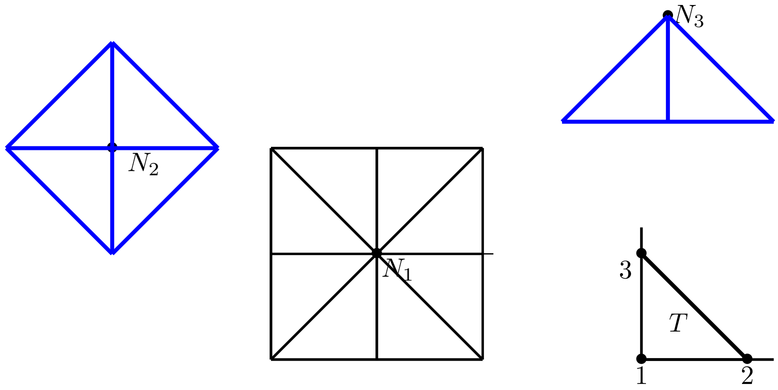
Making the "Ansatz" $U(x) = \sum_{j=1}^{3} \xi_j \varphi_j(x)$, where φ_i are the standard basis functions, we obtain the system of equations

$$\sum_{j=1}^{3} \xi_j \Big(\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j\,dx + \int_\Omega \varphi_i\varphi_i\,dx\Big) = \int_\Omega f\varphi_j\,dx, \quad i = 1, 2, 3,$$

or, in matrix form,

$$(S + M)\xi = F,$$

where $S_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix, $M_{ij} = (\varphi_i, \varphi_j)$ is the mass matrix, and $F_i = (f, \varphi_i)$ is the load vector.



We first compute the mass and stiffness matrix for the reference triangle T. The local basis functions are

$$\begin{aligned} \phi_1(x_1, x_2) &= 1 - \frac{x_1}{h} - \frac{x_2}{h}, & \nabla\phi_1(x_1, x_2) &= -\frac{1}{h}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ \phi_2(x_1, x_2) &= \frac{x_1}{h}, & \nabla\phi_2(x_1, x_2) &= \frac{1}{h}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \phi_3(x_1, x_2) &= \frac{x_2}{h}, & \nabla\phi_3(x_1, x_2) &= \frac{1}{h}\begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

Hence, with $|T| = \int_T dz = h^2/2$,

$$m_{11} = (\phi_1, \phi_1) = \int_T \phi_1^2\, dx = h^2 \int_0^1 \int_0^{1-x_2} (1 - x_1 - x_2)^2\, dx_1 dx_2 = \frac{h^2}{12},$$

$$s_{11} = (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2\, dx = \frac{2}{h^2}|T| = 1.$$

Alternatively, we can use the midpoint rule, which is exact for polynomials of degree 2 (precision 3):

$$m_{11} = (\phi_1, \phi_1) = \int_T \phi_1^2\, dx = \frac{|T|}{3}\sum_{j=1}^{3} \phi_1(\hat{x}_j)^2 = \frac{h^2}{6}\Big(0 + \frac{1}{4} + \frac{1}{4}\Big) = \frac{h^2}{12},$$

where $\hat{x}_j$ are the midpoints of the edges. Similarly we can compute the other elements and obtain

$$m = \frac{h^2}{24}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrices M and S from the local ones m and s:

$$\begin{aligned} M_{11} &= 8m_{22} = \frac{8}{12}h^2, & S_{11} &= 8s_{22} = 4, \\ M_{12} &= 2m_{12} = \frac{1}{12}h^2, & S_{12} &= 2s_{12} = -1, \\ M_{13} &= 0, & S_{13} &= 2s_{23} = 0, \\ M_{22} &= 4m_{11} = \frac{4}{12}h^2, & S_{22} &= 4s_{11} = 4, \\ M_{23} &= 2m_{12} = \frac{1}{12}h^2, & S_{23} &= 2s_{12} = -1, \\ M_{33} &= 2m_{22} = \frac{2}{12}h^2, & S_{33} &= 2s_{22} = 1. \end{aligned}$$

The remaining matrix elements are obtained by symmetry $M_{ij} = M_{ji}$, $S_{ij} = S_{ji}$. Hence,

$$M = \frac{h^2}{12}\begin{bmatrix} 8 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad S = \varepsilon \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} (1,\varphi_1) \\ (1,\varphi_2) \\ (1,\varphi_3) \end{bmatrix} = \begin{bmatrix} 8\cdot\frac{1}{3}\cdot\frac{1}{2} = \frac{4}{3} \\ 4\cdot\frac{1}{3}\cdot\frac{1}{2} = \frac{2}{3} \\ 2\cdot\frac{1}{3}\cdot\frac{1}{2} = \frac{1}{3} \end{bmatrix}.$$

5. For the formulation of the Lax-Milgram theorem see the book, Chapter 21.
As for the given case: $I = (0,1)$, $f \in L_2(I)$, $V = H^1(I)$ and

$$a(v,w) = \int_I (uw + v'w')\,dx + v(0)w(0), \quad L(v) = \int_I fv\,dx,$$

it is trivial to show that $a(\cdot,\cdot)$ is bilinear and $b(\cdot)$ is linear. We have that

$$(5) \quad a(v,v) = \int_I v^2 + (v')^2\,dx + v(0)^2 \geq \int_I (v)^2\,dx + \frac{1}{2}\int_I (v')^2\,dx + \frac{1}{2}v(0)^2 + \frac{1}{2}\int_I (v')^2\,dx.$$

Further

$$v(x) = v(0) + \int_0^x v'(y)\,dy, \quad \forall x \in I$$

implies

$$v^2(x) \leq 2\Big(v(0)^2 + (\int_0^x v'(y)\,dy)^2\Big) \leq \{C-S\} \leq 2v(0)^2 + 2\int_0^1 v'(y)^2\,dy,$$

so that

$$\frac{1}{2}v(0)^2 + \frac{1}{2}\int_0^1 v'(y)^2\,dy \geq \frac{1}{4}v^2(x), \quad \forall x \in I.$$

Integrating over x we get

$$(6) \quad \frac{1}{2}v(0)^2 + \frac{1}{2}\int_0^1 v'(y)^2\,dy \geq \frac{1}{4}\int_I v^2(x)\,dx.$$

Now combining (5) and (6) we get

$$\begin{aligned} a(v,v) &\geq \frac{5}{4}\int_I v^2(x)\,dx + \frac{1}{2}\int_I (v')^2(x)\,dx \\ &\geq \frac{1}{2}\Big(\int_I v^2(x)\,dx + \int_I (v')^2(x)\,dx\Big) = \frac{1}{2}||v||_V^2, \end{aligned}$$

so that we can take $\kappa_1 = 1/2$. Further

$$\begin{aligned} |a(v,w)| &\leq \Big|\int_I vw\,dx\Big| + \Big|\int_I v'w'\,dx\Big| + |v(0)w(0)| \leq \{C-S\} \\ &\leq ||v||_{L_2(I)}||w||_{L_2(I)} + ||v'||_{L_2(I)}||w'||_{L_2(I)} + |v(0)||w(0)| \\ &\leq \Big(||v||_{L_2(I)} + ||v'||_{L_2(I)}\Big)\Big(||w||_{L_2(I)} + ||w'||_{L_2(I)}\Big) + |v(0)||w(0)| \\ &\leq \sqrt{2}\Big(||v||_{L_2(I)}^2 + ||v'||_{L_2(I)}^2\Big)^{1/2} \cdot \sqrt{2}\Big(||w||_{L_2(I)}^2 + ||w'||_{L_2(I)}^2\Big)^{1/2} + |v(0)||w(0)| \\ &\leq \sqrt{2}||v||_V\sqrt{2}||w||_V + |v(0)||w(0)|. \end{aligned}$$

Now we have that

$$(7) \quad v(0) = -\int_0^x v'(y)\,dy + v(x), \quad \forall x \in I,$$

and by the Mean-value theorem for the integrals: $\exists \xi \in I$ so that $v(\xi) = \int_0^1 v(y)\,dy$. Choose $x = \xi$ in (7) then

$$\begin{aligned} |v(0)| &= \Big| -\int_0^\xi v'(y)\,dy + \int_0^1 v(y)\,dy\Big| \\ &\leq \int_0^1 |v'|\,dy + \int_0^1 |v|\,dy \leq \{C-S\} \leq ||v'||_{L_2(I)} + ||v||_{L_2(I)} \leq 2||v||_V, \end{aligned}$$

implies that

$$|v(0)||w(0)| \leq 4||v||_V||w||_V,$$

and consequently

$$|a(u,w)| \leq 2||v||_V||w||_V + 4||v||_V||w||_V = 6||v||_V||w||_V,$$

so that we can take $\kappa_2 = 6$. Finally

$$|L(v)| = \Big|\int_I fv\,dx\Big| \leq ||f||_{L_2(I)}||v||_{L_2(I)} \leq ||f||_{L_2(I)}||v||_V,$$

taking $\kappa_3 = ||f||_{L_2(I)}$ all the conditions in the Lax-Milgram theorem are fulfilled.

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