Mathematics Chalmers & GU

TMA372/MMG800: Partial Differential Equations, 2015–06–09, 8:30-12:30

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Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 6p. Valid bonus poits will be added to the scores.
Breakings: **3**: 15-21p, **4**: 22-28p och **5**: 29p- For GU students**G**:15-25p, **VG**: 26p-
For solutions the couse diary in: http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1415/

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1. Consider the Dirichlet problem (with $c_0 \le a(x) \le c_1, \ \forall x \in \Omega$, where c_0 and c_1 are constants)

$$-\nabla \cdot (a(x)\nabla u) = f(x), \quad x \in \Omega \subset \mathbb{R}^2, \qquad u = 0, \text{ for } x \in \partial\Omega.$$

Let $U = \sum_{j=1}^N \alpha_j w_j(x)$ be a Galerkin approximation of u in a finite dimensional subspace M of $H_0^1(\Omega)$. Prove the a priori error estimate below and specify C as best you can

$$||u - U||_{H_0^1(\Omega)} \le C \inf_{\chi \in M} ||u - \chi||_{H_0^1(\Omega)}.$$

2. Consider the following Neumann boundary value problem (n is the outward unit normal to Γ)

$$-\Delta u + u = f, \quad x \in \Omega \subset \mathbb{R}^d, \qquad n \cdot \nabla u = g, \quad \text{on} \quad \Gamma := \partial\Omega.$$

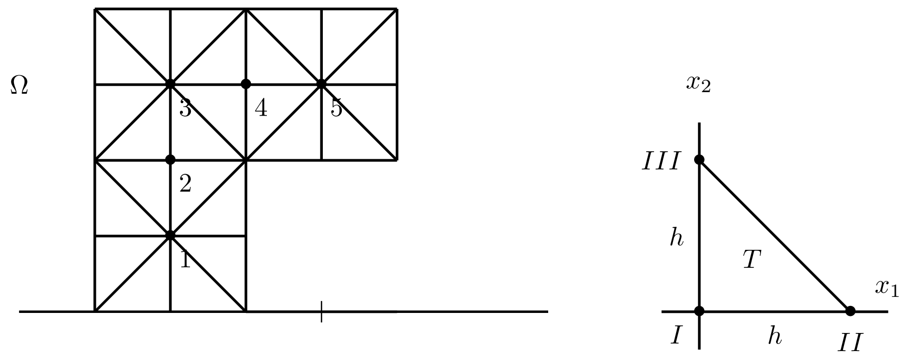
(a) Show the stability estimate: $\quad ||\nabla u||^2_{L_2(\Omega)} + ||u||^2_{L_2(\Omega)} \le C\Big(||f||^2_{L_2(\Omega)} + ||g||^2_{L_2(\Gamma)}\Big).$

(b) Formulate a finite element method for the $1D$-case and derive the resulting system of equations for $\Omega = [0, 1], \ f(x) = 1, \ g(0) = 3$ and $g(1) = 0$.

3. Formulate the cG(1) Galerkin finite element method for the Dirichlet boundary value problem

$$-\Delta u + u = f, \quad x \in \Omega; \qquad u = 0, \quad x \in \partial\Omega.$$

Write the matrices for the resulting equation system using the reference triangle-element T and the partition below (see fig.) with the nodes at $N_i, \ i = 1, \dots, 5$ and a uniform mesh size h.



4. Consider the boundary value problem

$$-\varepsilon u'' + \alpha(x)u' + u = f(x), \quad 0 < x < 1, \qquad u(0) = 0, \ u'(1) = 0,$$

where ε is a positive constant and α is a function satisfying $\alpha(x) \ge 0, \ \alpha'(x) \le 0$. Show that

$$\sqrt{\varepsilon}||u'|| \le C_1||f||, \quad ||\alpha u'|| \le C_2||f||, \quad \varepsilon||u''|| \le C_3||f||, \quad \text{where } ||v|| = \Big(\int_0^1 v^2\,dx\Big)^{1/2}$$

5. Consider the boundary value problem for the stationary heat flow (Poisson equation) in $1D$:

$$(BVP) \qquad -\,(a(x)u'(x))' = f(x), \quad 0 < x < 1, \qquad u(0) = u(1) = 0.$$

Formulate the corresponding variational formulation (VF), and show that: $\qquad (BVP) \Longleftrightarrow (VF).$

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TMA372/MMG800: Partial Differential Equations, 2015–06–09, 8:30-12:30. Solutions.

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1. Recall the continuous and approximate weak formulations:

$$(a\nabla u, \nabla v) = (f, v), \qquad \forall v \in H_0^1(\Omega), \tag{1}$$

and

$$(a\nabla U, \nabla v) = (f, v), \qquad \forall v \in M, \tag{2}$$

respectively, so that

$$(a\nabla(u-U), \nabla v) = 0, \qquad \forall v \in M. \tag{3}$$

We may write

$$u - U = u - \chi + \chi - U,$$

where χ is an arbitrary element of M, it follows that

$$\begin{aligned}(a\nabla(u-U), \nabla(u-U)) =& (a\nabla(u-U), \nabla(u-\chi))\\ &\le ||a\nabla(u-U)|| \cdot ||u-\chi||_{H_0^1(\Omega)}\\ &\le c_1 ||u-U||_{H_0^1(\Omega)} ||u-\chi||_{H_0^1(\Omega)},\end{aligned} \tag{4}$$

on using (3), Schwarz's inequality and the boundedness of a. Also, from the boundedness condition on a, we have that

$$(a\nabla(u-U), \nabla(u-U)) \ge c_0 ||u-U||^2_{H_0^1(\Omega)}. \tag{5}$$

Combining (4) and (5) gives

$$||u-U||_{H_0^1(\Omega)} \le \frac{c_1}{c_0} ||u-\chi||_{H_0^1(\Omega)}.$$

Since χ is an arbitrary element of M, we obtain the result.

2. a) Multiplying the equation by u and performing partial integration we get

$$\int_\Omega \nabla u \cdot \nabla u + uu - \int_\Gamma n \cdot \nabla u u = \int_\Omega fu,$$

i.e.,

$$||\nabla u||^2 + ||u||^2 = \int_\Omega fu + \int_\Gamma gu \le ||f|| ||u|| + ||g||_\Gamma C_\Omega (||\nabla u|| + ||u||) \tag{6}$$

where $||\cdot|| = ||\cdot||_{L_2(\Omega)}$ and we have used the inequality $||u|| \le C_\Omega(||\nabla u|| + ||u||)$. Further using the inequality $ab \le a^2 + b^2/4$ we have

$$||\nabla u||^2 + ||u||^2 \le ||f||^2 + \frac{1}{4}||u||^2 + C||g||_\Gamma^2 + \frac{1}{4}||\nabla u||^2 + \frac{1}{4}||u||^2$$

which gives the desired inequality.

b) Consider the variational formulation

$$\int_\Omega \nabla u \cdot \nabla v + uv = \int_\Omega fv + \int_\Gamma gv, \tag{7}$$

set $U(x) = \sum U_j \psi_j(x)$ and $v = \psi_i$ in (7) to obtain

$$\sum_{j=1}^N U_j \int_\Omega \nabla \psi_j \cdot \nabla \psi_i + \psi_j \psi_i = \int_\Omega f\psi_i + \int_\Gamma g\psi_i, \quad i = 1, \dots, N.$$

This gives $AU = b$ where $U = (U_1, \dots, U_N)^T$, $b = (b_i)$ with the elements

$$b_i = h,\ i = 2, \dots, N-1, \quad b(N) = h/2, \quad b(1) = h/2 + 3,$$

and $A = (a_{ij})$ with the elements

$$a_{ij} = \begin{cases} -1/h + h/6, & \text{for } i = j+1 \quad \text{and } i = j-1 \\ 2/h + 2h/3, & \text{for } i = j \quad\quad\ \ \text{and } i = 2, \dots, N-1 \\ 0, & \text{else.} \end{cases}$$

3. Let V be the linear function space defined by

$$V := \{v : v \ \text{ is continuous in } \Omega,\ \ v = 0,\ \text{ on } \partial\Omega\}.$$

Multiplying the differential equation by $v \in V$ and integrating over Ω we get that

$$-(\Delta u, v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Now using Green's formula we have that

$$-(\Delta u, \nabla v) = (\nabla u, \nabla v) - \int_{\partial\Omega} (n \cdot \nabla u) v \, ds = (\nabla u, \nabla v), \qquad \forall v \in V.$$

Thus, since $v = 0$ on $\partial\Omega$, the variational formulation is:

$$(\nabla u, \nabla v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Let now V_h be the usual finite element space consisting of continuous piecewise linear functions, on the given partition (triangulation), satisfying the boundary condition $v = 0$ on $\partial\Omega$:

$$V_h := \{v : v \ \text{ is continuous piecewise linear in } \Omega,\ \ v = 0,\ \text{ on } \partial\Omega\}.$$

The $cG(1)$ method is: Find $U \in V_h$ such that

$$(\nabla U, \nabla v) + (U, v) = (f, v) \qquad \forall v \in V_h$$

Making the "Ansatz" $U(x) = \sum_{j=1}^{5} \xi_i \varphi_j(x)$, where φ_j are the standard basis functions, we obtain the system of equations

$$\sum_{j=1}^{5} \xi_j \Big(\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx + \int_\Omega \varphi_i \varphi_j \, dx \Big) = \int_\Omega f \varphi_i \, dx, \quad i = 1, 2, 3, 4, 5$$

or, in matrix form,

$$(S + M)\xi = F,$$

where $S_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix, $M_{ij} = (\varphi_i, \varphi_j)$ is the mass matrix, and $F_j = (f, \varphi_j)$ is the load vector.

We first compute the mass and stiffness matrix for the reference triangle T. The local basis functions are

$$\phi_1(x_1, x_2) = 1 - \frac{x_1}{h} - \frac{x_2}{h}, \qquad \nabla\phi_1(x_1, x_2) = -\frac{1}{h}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\phi_2(x_1, x_2) = \frac{x_1}{h}, \qquad \nabla\phi_2(x_1, x_2) = \frac{1}{h}\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\phi_3(x_1, x_2) = \frac{x_2}{h}, \qquad \nabla\phi_3(x_1, x_2) = \frac{1}{h}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence, with $|T| = \int_T dz = h^2/2$,

$$m_{11} = (\phi_1, \phi_1) = \int_T \phi_1^2 \, dx = h^2 \int_0^1 \int_0^{1-x_2} (1 - x_1 - x_2)^2 \, dx_1 dx_2 = \frac{h^2}{12},$$

$$s_{11} = (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2 \, dx = \frac{2}{h^2}|T| = 1.$$

Alternatively, we can use the midpoint rule, which is exact for polynomials of degree 2 (precision 3):

$$m_{11} = (\phi_1, \phi_1) = \int_T \phi_1^2\, dx = \frac{|T|}{3}\sum_{j=1}^{3} \phi_1(\hat{x}_j)^2 = \frac{h^2}{6}\Big(0 + \frac{1}{4} + \frac{1}{4}\Big) = \frac{h^2}{12},$$

where $\hat{x}_j$ are the midpoints of the edges. Similarly we can compute the other elements and obtain

$$m = \frac{h^2}{24}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrices M and S from the local ones m and s:

$$\begin{aligned}
&M_{11} = M_{33} = M_{55} = 8m_{22} = 8 \times \frac{h^2}{12}, && S_{11} = S_{33} = S_{55} = 8s_{22} = 8 \times \frac{1}{2}8 = 4,\\
&M_{22} = M_{44} = 4m_{11} = 4 \times \frac{h^2}{12} = \frac{h^2}{3}, && S_{22} = S_{44} = 4s_{11} = 4 \times 1 = 4,\\
&M_{12} = M_{23} = M_{34} = M_{45} = 2m_{12} = \frac{1}{12}h^2, && S_{12} = S_{23} = S_{34} = S_{45} = 2s_{12} = -1,\\
&M_{13} = M_{14} = M_{15} = M_{24} = M_{25} = M_{35} = 0, && S_{13} = S_{14} = S_{15} = S_{24} = S_{25} = S_{35} = 0,
\end{aligned}$$

The remaining matrix elements are obtained by symmetry $M_{ij} = M_{ji}$, $S_{ij} = S_{ji}$. Hence,

$$M = \frac{h^2}{12}\begin{bmatrix} 8 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 1 & 1 & 8 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 8 \end{bmatrix}, \qquad S = \begin{bmatrix} 4 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & -1 & 4 \end{bmatrix}.$$

4. Multiplication by u gives

$$\varepsilon||u'||^2 + \int_0^1 \alpha u' u\, dx + ||u||^2 = (f, u) \le ||f||||u|| \le \frac{1}{2}||f||^2 + \frac{1}{2}||u||^2.$$

Here

$$\int_0^1 \alpha u' u\, dx = \frac{1}{2}\int_0^1 \alpha \frac{d}{dx} u^2\, dx = \frac{1}{2}\alpha(1)u(1)^2 - \frac{1}{2}\int_0^1 \alpha' u^2\, dx \ge 0, \tag{8}$$

and hence

$$\varepsilon||u'||^2 + \frac{1}{2}||u||^2 \le \frac{1}{2}||f||^2, \qquad \text{which implies} \qquad \sqrt{\varepsilon}||u'|| \le ||f||, \qquad ||u|| \le ||f||.$$

Multiply the equation by $\alpha u'$ and integrate over x to obtain

$$-\varepsilon\int_0^1 u''\alpha u'\, dx + ||\alpha u'||^2 + \int_0^1 \alpha u' u\, dx \le \frac{1}{2}||f||^2 + \frac{1}{2}||\alpha u'||^2.$$

Hence from the above estimates we get that

$$\begin{aligned}
||\alpha u'||^2 &\le ||f||^2 + \varepsilon\int_0^1 \alpha\frac{d}{dx}(u')^2\, dx = ||f||^2 - \varepsilon\alpha(0)u'(0)^2 - \varepsilon\int_0^1 \alpha'(u')^2\, dx\\
&\le ||f||^2 + ||\alpha'||\varepsilon||u'||^2 \le ||f||^2 + C\varepsilon||u'||^2.
\end{aligned}$$

This also yields

$$||\alpha u'|| \le C||f||. \tag{9}$$

Finally, by the differential equation and the estimates above we get

$$\varepsilon||u''|| = ||f - \alpha u' - u|| \le ||f|| + ||\alpha u'|| + ||u|| \le C||f||.$$

5. See the Lecture Notes.

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