Mathematic Chalmers & GU

TMA372/MMG800: Partial Differential Equations, 2013–03–13, 14:00-18:00 V Halls

Telephone: Oskar Hamlet: 0703-088304
Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 6p. Valid bonus poits will be added to the scores.
Breakings: **3**: 15-20p, **4**: 21-27p och **5**: 28p- For GU students **G**:15-24p, **VG**: 25p-
For solutions and gradings information see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1213/index.html

---

1. The dG(0) solution U for the scalar population dynamics, $\dot{u}(t) + au(t) = f$, $u(0) = u_0$, in the subinterval $I_n = (t_{n-1}, t_n]$ with $k_n = t_n - t_{n-1}$, $n = 1, 2, \dots N$, and $f \equiv 0$ is given by

$$ak_nU_n + (U_n - U_{n-1}) = 0, \qquad U_n = U|_{I_n} = U_n^- = U_{n-1}^+.$$

Let $a > 0$ and show the discrete stability estimate

$$U_N^2 + \sum_{n=0}^{N-1} |[U_n]|^2 \le U_0^2, \qquad [U_n] := U_n^+ - U_n^- = U_{n+1} - U_n.$$

2. Let α and β be positive constants. Give the piecewise linear finite element approximation procedure and derive the corresponding stiffness matrix, covection matrix and load vector using the uniform mesh with size $h = 1/3$ for the problem

$$-u''(x) + 2u'(x) = 3, \quad 0 < x < 1; \qquad u'(0) = \alpha, \quad u(1) = \beta.$$

3. Derive an a priori and an a posteriori error estimate in the energy norm: $\|u\|_E = \|u'\|_{L_2(0,1)}$, for the cG(1) finite element method for the problem

$$-u'' + 2xu' + u = f, \quad 0 < x < 1, \qquad u(0) = u(1) = 0.$$

4. Consider the convection-diffusion problem

$$-div(\varepsilon\nabla u + \beta u) = f, \text{ in } \Omega \subset \mathbb{R}^2, \quad u = 0, \text{ on } \quad \partial\Omega, \qquad \text{for } u \in H_0^1(\Omega),$$

where Ω is a bounded convex polygonal domain, $\varepsilon > 0$ is constant, $\beta = (\beta_1(x), \beta_2(x))$ and $f = f(x)$. Determine the conditions in the Lax-Milgram theorem that would guarantee existence of a unique solution for this problem. Prove a stability estimate for u i terms of $||f||_{L_2(\Omega)}$, ε and $diam(\Omega)$, and under the conditions that you derived.

5. Derive the variational formulation (VF) and formulate a minimization problem (MP) for the boundary value problrm:

$$-(a(x)u'(x))' = f(x), \quad 0 < x < 1, \qquad u(0) = u(1) = 0,$$

and show that (VF) $\Longleftrightarrow$ (MP).

MA

void!

TMA372/MMG800: Partial Differential Equations, 2013–03–13, 14:00-18:00 V Halls. Lösningar.

---

1. For dG(0) we have discontinuous, piecewise constant test functions, hence in the variational formulation below

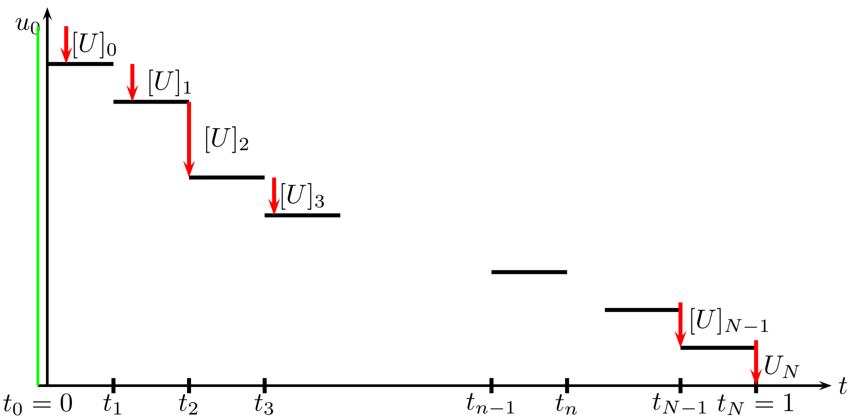
$$(\dot{u}, v) + (au, v) = (f, v),$$

we may take $v \equiv 1$ and hence we have for a single subinterval $I_n = (t_{n-1}, t_n]$ the dG(0) approximation

$$\int_{I_n} (\dot{U} + aU(t)dt + (U_n - U_{n-1})\, dt = \int_{I_n} f\, dt.$$

For $f = 0$ this yields (see als)o Fig below)

(1) $$aK_nU_n + (U_n - U_{n-1}) = 0.$$



Multiplying by U_n we get

$$ak_nU_n^2 + U_n^2 - U_nU_{n-1} = 0,$$

where $a > 0$, whence

$$U_n^2 - U_nU_{n-1} \le 0.$$

Now we use, for $n = 1, 2, \dots, N$,

$$U_n^2 - U_nU_{n-1} = \frac{1}{2}U_n^2 + \frac{1}{2}U_n^2 - U_nU_{n-1},$$

and sum over $n = 1, 2, \dots, N$ to write

$$\begin{aligned}\sum_{n=1}^{N}(U_n^2 - U_nU_{n-1}) &= U_N^2 - U_NU_{N-1} + U_{N-1}^2 - U_{N-1}U_{N-2} + - \dots U_1^2 - U_1U_0 \\ &= U_N^2 - U_NU_{N-1} + U_{N-1}^2 - U_{N-1}U_{N-2} + - \dots U_1^2 - U_1U_0 + \frac{1}{2}U_0^2 2 - \frac{1}{2}U_0^2 \\ &= \frac{1}{2}U_N^2 + \frac{1}{2}(U_N - U_{N-1})^2 + \frac{1}{2}U_{N-1}^2 + \dots + \frac{1}{2}U_1^2 + \frac{1}{2}(U_1 - U_0)^2 - \frac{1}{2}U_0^2 \le 0.\end{aligned}$$

Further by the definition $[U_n] = U_{n+1} - U_n$, hence the above inequality yields the desired result

$$U_N^2 + \sum_{n=0}^{N-1} |[U_n]|^2 \le U_0^2.$$

2. Since we have a Dirichlet boundary condition at $x = 1$, therefore, the test functions are chosen to be 0 at $x = 1$. Henve we multiply the pde by a test function v with $v(1) = 0$, integrate over $x \in (0, 1)$ and use partial integration to get

$$
\begin{aligned}
&- [u'v]_0^1 + \int_0^1 u'v'\,dx + 2\int_0^1 u'v\,dx = 3\int_0^1 v\,dx \qquad \Longleftrightarrow \\
(2)\qquad &- u'(1)v(1) + u'(0)v(0) + \int_0^1 u'v'\,dx + 2\int_0^1 u'v\,dx = 3\int_0^1 v\,dx \qquad \Longleftrightarrow \\
&+ \alpha v(0) + \int_0^1 u'v'\,dx + 2\int_0^1 u'v\,dx = 3\int_0^1 v\,dx.
\end{aligned}
$$

The continuous variational formulation is now formulated as follows: Find

$$(VF) \qquad u \in V := \{w : \int_0^1 \Big(w(x)^2 + w'(x)^2\Big)\,dx < \infty, \quad w(1) = \beta\},$$

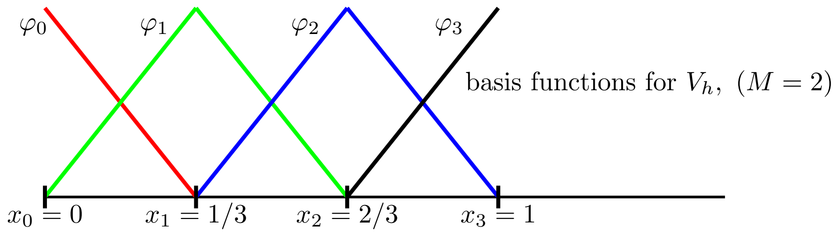
such that

$$\int_0^1 u'v'\,dx + 2\int_0^1 u'v\,dx = 3\int_0^1 v\,dx - \alpha v(0), \quad \forall v \in V^0,$$

where

$$V^0 := \{v : \int_0^1 \Big(v(x)^2 + v'(x)^2\Big)\,dx < \infty, \quad v(1) = 0\}.$$

For the discrete version we let $\mathcal{T}_h$ be a uniform partition: $0 = x_0 < x_1 < \ldots < x_{M+1}$ of $[0, 1]$ into the subintervals $I_n = [x_{n-1}, x_n],\ n = 1, \ldots M+1$. Here, we have M interior nodes: $x_1, \ldots x_M$, two boundary points: $x_0 = 0$ and $x_{M+1} = 1$ and hence $M + 1$ subintervals.



The finite element method (discrete variational formulation) is now formulated as follows: Find

$$(FEM) \qquad U \in V_h := \{w_h : w_h \text{ is piecewise linear, continuous on } \mathcal{T}_h,\ w_h(1) = \beta\},$$

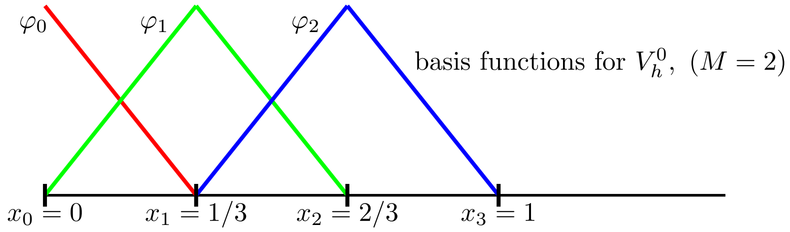
such that

$$(3) \qquad \int_0^1 U'v_h'\,dx + 2\int_0^1 U'v_h\,dx = 3\int_0^1 v_h\,dx - \alpha v_h(0), \quad \forall v \in V_h^0,$$

where

$$V_h^0 := \{v_h : v_h \text{ is piecewise linear, continuous on } \mathcal{T}_h,\ v_h(1) = 0\}.$$

Using the basis functions $\varphi_j,\ j = 0, \ldots M+1$, where $\varphi_1, \ldots \varphi_M$ are the usual *hat-functions*



whereas φ_0 and φ_{M+1} are *semi-hat-functions* viz;

$$(4) \qquad \varphi_j(x) = \begin{cases} 0, & x \notin [x_{j-1}, x_j] \\ \frac{x - x_{j-1}}{h} & x_{j-1} \le x \le x_j \\ \frac{x_{j+1} - x}{h} & x_j \le x \le x_{j+1} \end{cases}, \qquad j = 1, \ldots M.$$

and

$$\varphi_0(x)=\begin{cases}\frac{x_1-x}{h} & 0\le x\le x_1\\ 0, & x_1\le x\le 1\end{cases},\qquad \varphi_{M+1}(x)=\begin{cases}\frac{x-x_M}{h} & x_M\le x\le x_{M+1}\\ 0, & 0\le x\le x_M.\end{cases}$$

In this way we may write

$$V_h=[\varphi_0,\dots,\varphi_M]\oplus\beta\varphi_{M+1},\quad V_h^0=[\varphi_0,\dots,\varphi_M].$$

Thus every $U\in V_h$ can ve written as $U=v_h+\beta\varphi_{M+1}$ where $v_h\in V_h^0$, i.e.,

$$U=\xi_0\varphi_0+\xi_1\varphi_1+\ldots+\xi_{M+}\varphi_M+\beta\varphi_{M+1}=\alpha\varphi_0+\sum_{i=0}^M\xi_i\varphi_i+\beta\varphi_{M+1}\equiv\tilde U+\beta\varphi_{M+1},$$

where $\tilde U\in V_h^0$, and hence the problem (3) can be formulated as to find $\xi_0,\dots\xi_M$ such that

$$\int_0^1\Big(\sum_{j=0}^M\xi_j\varphi_j'+\beta\varphi_{M+1}'\Big)\varphi_i'\,dx+2\int_0^1\Big(\sum_{j=0}^M\xi_j\varphi_j'+\beta\varphi_{M+1}'\Big)\varphi_i\,dx=3\int_0^1\varphi_i\,dx-\alpha\varphi_i(0),\quad j=0,\dots,M,$$

which can be written as find $\xi_j, j=0,\dots,M$ such that for $i=0,\dots,M$,

$$\sum_{i=0}^M\Big(\int_0^1(\varphi_j'\varphi_i'+2\varphi_j'\varphi_i)\,dx\Big)\xi_j+=-\beta\int_0^1\varphi_{M+1}'\varphi_i'\,dx-2\beta\int_0^1\varphi_{M+1}'\varphi_i\,dx+3\int_0^1\varphi_i\,dx-\alpha\varphi_i(0),$$

or equivalently $A\xi=b$ where $A=S+2K$ where S is the stifness matrix and K is the convection matrix. For $h=1/3$ and recalling the half-hat function at $x=0$ we end up with

$$S=\frac{1}{h}\begin{bmatrix}1&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix},\quad K=\frac12\begin{bmatrix}-1&1&0\\-1&0&1\\0&-1&0\end{bmatrix},\quad \text{hence}\quad A=\begin{bmatrix}2&-2&0\\-4&6&-2\\0&-4&6\end{bmatrix},$$

and the unkown ξ and the data b are given by

$$\xi=\begin{bmatrix}\xi_0\\\xi_1\\\cdot\\\cdot\\\xi_{M-1}\\\xi_M\end{bmatrix},\qquad b=\begin{bmatrix}0+3h/2-\alpha\\0+3h-0\\\cdot\\\cdot\\0+3h-0\\-\beta(-1/h)-2\beta(1/2)+3h\end{bmatrix}=\{h=1/3\}=\begin{bmatrix}1/2-\alpha\\1\\2\beta+1\end{bmatrix}.$$

3. We multiply the differential equation by a test function $v\in H_0^1=\{v:||v||+||v'||<\infty,\ v(0)=v(1)=0\}$ and integrate over I. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u\in H_0^1(I)$ such that

$$\int_I(u'v'+2xu'v+uv)=\int_I fv,\quad\forall v\in H_0^1(I).\tag{5}$$

A *Finite Element Method* with *cG*(1) reads as follows: Find $U\in V_h^0$ such that

$$\int_I(U'v'+2xU'v+Uv)=\int_I fv,\quad\forall v\in V_h^0\subset H_0^1(I),\tag{6}$$

where

$$V_h^0=\{v: v\text{ is piecewise linear and continuous in a partition of }I,\ v(0)=v(1)=0\}.$$

Now let $e=u-U$, then (5)-(6) gives that

$$\int_I(e'v'+2xe'v+ev)=0,\quad\forall v\in V_h^0,\qquad\text{(Galerkin Orthogonality)}.\tag{7}$$

We note that using $e(0)=e(1)=0$, we get

$$2\int_I xe'e=\int_I x\frac{d}{dx}(e^2)=(xe^2)|_0^1-\int_I e^2=-\int_I e^2,\tag{8}$$

A priori error estimate: We use (7) and (8) to get

$$\begin{aligned}\|e\|_E^2 := \|e'\|^2_{L_2(0,1)} &= \int_I e'e' = \int_I (e'e' + 2xe'e + ee)\\ &= \int_I \Big(e'(u-U)' + 2xe'(u-U) + e(u-U)\Big) = \{v = U - \pi_h u \text{ in } (7)\}\\ &= \int_I \Big(e'(u-\pi_h u)' + 2xe'(u-\pi_h u) + e(u-\pi_h u)\Big)\\ &\le \|(u-\pi_h u)'\|\|e'\| + 2\|u-\pi_h u\|\|e'\| + \|u-\pi_h u\|\|e\|\\ &\le \{\|(u-\pi_h u)'\| + 3\|u-\pi_h u\|\}\|e'\|\\ &\le C_i\{\|hu''\| + 3\|h^2u''\|\}\|e\|_{H^1},\end{aligned}$$

where in the last step we used Poincare inequality $\|e\| \le \|e'\|$. This yielda the a priori error estimate:

$$\|e\|_{H^1} \le 2C_i\{\|hu''\| + 3\|h^2u''\|\}.$$

A posteriori error estimate:

$$\begin{aligned}\|e\|_E^2 := \|e'\|^2_{L_2(I)} &= \int_I (e'e' + 2xe'e + ee)\\ &= \int_I ((u-U)'e' + 2x(u-U)'e + (u-U)e) = \{v = e \text{ in } (5)\}\\ &= \int_I fe - \int_I (U'e' + 2xU'e + Ue) = \{v = \pi_h e \text{ in } (7)\} \qquad (9)\\ &= \int_I f(e-\pi_h e) - \int_I \Big(U'(e-\pi_h e)' + 2xU'(e-\pi_h e) + U(e-\pi_h e)\Big)\\ &= \{P.I. \text{ on each subinterval}\} = \int_I \mathcal{R}(U)(e-\pi_h e),\end{aligned}$$

where $\mathcal{R}(U) := f + U'' - 2xU' - U = f - 2xU' - U$, (for approximation with piecewise linears, $U \equiv 0$, on each subinterval). Thus (9) implies that

$$\|e\|_E^2 := \|e'\|^2_{L_2(I)} \le \|h\mathcal{R}(U)\|\|h^{-1}(e-\pi_h e)\|$$

where C_i is an interpolation constant, and hence we have with $\|\cdot\| = \|\cdot\|_{L_2(I)}$ that

$$\|e\|_E \le C_i\|h\mathcal{R}(U)\|.$$

4. Recall that $H_0^1(\Omega) := \{w : w \in L_2(\Omega),\ |\nabla w| \in L_2(\Omega),\ w = 0 \text{ on } \partial\Omega\}$. Consider the problem

$$-div(\varepsilon\nabla u + \beta u) = f, \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma = \partial\Omega. \qquad (10)$$

a) Multiply the equation (10) by $v \in H_0^1(\Omega)$ and integrate over Ω to obtain the Green's formula

$$-\int_\Omega div(\varepsilon\nabla u + \beta u)v\,dx = \int_\Omega (\varepsilon\nabla u + \beta u)\cdot\nabla v\,dx = \int_\Omega fv\,dx.$$

Thus the variational formulation for (10) is as follows: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = L(v), \qquad \forall v \in H_0^1(\Omega), \qquad (11)$$

where

$$a(u,v) = \int_\Omega (\varepsilon\nabla u + \beta u)\cdot\nabla v\,dx,$$

and

$$L(v) = \int_\Omega fv\,dx.$$

According to the Lax-Milgram theorem, for a unique solution for (11) we need to verify that the following relations are valid:

i)

$$|a(v,w)| \le \gamma||u||_{H^1(\Omega)}||w||_{H^1(\Omega)}, \qquad \forall v,\ w \in H_0^1(\Omega),$$

ii)

$$a(v,v) \geq \alpha ||v||^2_{H^1(\Omega)}, \qquad \forall v \in H^1_0(\Omega),$$

iii)

$$|L(v)| \leq \Lambda ||v||_{H^1(\Omega)}, \qquad \forall v \in H^1_0(\Omega),$$

for some $\gamma,\ \alpha,\ \Lambda > 0$.
Now since

$$|L(v)| = |\int_\Omega fv\,dx| \leq ||f||_{L_2(\Omega)}||v||_{L_2(\Omega)} \leq ||f||_{L_2(\Omega)}||v||_{H^1(\Omega)},$$

thus iii) follows with $\Lambda = ||f||_{L_2(\Omega)}$. Thus the first condition is that $f \in L_2(\Omega)$.
Further we have that

$$\begin{aligned}
|a(v,w)| &\leq \int_\Omega |\varepsilon\nabla v + \beta v||\nabla w|\,dx \leq \int_\Omega (\varepsilon|\nabla v| + |\beta||v|)|\nabla w|\,dx \\
&\leq \Big(\int_\Omega (\varepsilon|\nabla v| + |\beta||v|)^2\,dx\Big)^{1/2}\Big(\int_\Omega |\nabla w|^2\,dx\Big)^{1/2} \\
&\leq \sqrt{2}\max(\varepsilon, ||\beta||_\infty)\Big(\int_\Omega (|\nabla v|^2 + v^2)\,dx\Big)^{1/2}||w||_{H^1(\Omega)} \\
&= \gamma||v||_{H^1(\Omega)}||w||_{H^1(\Omega)},
\end{aligned}$$

which, with $\gamma = \sqrt{2}\max(\varepsilon, ||\beta||_\infty)$, gives i). Hence the second condition is that $\beta \in L_\infty(\Omega)$.
Finally, if $div\beta \leq 0$, then

$$\begin{aligned}
a(v,v) &= \int_\Omega \Big(\varepsilon|\nabla v|^2 + (\beta\cdot\nabla v)v\Big)\,dx = \int_\Omega \Big(\varepsilon|\nabla v|^2 + (\beta_1\frac{\partial v}{\partial x_1} + \beta_2\frac{\partial v}{\partial x_2})v\Big)\,dx \\
&= \int_\Omega \Big(\varepsilon|\nabla v|^2 + \frac{1}{2}(\beta_1\frac{\partial}{\partial x_1}(v)^2 + \beta_2\frac{\partial}{\partial x_2}(v)^2)\Big)\,dx = \text{Green's formula} \\
&= \int_\Omega \Big(\varepsilon|\nabla v|^2 - \frac{1}{2}(div\beta)v^2\Big)\,dx \geq \int_\Omega \varepsilon|\nabla v|^2\,dx.
\end{aligned}$$

Now by the Poincare's inequality

$$\int_\Omega |\nabla v|^2\,dx \geq C\int_\Omega (|\nabla v|^2 + v^2)\,dx = C||v||^2_{H^1(\Omega)},$$

for some constant $C = C(diam(\Omega))$, we have

$$a(v,v) \geq \alpha||v||^2_{H^1(\Omega)}, \qquad \text{with } \ \alpha = C\varepsilon,$$

thus ii) is valid under the *condition that* $div\beta \leq 0$.
From ii), (11) (with $v = u$) and iii) we get that

$$\alpha||u||^2_{H^1(\Omega)} \leq a(u,u) = L(u) \leq \Lambda||u||_{H^1(\Omega)},$$

which gives the *stability estimate*

$$||u||_{H^1(\Omega)} \leq \frac{\Lambda}{\alpha},$$

with $\Lambda = ||f||_{L_2(\Omega)}$ and $\alpha = C\varepsilon$ defined above.

5. See the Book and/or Lecture Notes.

`MA`