Mathematics Chalmers & GU

TMA372/MMG800: Partial Differential Equations, 2010–01–12; kl 8.30-13.30.

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Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 8p. Valid bonus poits will be added to the scores.
Breakings: **3**: 20-29p, **4**: 30-39p och **5**: 40p- For GU **G** students :20-35p, **VG**: 36p-
For solutions and gradings information see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/0809/index.html

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1. Prove that if $0 < b - a \le 1$, then $\|f\|_{L_1(a,b)} \le \|f\|_{L_2(a,b)} \le \|f\|_{L_\infty(a,b)}$.

2. $U(x) = C_1 \sin(x) + C_2 \sin(2x)$ is an approximate solution to the boundary value problem

$$-\Big(a(x)u'(x)\Big)' = f(x), \quad 0 < x < \pi, \qquad u(0) = u(\pi) = 0,$$

in two dimensions with basis functions $\sin(jx),\ j = 1, 2$. Let $a(x) = 1 + x$, $f(x) = \sin x$ and derive the linear system of equations for the coefficients C_1 and C_2 of U, using the orthogonality

$$\int_0^\pi R(x)\sin(jx)\,dx = 0, \quad j = 1, 2; \qquad \text{where} \quad R(x) := R(U(x)) \ \text{ is the residual.}$$

3. Consider the convection-diffusion problem

$$-\varepsilon u''(x) + a(x)u'(x) + u(x) = f(x), \qquad x \in I = (0, 1), \qquad u(0) = 0, \quad u'(1) = 0,$$

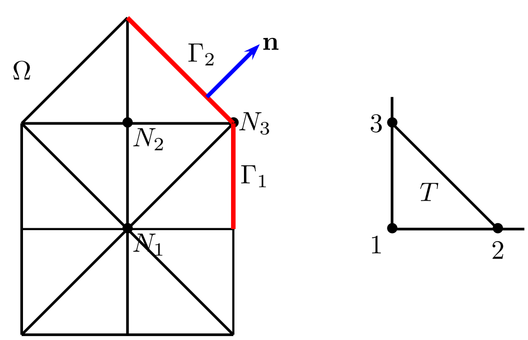
where ε is a positive constant and the function a satisfies $a(x) \ge 0$, $a'(x) \le 0$. Show that

$$(i)\ \sqrt{\varepsilon}||u'|| \le C||f||, \quad (ii)\ ||au'|| \le C||f||, \quad (iii)\ \varepsilon||u''|| \le C||f||, \quad \text{with } ||w|| = \Big(\int_0^1 w^2(x)\,dx\Big)^{1/2}.$$

4. Formulate the cG(1) piecewise continuous Galerkin method for the boundary value problem

$$-\Delta u + u = f, \quad x \in \Omega; \qquad u = 0, \quad x \in \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2), \quad \nabla u \cdot \mathbf{n} = 0, \quad x \in \Gamma_1 \cup \Gamma_2,$$

on the domain Ω, with outward unit normal $\mathbf{n}$ at the boundary (see fig.). Write the matrices for the resulting equation system using the following mesh with nodes at $N_1,\ N_2$ and N_3.



5. Prove an a posteriori error estimate for the cG(1) approximation of the two-point boundary value problem $-(a(x)u'(x))' = f,\ 0 < x < 1, \quad u(0) = u(1) = 0$: There is an interpolation constant C_i depending only on a such that the finite element approximation U satisfies

$$\|u' - U'\|_a \le C_i\|hR(U)\|_{a^{-1}}, \qquad \|w\|_q = \Big(\int_0^1 q(x)w^2(x)\,dx\Big)^{1/2}, \ (q \text{ is a weight function}).$$

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Lösningar/Solutions.

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1. Using the definition of L_p-norms we write

$$\begin{aligned}
\|f\|_{L_1(a,b)} &= \int_a^b |f(x)|\,bx = \int_a^b 1\cdot|f(x)|\,bx \le \{\text{C-S}\} \le \Big(\int_a^b 1^2\,dx\Big)^{1/2}\Big(\int_a^b f^2(x)\,dx\Big)^{1/2}\\
&= \sqrt{b-a}\Big(\int_a^b f^2(x)\,dx\Big)^{1/2} = \sqrt{b-a}\|f\|_{L_2(a,b)}\\
&\le \sqrt{b-a}\Big(\int_a^b \max_{x\in[a,b]} f^2(x)\,dx\Big)^{1/2} = \sqrt{b-a}\Big(\int_a^b \max_{x\in[a,b]} |f(x)|^2\,dx\Big)^{1/2}\\
&= \sqrt{b-a}\max_{x\in[a,b]}|f(x)|\cdot\Big(\int_a^b dx\Big)^{1/2} = (b-a)\|f\|_{L_\infty(a,b)}.
\end{aligned}$$

Thus, we have proved that

$$\|f\|_{L_1(a,b)} \le \sqrt{b-a}\|f\|_{L_2(a,b)} \le (b-a)\|f\|_{L_\infty(a,b)}.$$

If now $0<(b-a)\le 1$ then $0<\sqrt{b-a}\le 1$, then we get'

$$\|f\|_{L_1(a,b)} \le \|f\|_{L_2(a,b)} \le \|f\|_{L_\infty(a,b)}.$$

2. We insert $a(x)=1+x$ ans $f(x)=\sin x$ in the equation then we have
Variational formulation: Multiply the equation by $v\in H_0^1$ and integrate over $[0,\pi]$, where

$$v\in H_0^1 := H_0^1[0,\pi] := \{v : \int_0^1 \Big(v^2(x)+v'2(x)\Big)\,dx<\infty,\ v(0)=v(\pi)=0\}.$$

Partial integration gives that

$$-\int_0^\pi \Big((1+x)u'(x)\Big)'v(x)\,dx = -(1+x)u'(x)v(x)\Big|_0^\pi + \int_0^\pi \Big((1+x)u'(x)\Big)v'(x)\,dx.$$

With $v(0)=v(\pi)=0$, we obtain

$$(VF)\qquad \int_0^\pi \Big((1+x)u'(x)\Big)v'(x)\,dx = \int_0^\pi \sin x\ v(x)\,dx, \qquad \forall v\in H_0^1.$$

The corresponding Galerkin method in a finite dimensional space with base functions $\sin x$ and $\sin(2x)$ is given by

$$(GM)\qquad \int_0^\pi \Big((1+x)U'(x)\Big)\varphi_i'(x)\,dx = \int_0^\pi \sin x\,\varphi_i(x)\,dx, \qquad \varphi_i(x)=\sin(ix),\ i=1,2.$$

Now let $U(x)=C_1\sin(x)+C_2\sin(2x)=C_1\varphi_1(x)+C_2\varphi_2(x)$ then, $GM \quad\Longleftrightarrow$

$$\int_0^\pi \Big((1+x)(C_1\cos(x)+C_2 2\cos(2x)\Big)i\cos(ix)\,dx = \int_0^\pi \sin x\ \sin(ix)\,dx, \qquad i=1,2,$$

which corresponds to the system of equations: $A\xi=\mathbf{b}$, where $A=(a_{ij}),\ \mathbf{b}=(b_i),\ \xi=(C_i),\ i,j=1,2$, with

$$a_{ij} = ij\int_0^\pi (1+x)\cos(jx)\cos(ix)\,dx, \qquad b_i=\int_0^\pi \sin(ix)\sin(x)\,dx, \qquad i,j=1,2.$$

Now using partial integration we get

$$a_{11} = \int_0^\pi (1+x)\cos^2(x)\,dx = \int_0^\pi (1+x)\Big[\frac{1}{2} + \frac{1}{2}\cos(2x)\Big]\,dx$$

$$= (1+x)\Big[\frac{x}{2} + \frac{1}{4}\sin(2x)\Big]\Big|_0^\pi - \int_0^\pi \Big(\frac{x}{2} + \frac{1}{4}\sin(2x)\Big)\,dx$$

$$= (1+\pi)\frac{\pi}{2} - \Big[\frac{x^2}{4} - \frac{1}{8}\cos(2x)\Big]\Big|_0^\pi = (1+\pi)\frac{\pi}{2} - \frac{\pi^2}{4} = \frac{\pi}{2} + \frac{\pi^2}{4}.$$

$$a_{12} = a_{21} = 2\int_0^\pi (1+x)\cos(x)\cos(2x)\,dx = \int_0^\pi (1+x)\Big[\cos(x) + \cos(3x)\Big]\,dx$$

$$= (1+x)\Big[\sin(x) + \frac{1}{3}\sin(3x)\Big]\Big|_0^\pi - \int_0^\pi \Big(\sin(x) + \frac{1}{3}\sin(3x)\Big)\,dx$$

$$= \Big[\cos(x) + \frac{1}{9}\cos(3x)\Big|_0^\pi = \Big[(-1) + \frac{1}{9}(-1) - 1 - \frac{1}{9}\Big] = -2 - \frac{2}{9} = -\frac{20}{9}.$$

$$a_{22} = 4\int_0^\pi (1+x)\cos^2(2x)\,dx = 2\int_0^\pi (1+x)\Big[1 + \cos(4x)\Big]\,dx$$

$$= 2(1+x)\Big[x + \frac{1}{4}\sin(4x)\Big]\Big|_0^\pi - 2\int_0^\pi \Big(x + \frac{1}{4}\sin(4x)\Big)\,dx$$

$$= 2(1+\pi)\pi - 2\Big[\frac{x^2}{2} - \frac{1}{16}\cos(4x)\Big]\Big|_0^\pi = 2\pi + 2\pi^2 - \pi^2 = 2\pi + \pi^2.$$

Further by orthogonality of the set $\{\sin(jx)\}_j$ on the interval $[0,\pi]$ we have

$$b_1 = \int_0^\pi \sin^2(x)\,dx = \frac{\pi}{2}, \qquad b_2 = \int_0^\pi \sin(x)\sin(2x)\,dx = 0.$$

Hence the equation system for the coefficients C_1 and C_2 is given by

$$\begin{bmatrix} \frac{1}{4}(2\pi+\pi^2), & -\frac{20}{9} \\ -\frac{10}{9} & 2\pi+\pi^2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix}.$$

3. Multiply the equation by u and integrate over $[0,1]$ to get

$$\varepsilon||u'||^2 + \int_0^1 au'u\,dx + ||u||^2 = (f,u) \le ||f||||u|| \le \frac{1}{2}||f||^2 + \frac{1}{2}||u||^2.$$

Here

$$\int_0^1 au'u\,dx = \frac{1}{2}\int_0^1 a\frac{d}{dx}u^2\,dx = \frac{1}{2}a(1)u(1)^2 - \frac{1}{2}\int_0^1 a'u^2\,dx \ge 0, \tag{1}$$

therefore

$$\varepsilon||u'||^2 + \frac{1}{2}||u||^2 \le \frac{1}{2}||f||^2.$$

This gives that

$$\sqrt{\varepsilon}||u'|| \le ||f||, \qquad ||u|| \le ||f||. \tag{2}$$

Now we multiply the equation by au' and integrate over $x \in [0,1]$:

$$-\varepsilon\int_0^1 u''au'\,dx + ||au'||^2 + \int_0^1 au'u\,dx \le \frac{1}{2}||f||^2 + \frac{1}{2}||au'||^2.$$

Thus according to (1)

$$\begin{aligned}||au'||^2 &\le ||f||^2 + \varepsilon \int_0^1 a \frac{d}{dx}(u')^2\,dx \\ &= ||f||^2 - \varepsilon a(0)u'(0)^2 - \varepsilon \int_0^1 a'(u')^2\,dx \\ &\le ||f||^2 + ||a'||\varepsilon||u'||^2 \le ||f||^2 + C\varepsilon||u'||^2.\end{aligned}$$

Hence using (2) we get

$$||au'|| \le C||f||. \tag{3}$$

Finally, by the differential equation and (2), (3)

$$\varepsilon||u''||0||f - au' - u|| \le ||f|| + ||au'|| + ||u|| \le C||f||.$$

4. Let V be the linear function space defined by

$$V_h := \{v : v \text{ is continuous in } \Omega,\ \ v = 0,\ \text{ on } \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)\}.$$

Multiplying the differential equation by $v \in V$ and integrating over Ω we get that

$$-(\Delta u, v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Now using Green's formula we have that

$$\begin{aligned}-(\Delta u, \nabla v) &= (\nabla u, \nabla v) - \int_{\partial\Omega} (n \cdot \nabla u) v\,ds \\ &= (\nabla u, \nabla v) - \int_{\partial\Omega\setminus(\Gamma_1\cup\Gamma_2)} (n \cdot \nabla u) v\,ds - \int_{\Gamma_1\cup\Gamma_2} (n \cdot \nabla u) v\,ds \\ &= (\nabla u, \nabla v), \qquad \forall v \in V.\end{aligned}$$

Thus the variational formulation is:

$$(\nabla u, \nabla v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Let V_h be the usual finite element space consisting of continuous piecewise linear functions satisfying the boundary condition $v = 0$ on $\partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)$: The $cG(1)$ method is: Find $U \in V_h$ such that

$$(\nabla U, \nabla v) + (U, v) = (f, v) \qquad \forall v \in V_h$$

Making the "Ansatz" $U(x) = \sum_{j=1}^{3} \xi_j \varphi_j(x)$, where φ_i are the standard basis functions, we obtain the system of equations

$$\sum_{j=1}^{3} \xi_j \Big(\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j\,dx + \int_\Omega \varphi_i\varphi_i\,dx \Big) = \int_\Omega f\varphi_j\,dx, \quad i = 1, 2, 3,$$

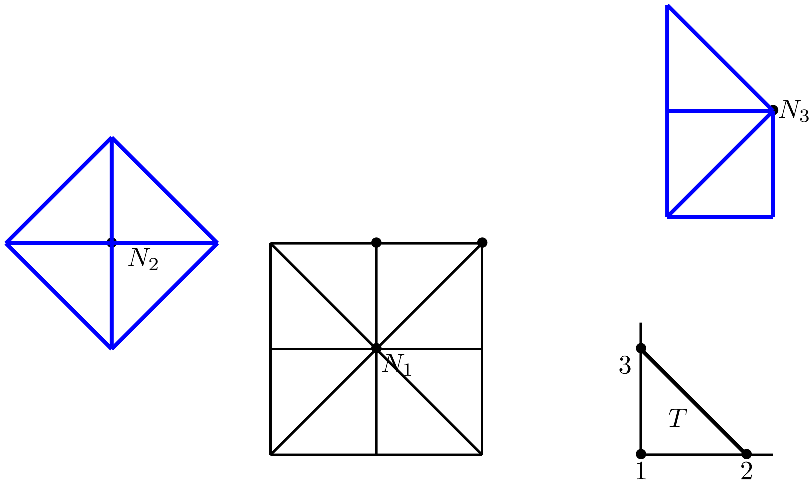
or, in matrix form,

$$(S + M)\xi = F,$$

where $S_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix, $M_{ij} = (\varphi_i, \varphi_j)$ is the mass matrix, and $F_i = (f, \varphi_i)$ is the load vector.

We first compute the mass and stiffness matrix for the reference triangle T. The local basis functions are

$$\begin{aligned}\phi_1(x_1, x_2) &= 1 - \frac{x_1}{h} - \frac{x_2}{h}, & \nabla\phi_1(x_1, x_2) &= -\frac{1}{h}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ \phi_2(x_1, x_2) &= \frac{x_1}{h}, & \nabla\phi_2(x_1, x_2) &= \frac{1}{h}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \phi_3(x_1, x_2) &= \frac{x_2}{h}, & \nabla\phi_3(x_1, x_2) &= \frac{1}{h}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.\end{aligned}$$



Hence, with $|T| = \int_T dz = h^2/2$,

$$m_{11} = (\phi_1, \phi_1) = \int_T \phi_1^2 \, dx = h^2 \int_0^1 \int_0^{1-x_2} (1 - x_1 - x_2)^2 \, dx_1 dx_2 = \frac{h^2}{12},$$
$$s_{11} = (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2 \, dx = \frac{2}{h^2}|T| = 1.$$

Alternatively, we can use the midpoint rule, which is exact for polynomials of degree 2 (precision 3):

$$m_{11} = (\phi_1, \phi_1) = \int_T \phi_1^2 \, dx = \frac{|T|}{3} \sum_{j=1}^{3} \phi_1(\hat{x}_j)^2 = \frac{h^2}{6}\Big(0 + \frac{1}{4} + \frac{1}{4}\Big) = \frac{h^2}{12},$$

where $\hat{x}_j$ are the midpoints of the edges. Similarly we can compute the other elements and obtain

$$m = \frac{h^2}{24} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad s = \frac{1}{2} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrices M and S from the local ones m and s:

$$\begin{aligned}
M_{11} &= 8m_{22} = \frac{8}{12}h^2, & S_{11} &= 8s_{22} = 4, \\
M_{12} &= 2m_{12} = \frac{1}{12}h^2, & S_{12} &= 2s_{12} = -1, \\
M_{13} &= 2m_{23} = \frac{1}{12}h^2, & S_{13} &= 2s_{23} = 0, \\
M_{22} &= 4m_{11} = \frac{4}{12}h^2, & S_{22} &= 4s_{11} = 4, \\
M_{23} &= 2m_{12} = \frac{1}{12}h^2,, & S_{23} &= 2s_{12} = -1, \\
M_{33} &= 3m_{22} = \frac{3}{12}h^2,, & S_{33} &= 3s_{22} = 3/2.
\end{aligned}$$

The remaining matrix elements are obtained by symmetry $M_{ij} = M_{ji}$, $S_{ij} = S_{ji}$. Hence,

$$M = \frac{h^2}{12} \begin{bmatrix} 8 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \qquad S = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 3/2 \end{bmatrix}.$$

5. See lecture notes

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