

TMA372/MAN660 Partiella differentialekvationer TM, E3, GU

OBS! Skriv namn och personnummer på samtliga inlämnade papper.

1. a) Derive the fundamental solution for the initial value problem

$$\dot{u}(t) + a(t)u(t) = f(t), \quad 0 < t \leq T, \quad u(0) = u_0.$$

- b) Prove the stability estimates

$$\begin{aligned} i) \quad a(t) \geq \alpha > 0 &\implies |u(t)| \leq e^{-\alpha t}|u_0| + \frac{1}{\alpha}(1 - e^{-\alpha t}) \max_{0 \leq s \leq t} |f(s)| \\ ii) \quad a(t) \geq 0 &\implies |u(t)| \leq |u_0| + \int_0^t |f(s)| ds. \end{aligned}$$

2. Show that for $a(t) > 0$, and for $N = 1, 2, \dots$, the piecewise linear approximate solution U for the problem 1 satisfies the a posteriori error estimate

$$|u(t_N) - U_N| \leq \max_{[0, t_N]} |k(\dot{U} + aU - f)|, \quad k = k_n, \text{ for } t_{n-1} < t \leq t_n.$$

3. Prove a *priori* and a *posteriori* error estimates, in the energy norm $\|v\|_E^2 = \|v'\|^2 + a\|v\|^2$, for the $cG(1)$ approximation of the boundary value problem

$$-u''(x) + u'(x) + au(x) = f(x), \quad 0 < x < 1, \quad u(0) = u(1) = 0, \quad a \geq 0.$$

4. a) Formulate a $cG(1)$ finite element method for the following system

$$\begin{cases} u(x) + v''(x) = f(x), & v(0) = v(1) = 0, & 0 < x < 1, \\ u''(x) - v(x) = 0, & u(0) = u(1) = 0, \end{cases}$$

and show how the approximate solution (U, V) can be computed from the load vector F , using mass- and stiffness matrices.

- b) Derive stability estimates for u and v , in terms of f , (e.g., through multiplying the first equation by u and the second by $-v$).

5. Consider the following *Schrödinger* equation

$$\dot{u} + i\Delta u = 0, \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega,$$

where $i = \sqrt{-1}$ and $u = u_1 + iu_2$. a) Show that the L_2 norm of the solution, i.e., $\int_{\Omega} |u|^2$ is time independent.

Hint: Multiply the equation by $\bar{u} = u_1 - iu_2$, integrate over Ω and consider the real part.

- b) Consider the corresponding *eigenvalue problem*, of finding $(\lambda, u \neq 0)$, such that

$$-\Delta u = \lambda u \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega.$$

Show that $\lambda > 0$, and give the relation between $\|u\|$ and $\|\nabla u\|$ for the corresponding eigenfunction u .

- c) What is the optimal constant C (expressed in terms of smallest eigenvalue λ_1), for which the inequality $\|u\| \leq C\|\nabla u\|$ can hold for all functions u , such that $u = 0$ on $\partial\Omega$?

**TMA372/MAN660 Partial Differential Equations TM, E3, GU
2001-12-18. Solutions**

1. a) Derive the fundamental solution for the initial value problem

$$\dot{u}(t) + a(t)u(t) = f(t), \quad 0 < t \leq T, \quad u(0) = u_0.$$

b) Prove the stability estimates

$$\begin{aligned} i) \quad a(t) \geq \alpha > 0 &\implies |u(t)| \leq e^{-\alpha t}|u_0| + \frac{1}{\alpha}(1 - e^{-\alpha t}) \max_{0 \leq s \leq t} |f(s)| \\ ii) \quad a(t) \geq 0 &\implies |u(t)| \leq |u_0| + \int_0^t |f(s)| ds. \end{aligned}$$

solution. See lecture notes (chapter 9).

2. Show that for $a(t) > 0$, and for $N = 1, 2, \dots$, the piecewise linear approximate solution U for the problem 1 satisfies the a posteriori error estimate

$$|u(t_N) - U_N| \leq \max_{[0, t_N]} |k(\dot{U} + aU - f)|, \quad k = k_n, \text{ for } t_{n-1} < t \leq t_n.$$

solution. See the course book; chapter 9 (a simple version of theorem 9.2).

3. Prove a priori and a posteriori error estimates, in the energy norm $\|v\|_E^2 = \|v'\|^2 + a\|v\|^2$, for the $cG(1)$ approximation of the boundary value problem

$$-u''(x) + u'(x) + au(x) = f(x), \quad 0 < x < 1, \quad u(0) = u(1) = 0, \quad a \geq 0.$$

Solution:

(a) The Variational formulation:

(Multiply the equation by $v \in V$, integrate by parts over $(0, 1)$ and use the boundary conditions.)

$$(1) \quad \text{Find } u \in V : \int_0^1 u'v' dx + \int_0^1 u'v dx + \int_0^1 auv dx = \int_0^1 fv dx, \quad \forall v \in V.$$

cG(1):

(2)

$$\text{Find } U \in V_h : \int_0^1 U'v' dx + \int_0^1 U'v dx + \int_0^1 aUv dx = \int_0^1 fv dx, \quad \forall v \in V_h,$$

where

$$V_h := \{v : v \text{ is continuous piecewise linear in } (0, 1), v(0) = v(1) = 0\}.$$

From (1)-(2), we find

The Galerkin orthogonality:

$$(3) \quad \int_0^1 \left((u - U)'v' + (u - U)'v + a(u - U)v \right) dx = 0, \quad \forall v \in V_h.$$

We define the inner product $(\cdot, \cdot)_E$ associated to the energy norm to be

$$(v, w)_E = \int_0^1 (v'w' + avw) dx, \quad \forall v, w \in V.$$

A priori error estimate:

$$\begin{aligned}
\|e\|_E^2 &= \int_0^1 (e'e' + aee) dx = \left\{ \int_0^1 e'e dx = \frac{1}{2} \int_0^1 \frac{d}{dx} (e^2) dx = \frac{1}{2} [e^2]_0^1 = 0 \right\} \\
&= \int_0^1 (e'e' + e'e + aee) dx = \int_0^1 \left(e'(u-U)' + e'(u-U) + ae(u-U) \right) dx \\
&= \{v \in V_h\} = \int_0^1 \left(e'(u-v)' + e'(u-v) + ae(u-v) \right) dx \\
&\quad + \int_0^1 \left(e'(v-U)' + e'(v-U) + ae(v-U) \right) dx = \{(3)\} \\
&= \int_0^1 \left(e'(u-v)' + e'(u-v) + ae(u-v) \right) dx \\
&= \int_0^1 \left(e'(u-v)' + ae(u-v) + e'(u-v) \right) dx \\
&\leq \|e'\|_{L_2} \cdot \|(u-v)'\|_{L_2} + a\|e\|_{L_2} \|u-v\|_{L_2} + \|e'\|_{L_2} \cdot \|u-v\|_{L_2} \\
&\leq \|e'\|_{L_2} \cdot \|(u-v)'\|_{L_2} + \|e\|_E \cdot \|u-v\|_{L_2} \\
&\leq \|e\|_E \cdot \|u-v\|_E + \|e\|_E \cdot \|u-v\|_{L_2} \\
&\leq \|e\|_E \left(\|u-v\|_E + \|u-v\|_{L_2} \right).
\end{aligned}$$

This gives the a priori error estimate:

$$\|e\|_E \leq \|u-v\|_E + \|u-v\|_{L_2}, \quad \forall v \in V_h.$$

A posteriori error estimate: We have that

$$\begin{aligned}
\|e\|_E^2 &= \int_0^1 (e'e' + aee) dx = \int_0^1 (e'e' + e'e + aee) dx \\
&= \int_0^1 (u'e' + u'e + aue) dx - \int_0^1 (U'e' + U'e + aUe) dx.
\end{aligned}$$

Thus using (1) we get

$$(4) \quad \|e\|_E^2 = \int_0^1 fe dx - \int_0^1 (U'e' + U'e + aUe) dx,$$

which by (2) can be written as

$$\begin{aligned}
\|e\|_E^2 &= \int_0^1 fe dx - \int_0^1 (U'e' + U'e + aUe) dx \\
&\quad + \int_0^1 \left(U'(\Pi_h e)' + U'\Pi_h e + aU\Pi_h e \right) dx - \int_0^1 f\Pi_h e dx.
\end{aligned}$$

Observe that the last line above is identically 0. Adding up we have

$$\begin{aligned}
\|e\|_E^2 &= \int_0^1 f(e - \Pi_h e) dx - \int_0^1 \left(U'(e - \Pi_h e)' + U'(e - \Pi_h e) + aU(e - \Pi_h e) \right) dx \\
&= \int_0^1 f(e - \Pi_h e) dx - \int_0^1 (U' + aU)(e - \Pi_h e) dx - \sum_{j=1}^{M+1} \int_{I_j} U'(e - \Pi_h e)' dx \\
&= \{\text{partial integration}\} \\
&= \int_0^1 f(e - \Pi_h e) dx - \int_0^1 (U' + aU)(e - \Pi_h e) dx + \sum_{j=1}^{M+1} \int_{I_j} U''(e - \Pi_h e) dx \\
&= \int_0^1 (f + U'' - U' - aU)(e - \Pi_h e) dx = \int_0^1 R(U)(e - \Pi_h e) dx \\
&= \int_0^1 hR(U)h^{-1}(e - \Pi_h e) dx \leq \|hR(U)\|_{L_2} \|h^{-1}(e - \Pi_h e)\|_{L_2} \\
&\leq C_i \|hR(U)\|_{L_2} \cdot \|e'\|_{L_2} \leq C_i \|hR(U)\|_{L_2} \cdot \|e\|_E.
\end{aligned}$$

This gives the a posteriori error estimate:

$$\|e\|_E \leq C_i \|hR(U)\|_{L_2},$$

with $R(U) = f + U'' - U' - aU = f - U' - aU$, on (x_{i-1}, x_i) , $i = 1, \dots, M + 1$.

4. a) Formulate a $cG(1)$ finite element method for the following system

$$\begin{cases} u(x) + v''(x) = f(x), & v(0) = v(1) = 0, \quad 0 < x < 1, \\ u''(x) - v(x) = 0, & u(0) = u(1) = 0, \end{cases}$$

and show how the approximate solution (U, V) can be computed from the load vector F , using mass- and stiffness matrices.

b) Derive stability estimates for u and v , in terms of f , (e.g., through multiplying the first equation by v and the second by u).

Solution: a) Multiplying the equations in the system by the test functions φ and ψ , with $\varphi = \psi = 0$ for $x = 0$ and $x = 1$, and integrating by parts gives that

$$(5) \quad \begin{cases} \int_0^1 (\varphi u - \varphi' v') = \int_0^1 \varphi f, \\ \int_0^1 (-\psi' u' - \psi v) = 0. \end{cases}$$

Partitioning of $[0, 1]$ into subintervals (elements) $I_j = [x_{j-1}, x_j]$, $x_j = j/(m + 1)$, the linear approximations $U(x) = \sum_{j=1}^m U_j \varphi_j(x)$ and $V(x) = \sum_{j=1}^m V_j \varphi_j(x)$, with $\varphi_j(x)$ s being the usual piecewise linear basis functions, the $cG(1)$ approximation of the above system (4) can be formulated as: Find the nodal values U_j and V_j such that

$$(6) \quad \begin{cases} \int_0^1 (\varphi_i U - \varphi_i' V') = \int_0^1 \varphi_i f, & i = 1, \dots, m, \\ \int_0^1 (-\varphi_i' U' - \varphi_i V) = 0, & i = 1, \dots, m. \end{cases}$$

This gives $2m$ equations with the $2m$ unknown nodal values $U = [U_1, \dots, U_m]^T$ and $V = [V_1, \dots, V_m]^T$, which can be written in the matrix form as

$$(7) \quad \begin{cases} MU - SV = F, \\ -SU - MV = 0, \end{cases}$$

where M and S are the usual, 3-diagonal, mass- and stiffness matrices, respectively: M has $2h/3$ diagonal elements and $h/6$ super and subdiagonal elements. Corresponding elements for S are $2/h$ diagonal elements and $-1/h$ sub and superdiagonal elements. All other elements are zeros. F is the load vector with elements $\int_0^1 \varphi_i f$. From the second equation above we get that $V = -M^{-1}SU$, which inserting in the first equation gives $U = (M + SM^{-1}S)^{-1}F$.

b) Multiply the first equation by u and the second equation by $-v$, add the two resulting equations and integrate over $[0, 1]$. By partial integration we have then

$$\int_0^1 u^2 + v^2 = \int_0^1 uf,$$

Using Cauchy-Schwartz inequality we get

$$\|u\|^2 + \|v\|^2 = \int_0^1 uf \leq \|u\| \|f\| \leq \frac{1}{2}\|u\|^2 + \frac{1}{2}\|f\|^2$$

This gives that $\|u\| \leq \|f\|$, and consequently even $\|v\| \leq \|f\|$.

We could alternatively multiply the first equation by $-v$ and the second by $-u$, add the two resulting equations and integrate over $[0, 1]$. By partial integration we have this time

$$\int_0^1 (u')^2 + (v')^2 = \int_0^1 (-v)f,$$

Using, first Poincaré', and then the Cauchy-Schwartz inequality we get

$$\|u'\|^2 + \|v'\|^2 \leq \|v\| \|f\| \leq \|v'\| \|f\| \leq \frac{1}{2}\|v'\|^2 + \frac{1}{2}\|f\|^2,$$

so that we have now $\|v'\| \leq \|f\|$, and consequently even $\|u'\| \leq \|f\|$.

We could obviously continue in this manner and get basic stability estimates for the, e.g., moment $v = u''$, through

$$\|u''\| = \|v\| \leq \|f\|,$$

and for v'' :

$$\|v''\| = \|f - u\| \leq \|f\| + \|u\| \leq 2\|f\|.$$

5. Consider the following *Schrödinger* equation

$$\dot{u} + i\Delta u = 0, \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega,$$

where $i = \sqrt{-1}$ and $u = u_1 + iu_2$. a) Show that the *total probability* $\int_{\Omega} |u|^2$ is time independent.

Hint: Multiply the equation by $\bar{u} = u_1 - iu_2$, integrate over Ω and consider the real part.

b) Consider the corresponding *eigenvalue problem*, of finding $(\lambda, u \neq 0)$, such that

$$-\Delta u = \lambda u \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega.$$

Show that $\lambda > 0$, and give the relation between $\|u\|$ and $\|\nabla u\|$ for the corresponding eigenfunction u .

c) What is the optimal constant C (expressed in terms of smallest eigenvalue λ_1), for which the inequality $\|u\| \leq C\|\nabla u\|$ can fulfil for all functions u , such that $u = 0$ on $\partial\Omega$?

Solution: a) We multiply the Schrödinger equation by \bar{u} and integrate over Ω to obtain

$$\int_{\Omega} \bar{u} \dot{u} + i \int_{\Omega} \bar{u} \nabla u = \int_{\Omega} (u_1 \dot{u}_1 + u_2 \dot{u}_2) + i \int_{\Omega} (u_1 \dot{u}_2 - u_2 \dot{u}_1 - \nabla \bar{u} \cdot \nabla u) = 0.$$

Now both real and imaginary part of the above expression is 0. Thus, considering the real part, we have

$$\int_{\Omega} (u_1 \dot{u}_1 + u_2 \dot{u}_2) = \frac{1}{2} \frac{\partial}{\partial t} \int_{\Omega} (u_1^2 + u_2^2) = 0,$$

therefore $\int_{\Omega} |u|^2$ is independent of the time.

b) Multiplying the eigenvalue equation $-\Delta u = \lambda u$ by u , integrating over Ω , and using partial integration we get

$$\lambda \int_{\Omega} u^2 = \int_{\Omega} u(-\Delta u) = \int_{\Omega} |\nabla u|^2,$$

which gives $\lambda \geq 0$ (and also $\lambda > 0$, for $u \neq 0$). Further $\|u\| = \frac{1}{\sqrt{\lambda}} \|\nabla u\|$. This indicates that the constant in the estimate $\|u\| \leq C\|\nabla u\|$, satisfying for all functions u with $u = 0$ on $\Gamma := \partial\Omega$, can not be smaller than $\frac{1}{\sqrt{\lambda_1}}$, with $\lambda_1 > 0$ being the smallest eigenvalue. As a matter of fact we have the inequality $\|u\| \leq \frac{1}{\sqrt{\lambda_1}} \|\nabla u\|$, for all u with $u = 0$ on Γ . This is due to the fact that we can represent u in terms of orthogonal eigenfunctions both “with and without gradient”, i.e. $\int_{\Omega} u_i u_j = \int_{\Omega} \nabla u_i \cdot \nabla u_j = 0$, for $i \neq j$.

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