

Föreläsning 20/1:

Goal: PDE

(IVP) \rightarrow Dynamical systems
 (BVP) \rightarrow Poisson's eqn
 (IBVP) \rightarrow Heat & Wave eqn

Theoretical approach

Weak formulation

Variational formulation

(VF)

$$\int_{\Omega} ((DE) \text{ test function}) dx$$

$\Omega \subseteq \mathbb{R}^n$

↑ contains
 ∞ many points

minimization problem (MP) \Leftrightarrow Show that $\exists!$ sol for MP

Well posedness

stability

Approximation procedure: (FEM) = discrete version of (VF)

By piecewise continuous/discontinuous subdomains $\Omega_1, \dots, \Omega_n \subset \Omega \Leftrightarrow$ $\cup_i \Omega_i = \Omega$

continuous Galerkin

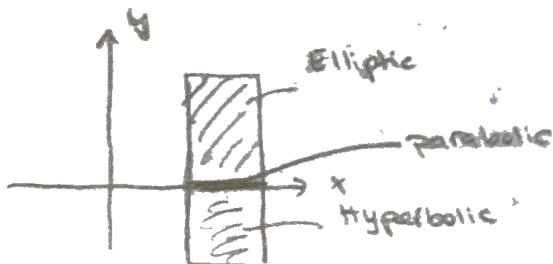
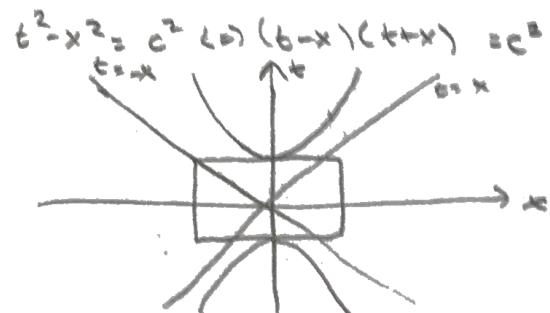
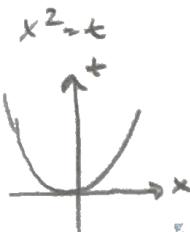
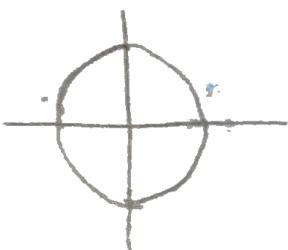
discontinuous G

polynomials in Ω

Polynomial interpolation

char equation:

$$x^2 + y^2 = c$$



F-transform of a function f: $\hat{f}(g) = \int_{\mathbb{R}^n} [e^{-ix \cdot \xi}] f(x) dx$

Method of Fundamental solution:

DE: $\begin{cases} LHS = f \\ B.C \\ I.C \end{cases}$ $(LHS)^T = g \Rightarrow u_g \Rightarrow u = u_g * f$

Dynamical system:

$$\begin{cases} \dot{u}(t) = \lambda u(t) + f(t) \\ u(0) = u_0 \end{cases} \quad \text{given} \quad \Rightarrow \frac{\dot{u}(t)}{u(t)} = \lambda \Rightarrow \ln u(t) = \lambda t \Rightarrow u(t) = e^{\lambda t}$$

Remedy

if $\lambda > 0$, then $u(t) \rightarrow \infty$ when $t \rightarrow \infty$
 if $\lambda < 0$, then $u(t) \rightarrow 0$ when $t \rightarrow \infty$

General form: $\begin{cases} \dot{u}(t) = F(u, t), \\ F: \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n \end{cases}$

$$u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix}, \quad \dot{u}_i = \frac{\partial u_i}{\partial t} = \begin{pmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \\ \vdots \\ \dot{u}_n(t) \end{pmatrix}$$

Galerkin method: (for 1D DS)

For a solution of ⑤ we have

$$\int_0^T \dot{u}(t)v(t)dt = \lambda \int_0^T u(t)v(t)dt \quad \forall v(t)$$

$$(*) \int_0^T (\dot{u}(t) - \lambda u(t))v(t)dt = 0 \quad \forall v(t)$$

Def: If w is an approximation of a solution, then $R(w(t)) := w(t) - \lambda w(t)$ is called the residual of approximate solution $w(t)$

$$(*) \Rightarrow (\dot{u}(t) - \lambda u(t)) \perp v(t) \quad \forall v(t)$$

Approx: we require that $R(w(t)) \perp V(t) \quad V(t) \in ?$

Want to look at approx with polynomials of degree $\leq q$.

$$V^{(q)} = \{ \xi_0 + \xi_1 t + \dots + \xi_q t^q \}$$

already known

$$\text{Let } V_0^{(q)} = \{ v \in V^q : v(0) = 0 \}$$

Then the Galerkin method is:

$$\text{Find } u(t) \in V^{(q)} \text{ such that } \int_0^1 R(u(t)) v(t) dt = \int_0^1 (f(t) - u(t)) v(t) dt = 0$$

$$\forall v \in V_0^{(q)}$$

The space of test functions
+
The space of solution = trial function

OBS! $u(t) = \sum_{j=0}^q \xi_j t^j \Rightarrow \dot{u}(t) = \sum_{j=1}^q j \xi_j t^{j-1}$ & $v(t) \in \text{span}[t, t^2, \dots, t^q]$

$$(*) \Leftrightarrow \int_0^1 \left(\sum_{j=1}^q j \xi_j t^{j-1} - \sum_{j=1}^q \xi_j t^j - \xi_0 \right) t^i dt = 0 \quad i = 1, 2, \dots, q$$

$$\Leftrightarrow \sum_{j=1}^q \int_0^1 (j t^{i+j-1} - t^{i+j}) \xi_j dt = \int_0^1 \lambda \xi_0 t^i dt \quad i = 1, 2, \dots, q$$

$\downarrow \quad \xi_0 \text{ known}$

$$\Leftrightarrow \sum_{j=1}^q \left(\frac{(j t^{i+j-1})|_0^1}{i+j} - \frac{\lambda t^{i+j}|_0^1}{i+j+1} \right) \xi_j = \lambda \xi_0 \frac{t^{i+1}|_0^1}{i+1}$$

$$\sum_{j=1}^q \left(\frac{1}{i+j} - \frac{1}{i+j+1} \right) \xi_j = \lambda \xi_0 \frac{1}{i+1}, \quad i = 1, 2, \dots, q \quad \Leftrightarrow A\xi = b$$

$\underbrace{a_{ij}}_{\text{a}_{ij}}$

$$A = (a_{ij})_{i,j=1}^q, \quad \xi = (\xi_1, \xi_2, \dots, \xi_q)^T, \quad \text{No } (b_i)_{i=1}^q = \frac{\lambda \xi_0}{i+1}$$

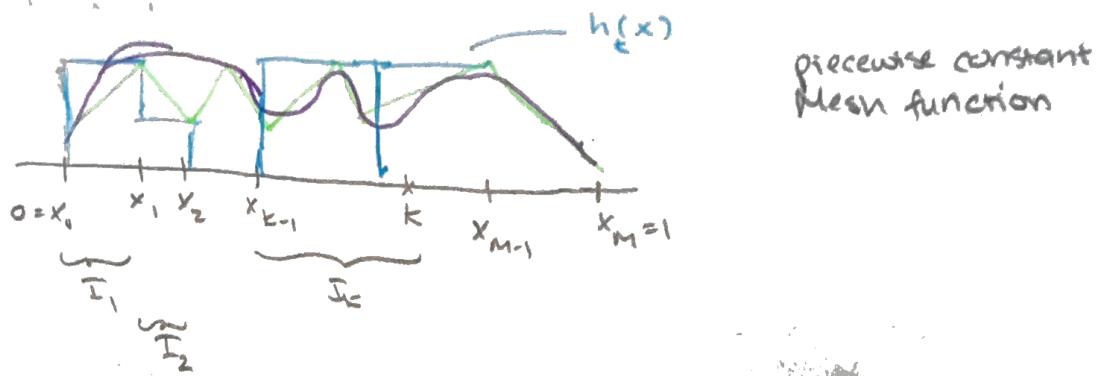
OBS! A 's invertible, however ill-conditioned. if q is large then the last q -ous (columns) are close to 0.

The problem is that monomials are NOT orthogonal.



Remedy is to construct basic functions that are almost orthogonal

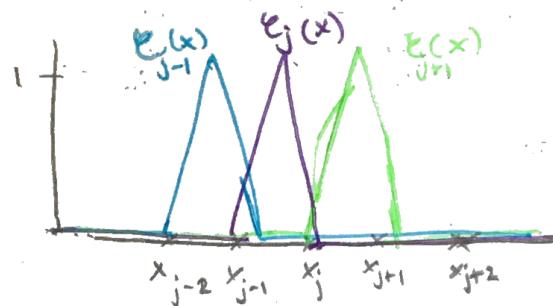
Let $\mathcal{T}_h: 0 = x_0 \leq x_1 \leq \dots \leq x_M = 1$ be a partition of $[0, 1]$
 then a piecewise linear function on \mathcal{T}_h looks like:



$$h_k = |I_k| = x_k - x_{k-1} \quad \text{Define } h(x) = h_k, \quad x \in I_k$$

The basic functions on \mathcal{T}_h

$$e_j(x) = \begin{cases} \frac{x - x_j}{h_j} & x_{j-1} \leq x \leq x_j \\ \frac{x_{j+1} - x_j}{h_{j+1}} & x_j \leq x \leq x_{j+1} \\ 0 & \text{else} \end{cases}$$



$$e_{j+1}(x) = \begin{cases} \frac{x - x_j}{h_{j+1}} & x_j \leq x \leq x_{j+1} \\ \frac{x_{j+2} - x}{h_{j+2}} & x_{j+1} \leq x \leq x_{j+2} \\ 0 & \text{else} \end{cases}$$

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(D) $\begin{cases} \dot{u}(t) = \lambda u(t) \Rightarrow u(t) = u_0 e^{\lambda t}, \\ u(0) = u_0. \end{cases}$ NOT GOOD WHEN $\lambda \uparrow$.
 (continuous solution)

Approximate by polynomial of order q : Take $u(t) = \xi_0 + \xi_1 t + \dots + \xi_q t^q$

and decide $\xi_1, \xi_2, \dots, \xi_q$ ($\xi_0 = u_0$ given)

$$\Rightarrow \dots \Rightarrow u \text{ in (D)} \Rightarrow \dots \Rightarrow A\xi = b, \quad A = (\alpha_{ij})_{i,j=1}^q$$

$\Rightarrow A$ is invertible but ill-conditioned

$$\begin{cases} \alpha_{ij} = \frac{1}{i+j} \cdot \frac{1}{1+j+1} \\ b_i = \frac{\lambda \xi_0}{i+1} \end{cases}$$

Galerkin Method:

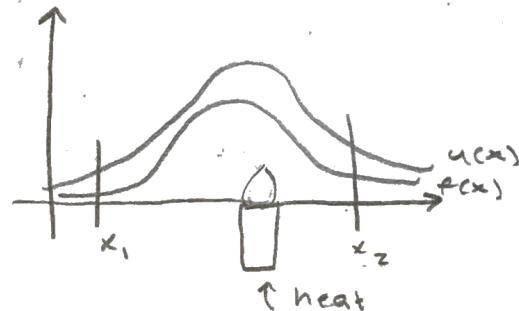
Ex 1: stationary heat equation

$u(x)$ is a temperature (independent of t), $x \in (0,1)$.

$q(x)$ is a heat flux

$f(x)$ is a source term

$\alpha(x)$ is the heat conductivity



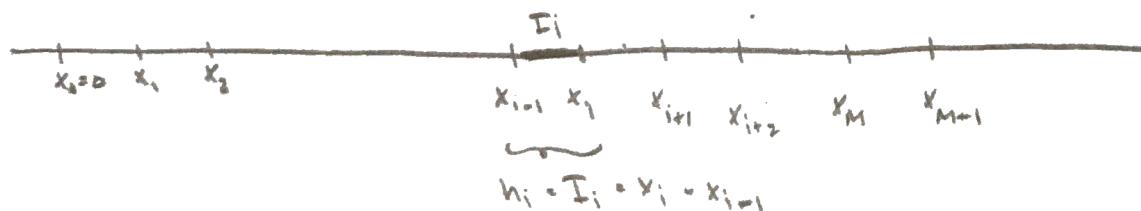
The heat through endpoints x_1 and x_2

$$\Leftrightarrow q(x_2) - q(x_1) = \int_{x_1}^{x_2} f(x) dx \Rightarrow f(x) = q'(x) \quad (1)$$

Fourier's law: $q(x) = -\alpha(x) \cdot u'(x) \quad (2)$

$$(2) \Rightarrow (1): \begin{cases} (-\alpha(x) \cdot u'(x))' = f(x) & 0 < x < 1 \\ u(0) \cdot u(1) = 0 \end{cases} \quad (DE) \quad (BC)$$

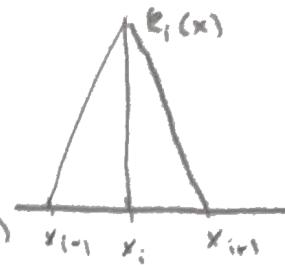
$T_h = \{0 = x_0 < x_1 < \dots < x_M < x_{M+1} = 1\}$ is a partition of $[0,1]$



Let $V_h = \{v : v \text{ continuous, piecewise linear on } T_h \text{ and } v(0) = v(1) = 0\}$

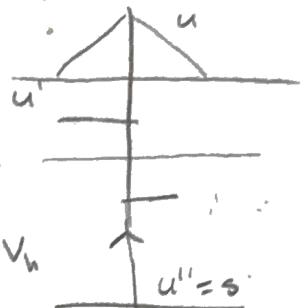
Define the base functions $\{\epsilon_i\}$ of V_h .

$$h(x) = h_i(x), \quad x \in I_i$$



Galerkin Method: Find $u_h \in V_h$ s.t. (take $a=1$) $\int_0^1 (-u'' - f)v dx = 0 \quad \forall v \in V_h$, doesn't work (if $v \in V_h \Rightarrow u'' \neq 0$)

$$\text{But } \int_0^1 -u'' v dx = + \left[u' v - [u(x)v(x)] \right]_{x=0}^{x=1}$$



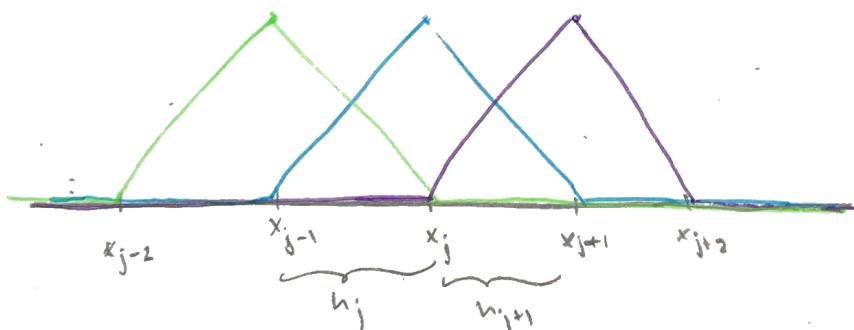
So the finite element is:

Finite element Method (FEM): $\boxed{\int_0^1 u_h' v' dx = \int_0^1 f v dx \quad \forall v \in V_h}$

$\Rightarrow u_h(x) = \sum_{j=1}^M \epsilon_j(x)$ $\{\epsilon_j(x)\}_{j=1}^M$ are the basis function for V_h

$$\Rightarrow \int_0^1 u_h(x) v'(x) dx = \int_0^1 f(x) v(x) dx \quad \forall x \in V_h \Leftrightarrow \int_0^1 \left(\sum_{j=1}^M \epsilon_j(x) \epsilon'_j(x) \right) dx = \int_0^1 f(x) \epsilon'(x) dx \quad \text{if } i \leq M \Leftrightarrow \sum_{j=1}^M \delta_{ij} = 10, \quad S = \{ \delta_{ij} \}_{i,j=1}^M, \quad \delta_{ij} = \int \epsilon'_i \epsilon_j$$

$$10 = (b_i)_{i=1}^M; \quad b_i = \int_0^1 \epsilon_i(x) f(x) dx$$



$$\delta_{ij} ? \quad \text{Note: } \delta_{ij} = 0 \text{ if } |i-j| > 1$$

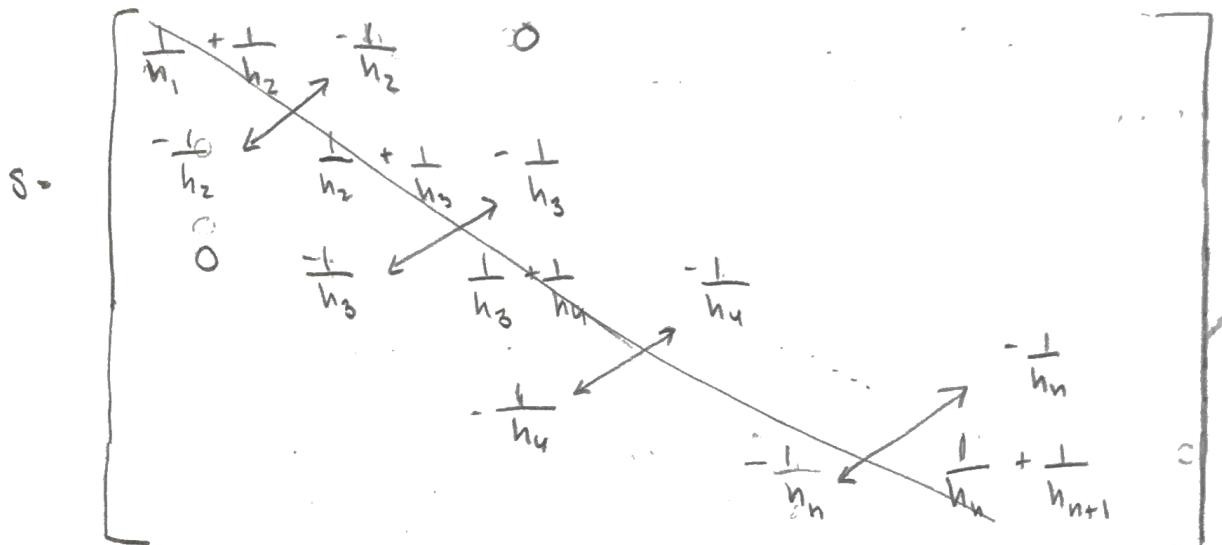
Obs! $\epsilon_i(x) = \begin{cases} \frac{x-x_{i-1}}{h_i} & x_{i-1} \leq x \leq x_i \\ x_{i+1}-x & x_i \leq x \leq x_{i+1} \end{cases}$

$$\epsilon_{j+1}(x) = \begin{cases} \frac{x-x_j}{h_{j+1}} & x_j \leq x \leq x_{j+1} \\ \frac{x_{j+2}-x}{h_{j+2}} & x_{j+1} \leq x \leq x_{j+2} \end{cases} \quad \text{else.}$$

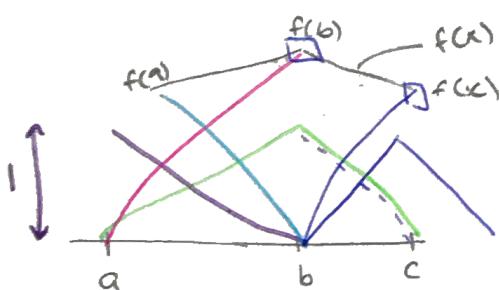
$$\boxed{\delta_{ij}} = \int_0^1 e_i'(x) e_j'(x) dx = \int_{x_{j-1}}^{x_j} \left(\frac{1}{h_j}\right)^2 dx + \int_{x_j}^{x_{j+1}} \left(\frac{1}{h_{j+1}}\right)^2 dx = \frac{1}{h_j} + \frac{1}{h_{j+1}}$$

$$\delta_{j,j+1} = \int_{x_j}^{x_{j+1}} \left(\frac{-1}{h_{j+1}}\right) \left(\frac{1}{h_{j+1}}\right) dx \Rightarrow \boxed{\delta_{j,j+1} = -\frac{1}{h_{j+1}}}$$

$\Rightarrow \delta_{j+1,j} = \delta_{j,i(i)}$



$$\Rightarrow \text{Sumform} = \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \vdots & & & & \ddots & \vdots \end{bmatrix} \quad h = h_i \forall i$$



$$\lambda_a(x) = \frac{b-x}{b-a}$$

Stiffness matrix

$$\lambda_b(x) = \begin{cases} \frac{x-a}{b-a} & a \leq x \leq b \\ 0 & b < x \leq c \end{cases}$$

S is positive definite

S is almost \perp , tridiagonal, symmetric.

$$S = \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \vdots & & & & \ddots & \vdots \end{bmatrix}$$

Def: A matrix S is p.d. if for $\eta^T \in \mathbb{R}^M$

$$\eta^T S \eta > 0 \Leftrightarrow \sum_{i,j=1}^M \eta_i \delta_{ij} \eta_j > 0$$

$\Leftrightarrow S^{-1} \exists$ (S is invertible)

$\Leftrightarrow S^{-1} = 1B$ has a unique solution $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_M)^T$

Let $S = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $u = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow u^T S u = (x, y) \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}^T = (x, y) \begin{pmatrix} 2x - y \\ -x + 2y \end{pmatrix}$

$$= x^2 + y^2 + (x - y)^2 \geq 0 \quad (= 0 \text{ only if } x = y = 0)$$

Why invertible?

Suppose $Sx = \emptyset \Rightarrow x^T S x = \emptyset \stackrel{S}{\Rightarrow} x \in \emptyset$
pos. definite

$$\Rightarrow N(S) = \{\emptyset\}$$

$\Rightarrow S$ has full range

$\Rightarrow S$ is invertible

Föreläsning 23/1:

Conclusion so far:

(i) we need to approximate functions by polynomials agreeing with them at certain (partition) points: Interpolation

(ii) We need to integrate/approx. integrals over subdomains & then sum:

Gauss quadrature

(iii) We need to solve linear system of equations: $\begin{cases} \text{Gauss-elimination} \\ \text{Gauss-Jacobi} \\ \text{Gauss-Seidel} \end{cases}$ (\leftarrow S-O-R)

(i)-(iii) are approx. procedure for dynamical equations (DE).

Dirichlet problem: We study the equation: $\begin{cases} -(a(x)u'(x))' = f(x) & 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$ (BVP)

Assume that $a(x) > 0$ & $a: \text{const.}$

Let $H_0^1 = \{v: \int_0^1 (v^2(x) + v'^2(x)) dx < \infty\}$. Then the (VF) for (BVP) is:

Find $u \in H_0^1$ s.t.

$$(\text{VF}) \quad \int_0^1 a(x) u'(x) v'(x) dx = \int_0^1 f(x) v(x) dx \quad \forall v \in H_0^1$$

(BVP) \Leftrightarrow (VF)

$$\begin{aligned} \text{PF: } & \Rightarrow [\text{PI}] \Rightarrow \int_0^1 (-(a(x)u'(x))' v(x) dx \stackrel{?}{=} \int_0^1 [a(x)u'(x)v'(x) - [a(x)u'(x)]v(x)]' \\ & \Leftrightarrow \int_0^1 a(x)u'(x)v'(x) dx = \int_0^1 f(x)v(x) dx: \text{VF} \end{aligned}$$

$\Leftarrow \text{VF} \Rightarrow \text{BVP}$

$$\int_0^1 a(x)u'(x)v'(x) dx = \int_0^1 f(x)v(x) dx \stackrel{\text{PI}}{\Rightarrow} - \int_0^1 [(a(x)u'(x))'v(x) + [a(x)u'(x)]v'(x)] dx$$

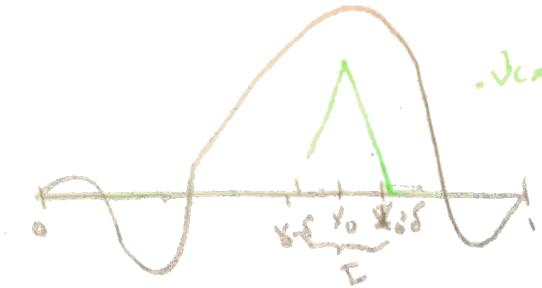
$$= \int_0^1 f(x)v(x) dx \Leftrightarrow - \int_0^1 (a(u(x))' + f(x))v(x) dx = 0 \quad \forall v \in H_0^1 \quad (\Leftrightarrow)$$

Claim $\Leftrightarrow - (a(x)u'(x))' + f(x) = 0$

Assume w-log
 > 0

Suppose not: Then $\exists x_0 \in I = (0, 1)$ such that $a(x_0)u'(x_0) + f(x_0) \neq 0$

continuity: $\Rightarrow \exists \delta > 0: g(x) = (a(x)u'(x))' + f(x) > 0$ for $x \in I_\delta = (x_0 - \delta, x_0 + \delta) \subset I$



Let \hat{v} be a "hat"-function on I_f .

OBS! that $\hat{v} \in H_0^1 \Rightarrow \int g(u)\hat{v}(x) dx > 0$

An equivalent minimization problem (MF):

Find $u \in H_0^1(0,1)$ such that

$$(MP) \quad F(u) \leq F(w) \quad \forall w \in H_0^1 \quad \text{with} \quad F(w) = \frac{1}{2} \underbrace{\int_0^1 a(x) w'(x)^2 dx}_{\text{Internal energy}} - \underbrace{\int_0^1 f(x) w(x) dx}_{\text{Load potential}}$$

shall show: $V(F) \Leftarrow (MP)$

Claim $(NF) \Leftarrow (MP)$

Pf: \Rightarrow Let $w \in H_0^1$, set $v = w - u \in H_0^1$

$$\begin{aligned} F(w) = F(u+v) &= \frac{1}{2} \int_0^1 a(x) (u'(x) + v'(x))^2 dx - \int_0^1 f(x) (u(x) + v(x)) dx \\ &= \frac{1}{2} \underbrace{\int_0^1 a(x) (u'(x))^2 dx}_{= F(u)} + \frac{1}{2} \underbrace{\int_0^1 a(x) (v'(x))^2 dx}_{\geq 0} + \boxed{\int_0^1 a(x) u'(x) v'(x) dx - \int_0^1 f(x) v(x) dx} \\ &\quad - \underbrace{\int_0^1 f(x) u(x) dx}_{= 0} \\ &= F(u) + \frac{1}{2} \underbrace{\int_0^1 a(x) v'(x)^2 dx}_{\geq 0} \geq F(u) \end{aligned}$$

\Leftarrow Assume $F(u) \leq F(w) \quad \forall w \in H_0^1 \quad \& \quad g(\varepsilon, w) = F(u + \varepsilon v)$

$g(\varepsilon)$ has a minimum at $\varepsilon = 0$ i.e. $\boxed{g'(\varepsilon)|_{\varepsilon=0} = 0}$

$$\begin{aligned} g(\varepsilon, w) = F(u + \varepsilon v) &= \frac{1}{2} \int_0^1 a(x) (u'(x) + \varepsilon v'(x))^2 dx - \int_0^1 f(x) (u(x) + \varepsilon v(x)) dx \\ &= \frac{1}{2} \int_0^1 a(x) u'(x)^2 dx + \frac{1}{2} \int_0^1 a(x) \varepsilon^2 v'(x)^2 dx + \int_0^1 a(x) \varepsilon u'(x) v'(x) dx - \int_0^1 f(x) u(x) dx \\ &\quad - \varepsilon \int_0^1 f(x) v(x) dx \Rightarrow g'_\varepsilon|_{\varepsilon=0} = \int_0^1 a(x) \varepsilon v'(x)^2 dx + \int_0^1 a(x) u'(x) v'(x) dx - \int_0^1 f(x) v(x) dx \\ &= 0 \Rightarrow (NF) \end{aligned}$$

Poincaré inequality in 1D:

If $u(0) = u(L) = 0$, Then $\int_0^L |u'(x)|^2 dx \leq c \int_0^L |u(x)|^2 dx$

Pf: $u(x) = \int_0^x u'(y) dy \leq \int_0^x |u'(y)| dy \leq \int_0^L |u'(y)| dy \leq \{ \text{Cauchy-Schwarz} \}$

$$\leq \left(\int_0^L 1^2 dy \right)^{1/2} \cdot \left(\int_0^L |u'(y)|^2 dy \right)^{1/2} = \left(\int_0^L |u'(y)|^2 dy \right)^{1/2}$$

$$\Rightarrow \int_0^L |u'(x)|^2 dx \leq \int_0^L \left(\int_0^L |u'(y)|^2 dy \right) dx = \int_0^L \int_0^L |u'(y)|^2 dy dx$$

$$\Rightarrow \|u'\|_{L_2(0,L)} \leq L \|u'\|_{L_2(0,L)} \quad \text{important to have a bounded interval! this inequality is useless otherwise.}$$

Power of abstraction:

(V) Find $u \in X_0'$ such that

$$\boxed{\begin{array}{c} \int u' v' \\ \text{bilinear form} \\ (u, v) = l(v) \end{array}} \quad \forall v \in X_0$$

(M) Find $u \in X_0'$ such that

$$F(u) = \min_{v \in X_0'} F(v), \quad F(v) = \frac{1}{2} \|v\|^2 - \ell(v)$$

Want to show that $\exists!$ solution for $\forall k \in M \quad (V \Leftrightarrow M)$

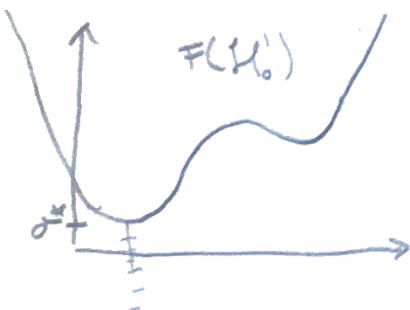
Pf:

1st note that $\exists \sigma: F(v) > \sigma \quad \forall v \in X_0'$ (otherwise it is not possible to minimize)

$$\text{Namely: } F(v) = \frac{1}{2} \|v\|^2 - \ell(v) \geq \frac{1}{2} \|v\|^2 - \gamma \|v\|$$

$$\uparrow \text{ obs } \ell(v) \leq |\ell(v)| \leq \gamma \|v\| \Rightarrow -\ell(v) \geq -\gamma \|v\|$$

$$\text{But } \frac{1}{2} \|M\|^2 - \gamma \|M\| \geq \frac{1}{2} \gamma^2 \Leftrightarrow \frac{1}{2} (a^2 + b^2) \geq ab$$

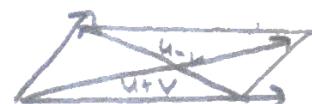


Let σ^* be the largest real number s.t.

$$\textcircled{1} \quad F(v) > \sigma^* \quad \forall v \in X_0'$$

Take the sequence $\{u_n\} \subset X_0'$ s.t.

$$\textcircled{2} \quad F(u_n) \rightarrow \sigma^* \quad (\text{Completeness of } \mathbb{R} \text{, Axiom of Choice or Zorn's Lemma})$$



Recall parallelogram law:

$$\textcircled{3} \quad \|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

$$\begin{aligned}
 ③ \rightarrow \|u_k - u_j\|^2 &= 2\|u_k\|^2 + 2\|u_j\|^2 - \|u_k + u_j\|^2 - \cancel{4\lambda(u_k) - 4\lambda(u_j)} \\
 + 4\lambda(u_k + u_j) &= 4F(u_k) + F(u_j) - 8F\left(\frac{u_k + u_j}{2}\right) \\
 &\quad - 8\left[\underbrace{\frac{1}{2}\|u_k + u_j\|^2}_{\sigma^*} - \lambda\left(\frac{u_k + u_j}{2}\right)\right]
 \end{aligned}$$

$$\leq 4F(u_k) + 4F(u_j) - 8\sigma^* \xrightarrow{k_{ij} \rightarrow \infty} 0$$

$$\Rightarrow \|u_k - u_j\|^2 \rightarrow 0 \text{ as } k_{ij} \rightarrow \infty$$

$\because \{u_k\}$ is a Cauchy sequence $\Rightarrow \exists u \in H_0 : u_k \rightarrow u \text{ as } k \rightarrow \infty$

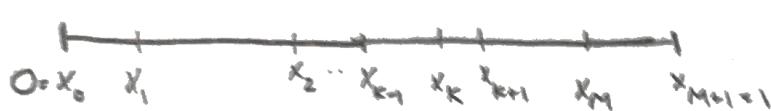
$$\begin{aligned}
 F \Rightarrow F(u_k) &\rightarrow F(u) \text{ as } k \rightarrow \infty \\
 F \text{ cont}
 \end{aligned}$$

$$\Rightarrow \underline{F(u) = \sigma^*} < F_v \quad \forall v \in H_0$$

Föreläsning 27/1:

Polynomial approximation: INTERPOLATION

Def: Assume that $\mathcal{P}_h := \{0 = x_0 < x_1 < \dots < x_M < x_{M+1} = 1\}$ be a partition of $[0,1]$ into $M+1$ subintervals $I_k := (x_{k-1}, x_k)$ $k=1, \dots, M+1$



with a mesh function
 $h(x) = h_k = x_k - x_{k-1}$ if $x \in I_k$

Then $\Pi_h f(x)$ is said to be an interpolant of $f(x)$ defined on I
 If $\Pi_h f(x_i) = f(x_i)$, $i=0, \dots, M+1$ & $\Pi_h f(x)$ is a polynomial of degree $M+1$



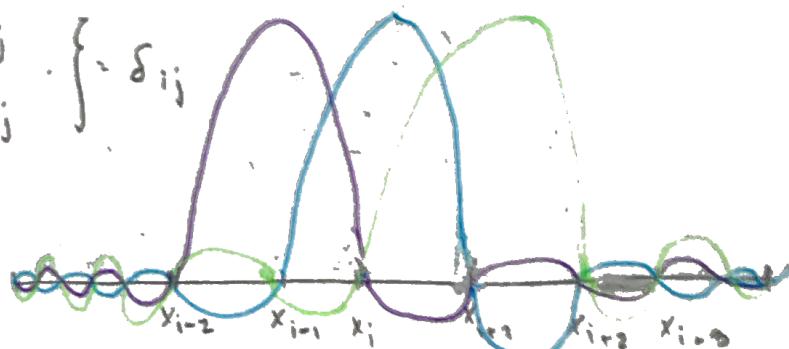
Consider vectorspace of polynomials on (a,b) : Basis for $P(a,b); 1, x, \dots, x^q$

Lagrange basis. (cardinal functions): For a partition $a = x_0 < x_1 < \dots < x_q = b$ of $[a,b]$ we define:

$$\ell_i(x) = \frac{(x-x_0)(x-x_1) \dots (x-x_{i-1})(x-x_{i+1}) \dots (x-x_{q-1})(x-x_q)}{(x_i-x_0)(x_i-x_1) \dots (x_i-x_{i-1})(x_i-x_{i+1}) \dots (x_i-x_{q-1})(x_i-x_q)} =$$

$$= \prod_{\substack{j=0 \\ j \neq i}}^q \frac{x-x_j}{(x_i-x_j)}$$

obs! $\ell_i(x_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \quad \sum_j \delta_{ij}$



Taylor about $x = \bar{x}$

$$f(x) = f(\bar{x}) + (x-\bar{x})f'(\bar{x}) + \frac{1}{2}(x-\bar{x})^2 f''(\bar{x}) + \dots + \frac{1}{q!}(x-\bar{x})^q f^{(q)}(\bar{x})$$

$$\boxed{\frac{1}{(q+1)!} (x-\bar{x})^{q+1} f^{(q+1)}(\bar{x})} := T_q f(\bar{x}) + R_q(f)$$

$\bar{x} \in I_{x,\bar{x}}$

Lagrange Interpolation: $\pi_q f(x) = \sum_{i=0}^q f(x_i) L_i(x)$

Interpolation error:

Taylor: $|f(x) - \pi_q f(x)| \leq \frac{1}{(q+1)!} |(x-\bar{x})^{q+1}| \max_{\bar{x} \in [a,b]} |f^{(q+1)}(\bar{x})|$

is accurate of degree $q+1$ near $x=\bar{x}$

Lagrange: $|f(x) - \pi_q f(x)| \leq \frac{1}{(q+1)!} \left| \prod_{i=0}^q (x-x_i) \right| \max_{a \leq x \leq b} |f^{(q+1)}(x)|$

is accurate of degree 1 at $x=x_0, x=x_1, \dots, x=x_q$

Need to prove this

Proof of *: We have $f(x_i) = \pi_q f(x_i)$ for $i = 0, 1, \dots, q \Rightarrow$

$$f(x) - \pi_q f(x) = (x-x_0)(x-x_1) \dots (x-x_q) g(x)$$

Need to find $g(x)$

Def: $e(t) := f(t) - \pi_q f(t) = (t-x_0)(t-x_1) \dots (t-x_q) g(x)$

t not! x

x not! t

Then $e(x_0) = e(x_1) = \dots = e(x_q) = e(x) = 0$

Thus e has $q+2$ zeros in interval $[a,b]$

Generalized Rolles theorem $\Rightarrow \exists \bar{x} \in [a,b]$ such that

$$e^{(q+1)}(\bar{x}) = 0 \quad \text{derivative} \quad \Rightarrow e^{(q+1)}(t) = f^{(q+1)}(t) \cdot (q+1)! g(x)$$

[because of $\deg[\pi_q f] = q$ & $(t-x_0) \dots (t-x_q) = t^{q+1} + \text{lower terms}$]

$$e^{(q+1)}(t) = 0 \Rightarrow f^{(q+1)}(t) - (q+1)! g(x) = 0 \quad **$$

$$\Rightarrow g(x) = \frac{f^{(q+1)}(t)}{(q+1)!} \quad \blacksquare$$



2 zeros

3 zeros

Thm 1: There are interpolation constants C_i , independent of the function f and the interval $[a, b]$ such that

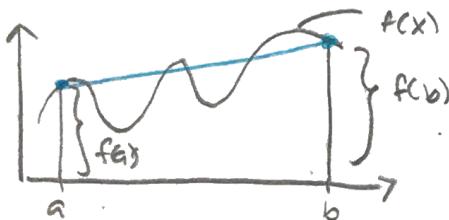
$$(1) \|\pi_1 f - f\|_{L^p([a,b])} \leq C_1 (b-a)^2 \|f''\|_{L^p([a,b])}$$

$$(2) \|\pi_1 f - f\|_{L^\infty([a,b])} \leq C_2 (b-a) \|f'\|_{L^\infty([a,b])},$$

$$(3) \|(\pi_1 f)' - f'\|_{L^p([a,b])} \leq C_3 (b-a) \|f''\|_{L^p([a,b])},$$

Shall give the proof for (1) only for $p = \infty$

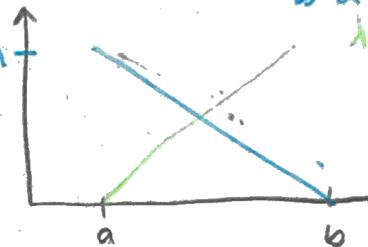
Proof: We consider a single interval $I = [a, b]$



$$\begin{aligned} \pi_1 f(x) \\ \text{basis} \\ \text{fcts} \end{aligned}$$

$$\lambda_a(x) = \frac{b-x}{b-a}$$

$$\lambda_b(x) = \frac{x-a}{b-a}$$



Every linear function on $[a, b]$ can be written as a linear combination of $\lambda_a(x)$ & $\lambda_b(x)$

$$\text{OBS! } \begin{cases} \lambda_a(x) + \lambda_b(x) = 1 \\ a\lambda_a(x) + b\lambda_b(x) = x \end{cases} \parallel \begin{array}{l} \text{other basis of} \\ \text{degree } 1, x \\ x, 1-x \text{ in } (0,1) \end{array}$$

$$\pi_1 f(x) = f(a) \lambda_a(x) + f(b) \lambda_b(x)$$

Now Taylor expand $f(a)$ & $f(b)$ about x :

$$[f(a) = f(x) + (a-x)f'(x) + \frac{1}{2}(a-x)^2 f''(\zeta_a)], \quad a \leq \zeta_a \leq x$$

$$[f(b) = f(x) + (b-x)f'(x) + \frac{1}{2}(b-x)^2 f''(\zeta_b)], \quad x \leq \zeta_b \leq b$$

$$\begin{aligned} f(a)\lambda_a(x) + f(b)\lambda_b(x) &= (\underbrace{\lambda_a(x) + \lambda_b(x)}_1) f(x) + (\underbrace{(a\lambda_a(x) + b\lambda_b(x))}_x) f'(x) - x(\underbrace{(\lambda_a(x) + \lambda_b(x))}_{f'(x)}) \\ &+ \frac{\lambda_a(x)}{2}(a-x)^2 f''(\zeta_a) + \frac{\lambda_b(x)}{2}(b-x)^2 f''(\zeta_b) \end{aligned}$$

OBS! $C_1 = 1$
The optimal $C_1 = \frac{1}{2}$ is an exercise

$$\Rightarrow \pi_1 f(x) - f(x) = \frac{\lambda_a(x)}{2}(a-x)^2 f''(\zeta_a) + \frac{\lambda_b(x)}{2}(b-x)^2 f''(\zeta_b)$$

$$\Rightarrow |\pi_1 f(x) - f(x)| \leq (b-a)^2 \max_{x \in [a,b]} |f''(x)| \Rightarrow \|\pi_1 f - f\|_{L^\infty([a,b])} \leq (b-a)^2 \|f''\|_{L^\infty([a,b])} \quad \square$$

Have also the following theorem, but skip the proof.

Thm 2: Let $\Omega = x_0 \cup x_1 \cup \dots \cup x_{M+1} = I$ be a partition of $[a, b]$

Let $\Pi_h v$ be piecewise linear interpolant of $v(x)$. Then there exists a constant C_1 :

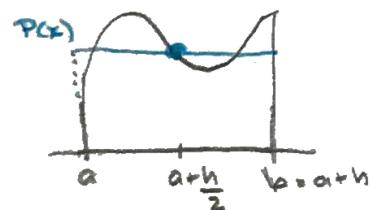
$$(i) \quad \|\Pi_h v - v\|_{L^p} \leq \tilde{C}_1 \|h^2 v''\|_{L^p}, \quad p \in \{1, 2, \infty\}$$

$$(ii) \quad \|\Pi_h v - v\|_{L^p} \leq \tilde{C}_2 \|hv'\|_{L^p}, \quad p = 1, 2, \infty$$

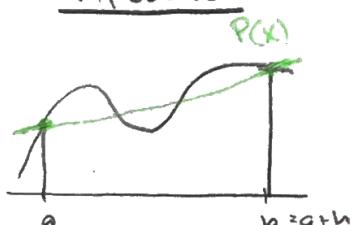
$$(iii) \quad \|\Pi_h v - v\|_{L^p} \leq \tilde{C}_3 \|hv''\|_{L^p}, \quad p = 1, 2, \infty$$

Numerical Integration Method:

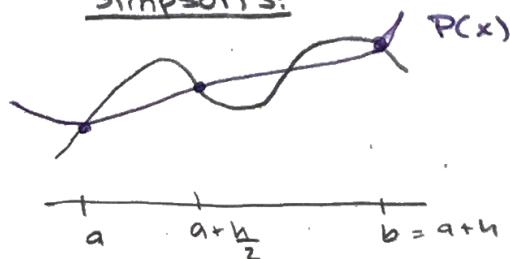
Midpoint:



Trapezoidal:



Simpson's:



All use the values of the function $f(x)$ at equally spaced points (uniform partition)

Gauss quadratic rule: Choose the points of evaluation in an optimal manner.

\Rightarrow not equally spaced

Problem: choose nodes $x_i \in [a, b]$ and the coefficients c_i , $1 \leq i \leq n$, minimizing the error of integration

$$E(f) := \int_a^b f(x) dx - \sum_{i=1}^n c_i f(x_i) \quad \text{for an arbitrary } f$$

We have $2n$ parameters produce quadrature rule which is exact (error is "0") for polynomials of degree $2n-1$.

Ex: $n=2$, $[a, b] = [-1, 1]$

$$\int_{-1}^1 f(x) dx \approx c_1 f(x_1) + c_2 f(x_2) *$$

gives exact approx for polynomials of degree ≤ 3

The basis for $P^3(-1,1)$: $1, x, x^2, x^3$

$$f(x)=1, \text{ equality in } \Rightarrow c_1 + c_2 = \int_{-1}^1 1 dx = 2$$

$$f(x)=x, \quad \Rightarrow c_1 x + c_2 x^2 \int_{-1}^1 x dx = 0$$

$$f(x)=x^2, \quad \Rightarrow c_1 x_1^2 + c_2 x_2^2 = \int_{-1}^1 x^2 dx = \frac{2}{3}$$

$$f(x)=x^3, \quad \Rightarrow c_1 x_1^3 + c_2 x_2^3 = \int_{-1}^1 x^3 dx = 0$$

4 eqns with 4
unknowns,
But non-linear

But solvable
 $c_1, c_2 \neq 0$

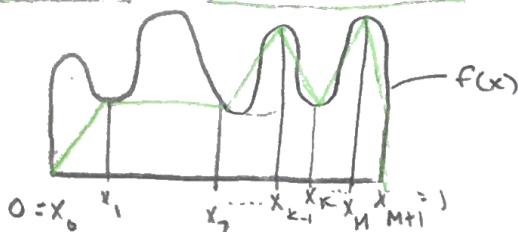
$$x_1 = -\frac{1}{\sqrt{2}}, \quad x_2 = +\frac{1}{\sqrt{2}}$$

Föreläsning 29/1:

Interpolation: Given a function $f(x)$, we approximate $f(x)$ by piecewise polynomials "agreeing with f " in certain partition points

- The partition function "mesh" (here denoted with Π_h) & the Interpolant

$$\Pi_h f(x) : \Pi_h f(x_k) = f(x_k) \text{ for } x_k \in \Pi_h = \{x_k\}_{k=0}^{M+1}$$



Quadrature: Approx integrals by finite sum over a partition:

$$(\ast\ast) \int_a^b f(x) dx \approx \sum_{i=1}^n c_i f(x_i)$$

(**) is exact for polynomials (f being a polynomial) of degree $\leq 2n-1$

Last time:

Example: $\int_{-1}^1 f(x) dx \approx f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}})$ {if f is a polynomial of degree 3}

$$\text{i.e. } c_1 = c_2 = 1, \quad x_1 = -\frac{1}{\sqrt{3}}, \quad x_2 = \frac{1}{\sqrt{3}}$$

To generalize: we introduce Legendre polynomials $\{P_n\}_{n=0}^{\infty}$:

i) For each n , P_n is a polynomial of degree n .

ii) $P_n \perp P$ ($\forall P$, P : Polynomial of degree $< n$)

The first few Legendre polynomials:

(Obs! (ii) $\Leftrightarrow \int_{-1}^1 P_n(x) P(x) dx = 0$, $\deg P(x) < n$)

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = x^2 - \frac{1}{3}, \quad P_3(x) = x^3 - \frac{3}{5}x, \quad \dots$$

What are the optimal choice of x_i & c_i in (**) for $(a, b) = (-1, 1)$?

The root of Legendre polynomials ($P_0(x) \equiv 1$ is an exception) one distinct, symmetric and the correct choice do quadrature points.

If $\{x_i\}_{i=1}^n$ are the roots of polynomial $P_n(x)$, then $\int_{-1}^1 f(x) dx \approx \sum_{i=1}^n c_i f(x_i)$

Thm: Let x_i , $i=1 \dots n$, be the roots to the n -th Legendre polynomial $P_n(x)$ and

$$c_i = \int_{-1}^1 \prod_{j \neq i}^{n-1} \frac{(x-x_j)}{(x_i-x_j)} dx = \underbrace{\int_{-1}^1 e_i(x) dx}_{\text{Lagrange interpolation polynomials}}$$

Then if $p(x)$ is a polynomial of degree $\leq 2n$

$$\text{Then } \int_{-1}^1 p(x) dx = \sum_{i=1}^n c_i p(x_i)$$

Pf: Let $R(x)$ be a polynomial of degree $\leq n$

Rewrite $R(x)$ as $(n-1)^{\text{st}}$ Lagrange polynomials (i.e. $e_i(x)$) with the nodes at the roots of the n -th Legendre polynomial P_n

This representation is exact:

$$\text{Error: } = \frac{1}{n!} \underbrace{\int_{-1}^1}_{\downarrow} \underbrace{w_n(x)}_{\prod_{i=1}^n (x-x_i)} \underbrace{\frac{R''(x)}{2!}}_{=0} = 0$$

$$\int_{-1}^1 R(x) dx = \int_{-1}^1 \left[\underbrace{\sum_{j=1}^n e_j(x)}_{e_i(x)} \right] R(x) dx \stackrel{R(x) = \sum_{i=1}^n c_i P_n(x_i)}{=} \sum_{i=1}^n \left[\int_{-1}^1 \prod_{j \neq i}^{n-1} \frac{(x-x_j)}{(x_i-x_j)} R(x_i) dx \right] R(x_i)$$

Now consider a polynomial $p(x)$ degree $\leq 2n$

Divide $p(x)$ by the n -th Legendre polynomial $P_n(x)$:

$$(I) \quad \underbrace{p(x)}_{\deg \leq 2n} = \underbrace{q(x)}_{\deg \leq n} \underbrace{P_n(x)}_{\deg = n} + \underbrace{R_n(x)}_{\deg \leq n}$$

$$(II) \quad \int_{-1}^1 q(x) P_n(x) dx = 0 \quad (P_n \perp \forall q \text{ of deg. } n)$$

$$(III) \quad p(x_i) = q(x_i) P_n(x_i) + R_n(x_i)$$

$$(I) - (III) \Rightarrow \int_{-1}^1 p(x) dx = \cancel{\int_{-1}^1 q(x) P_n(x) dx} + \int_{-1}^1 R_n(x) dx = \sum_{i=1}^n q(x_i) + R_n(x_i) = \sum_{i=1}^n c_i p(x_i)$$

Remember the L_2 -interpolation error: $\|u - (\pi_h u)\|_{L_2} \leq C_h \|u'\|_{L_2}$

Error estimates for FEM in 1D two points boundary problem

1. Dirichlet problems

$$(BVP) \begin{cases} -(a(x)u'(x))' = f(x), & 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$$

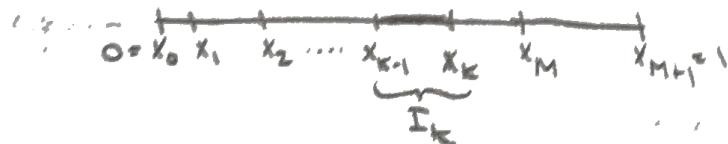
Assume that $a(x) > 0$, piecewise continuous and bounded in $(0,1)$

$$\text{Let } H_0^1 = \left\{ V : \int_0^1 V'(x)^2 dx < \infty, \quad V(0) = V(1) = 0 \right\}$$

The variational formulation: Find $u \in H_0^1$

$$(VF) \int_0^1 a(x)u'(x)V'(x) dx = \int_0^1 f(x)V(x) dx \quad \forall v \in H_0^1$$

The finite element method (FEM): Let $T_h := \{0 = x_0 < x_1, \dots < x_M < x_{M+1} = 1\}$ be a partition of $(0,1)$ into subintervals, $I_k := [x_{k-1}, x_k]$ & set $h_k := x_k - x_{k-1}$. We may define the piecewise constant function $h(x) = 1_{I_k}(x) = x_k - x_{k-1}$, if $x \in I_k$.



Let $\hat{V}_h := \{v : v(x) \text{ is continuous linear on each } I_k \text{ & } v(0) = v(1) = 0\} \subset H_0^1$

A FEM for (BVP) is: Find $u_h \in \hat{V}_h$ such that

$$(FEM) \int_0^1 a(x)u'(x)v'(x) dx = \int_0^1 f(x)v(x) dx \quad \forall v \in \hat{V}_h$$

If in (VF), we restrict v to $\hat{V}_h \subset H_0^1$, then (VF)-(FEM) \Rightarrow

$$\Rightarrow (G^\perp) \int_0^1 a(x)(u'(x) - u'_h(x))v'(x) dx = 0 \quad \text{Raviart-Thomas orthogonality.}$$

Measuring environments (Norms)

$$\|v\|_a = \left(\int_0^1 a(x)|v'(x)|^2 dx \right)^{1/2} \quad (\text{weighted } L_2 \text{ norm}) \quad \left. \begin{array}{l} \\\end{array} \right\} \|v\|_a = \|v'\|_a$$

$$\|v\|_E = \left(\int_0^1 a(x)|v'(x)|^2 dx \right)^{1/2} \quad (\text{energy norm})$$

Thm 1: Let u & u_h be solutions of (BVP) & (FEM) respectively.

$$\text{Then } \|u - u_h\|_E \leq \|u - v\|_E \quad \forall v \in \tilde{V}_h$$

Pf: $\|u - u_h\|_E^2 = \int_a^b a(u - u_h)^2 dx =$

i.e. FE solution is the best approx for solutions to BVP in V_h in Energynorm

$$= \int_a^b a(u - u_h)(u - v + v - u_h) dx = \{v \in \tilde{V}_h\} = \int_a^b a(u - u_h)(u - v) dx + \int_a^b a(u - u_h)(v - u_h) dx$$

$$\geq \int_a^b a(u - u_h)(u - v) dx = \int_a^b a''(u - u_h) a^{1/2}(u - v) dx \leq [c - c] \leq \left(\int_a^b a(x)(u'(x) - u'_h(x))^2 dx \right)^{1/2}$$

$$\cdot \left(\int_a^b a(u - v)^2 dx \right)^{1/2} = \cancel{\|u - u_h\|_E} \cdot \|u - v\|_E \Rightarrow \blacksquare$$

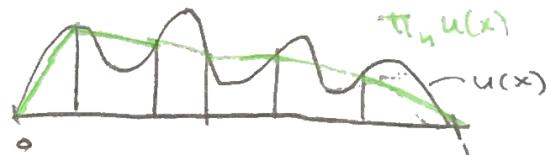
~~$\int_a^b a(u - u_h)(v - u_h) dx = 0$~~
Cauchy-Schwarz $\Rightarrow 0$

The next step is to show that $\exists v \in \tilde{V}_h: \|u - v\|_E$ is not "too large":

Here $(\pi_h u)$ is the candidate.

Thm 2: (a-priori error estimate for Dirichlet BVP)

u & u_h are solutions of (BVP) & (FEM) respectively. Then $\|u - u_h\|_E \leq C_1 \|h u''\|_a$
(obs! smaller $h \Rightarrow$ error)



Pf: Note that $(\pi_h u) \in \tilde{V}_h$

By the Thm 1:

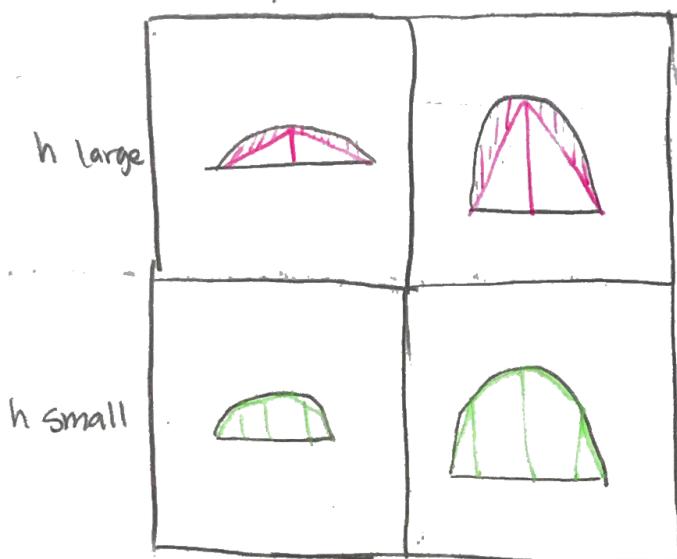
$$\begin{aligned} \|u - u_h\|_E &\leq \|u - \pi_h u\|_E \geq \|u' - (\pi_h u)'\|_a = \left(\int_a^b a(x)(u'(x) - (\pi_h u)'(x))^2 dx \right)^{1/2} \leq \\ &\leq \max_x a''(x) \underbrace{\left\{ \left(\int_a^b u'(x) - (\pi_h u)'(x) dx \right)^2 \right\}}_{L_2 \text{ norm of } u' - (\pi_h u')}^{1/2} \leq \max_x a(x)^{1/2} C_3 \left(\int_a^b h(x) u''(x)^2 dx \right)^{1/2} \end{aligned}$$

$$\leq C_3 \max_x a(x)^{1/2} \left(\int_a^b \frac{a(x)}{\min a(x)} (h(x) u''(x))^2 dx \right)^{1/2} = C_3 \frac{\max \sqrt{a(x)}}{\min \sqrt{a(x)}} \left(\int_a^b a(x) (h(x) u''(x))^2 dx \right)^{1/2}$$

$$= C_1 \|h u''\|_a \blacksquare$$

← This is "unknown known" i.e. knowing the unknown.

u'' small u'' large



This is the basis for the concepts of ADAPTIVITY

h large \leftrightarrow u'' small

h small \leftrightarrow u'' large

Föreläsning 30/1:

Dirichlet:

$$\begin{aligned} \text{(BVP)} \quad & \left\{ \begin{array}{l} (a(x)u'(x))' = f(x) \quad 0 < x < 1 \\ u(0) = u(1) = 0 \end{array} \right. \\ & \Downarrow \end{aligned}$$

$$\text{Find } u \in H_0^1 : \int_0^1 a u' v' dx = \int_0^1 f v dx \quad \forall v \in H_0^1 = \{ w : \int_0^1 (w')^2 dx < \infty \quad w(0) = w(1) = 0 \}$$

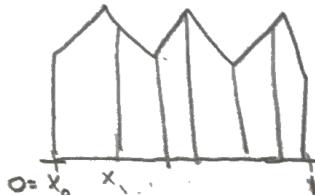
□

$$\text{Find } u \in H_0^1 : F(u) \leq F(v) \quad \forall v \in H_0^1 \quad F(w) = \frac{1}{2} \|w\|_a^2 - \int_0^1 f w$$

Find $u_h \in V_h$

$$\int_0^1 a(x) u'_h(x) v'(x) dx = \int_0^1 f(x) v(x) dx \quad \forall v \in V_h = \{ w : w \text{ piecewise continuous on } \mathcal{T}_h : w(0) = w(1) = 0 \}$$

OBS! $V_h \subset H_0^1$



$$\text{Thm 1+2: } \|u - u_h\|_E \leq C_1(a) \|h u'\|_a$$

$$\|u - u_h\|_E \leq \|u - v\|_E \quad \forall v \in V_h$$

a priori (theoretical error estimate)

$$\|w\|_a = \left(\int_0^1 |a(x) w(x)|^2 dx \right)^{1/2} \quad \& \quad \|w\|_E = \|w\|_a$$

▷ interpolation

Error estimate for the FEM in 1D.

Dirichlet problem: [The a posteriori approach (computational)]

Recall the definition of the residual [for our BVP $\underline{R(u_h(x)) = f(x) + (a(x)u'_h(x))'}$]

Thm 3: [A posteriori error estimate]:

$$\text{Let } e(x) = u(x) - u_h(x) \in H_0^1, \text{ then } \|e\|_E \leq C_1(a) \left(\int_0^1 \frac{1}{a(x)} h^2(x) |R(u_h(x))|^2 dx \right)^{1/2}$$



$$\text{Pf: } \|e\|_E^2 = \int_0^1 a(x) e'(x)^2 dx = \int_0^1 a(x) (u'(x) - u_h'(x)) e'(x) dx$$

$$= \int_0^1 a(x) u'(x) e'(x) dx - \int_0^1 a(x) u_h'(x) e'(x) dx \stackrel{\text{(VP)}}{=} \int_0^1 f(x) e(x) dx - \int_0^1 a(x) u_h'(x) e'(x) dx$$

$$= \left\{ \begin{array}{l} \text{add and subtract} \\ \dots \end{array} \right\} = \int_0^1 f(x) (e(x) - \pi_h e(x)) dx + \int_0^1 f(x) \underbrace{\pi_h e(x)}_{\checkmark} dx$$

$$- \int_0^1 a(x) u_h'(x) (e'(x) - (\pi_h e)'(x)) dx - \int_0^1 a(x) u_h'(x) (\pi_h e)'(x) dx$$

\checkmark

$$= 0 \quad (\text{FEM})$$

$$\Rightarrow \int_0^1 f(x) (e(x) - \pi_h e(x)) dx - \sum_{k=1}^{M+1} \int_{x_{k-1}}^{x_k} a(x) u_h'(x) (e'(x) - (\pi_h e)'(x)) dx$$

$$\text{Pf: 2nd term} \quad \int_0^1 f(x) \underbrace{(e(x) - (\pi_h e)(x))}_{=0} + \sum_{k=1}^{M+1} \int_{x_{k-1}}^{x_k} (a(x) u_h'(x))' (e(x) - \pi_h e(x)) -$$

$$\sum_{k=1}^{M+1} \underbrace{[a(x) u_h'(x) (e(x) - \pi_h e(x))]}_{=0} \Big|_{x_{k-1}}^{x_k} = \int_0^1 \underbrace{(f(x) + (a(x) u_h'(x))')}_{R(u_h(x))} (e(x) - \pi_h e(x))$$

$$= \int_0^1 \frac{1}{\sqrt{a(x)}} h(x) R(u_h(x)) \sqrt{a(x)} \frac{e(x) - \pi_h e(x)}{h(x)} dx \leq \{\text{Cauchy-Schwarz}\}$$

$$\leq \left(\int_0^1 \frac{1}{a(x)} h^2(x) R^2(u_h(x)) dx \right)^{1/2} \underbrace{\| \frac{e - \pi_h e}{h} \|_a}_{:= \gamma}$$

$$\gamma = \left(\int_0^1 a(x) \left(\frac{e(x) - \pi_h e(x)}{h(x)} \right)^2 dx \right)^{1/2} \leq \max_x \sqrt{a(x)} \left(\int_0^1 \left(\frac{e(x) - \pi_h e(x)}{h(x)} \right)^2 dx \right)^{1/2}$$

$$\leq \left\{ \begin{array}{l} \|e - \pi_h e\|_L \leq C_i \|h e'\|_{L_2(0,1)} \\ L_2(0,1) \end{array} \right\} \leq C_i \max_x \sqrt{a(x)} \|e'\|_{L_2(0,1)} \leq C_i \max_x \sqrt{a(x)}.$$

$$\cdot \left(\int_0^1 \frac{a(x)}{\min a(x)} e'(x)^2 dx \right)^{1/2} = C_i \underbrace{\frac{\max \sqrt{a(x)}}{\min \sqrt{a(x)}}}_{C_i \|e'\|_E} \left(\int_0^1 a(x) e'^2(x) dx \right)^{1/2}$$

$$\Rightarrow \|e\|_E^2 \stackrel{(\star\star\star)}{=} \widehat{C}_i \left\{ \int_0^1 \frac{1}{a(x)} (h^2(x) R(u_h)^2(x)) dx \right\}^{1/2} \|h\|_E \Rightarrow \square$$

Obs! as a remark that $(\star\star\star)$ can be used for mesh-refinement

Adaptivity steps:

Assume that we seek an error bound $\|e\|_E \leq \text{TOL}$

error tolerance

- (i) Make an initial partition of the interval
- (ii) Compute the corresponding FEM solution u_h & the residual $R(u_h)$
- (iii) Use (****) if $\|e\|_E > \text{TOL}$, refine the places for which $\frac{1}{\Delta x} R(u_h)|_{\Delta x}$ is large and then perform (ii) - (iii) in this new mesh.

A mixed boundary value problem:

$$(BVP)_2 \quad \begin{cases} -(au')' = f & \text{in } (0,1) \\ u(0) = 0 \\ u'(1) = g_1 \\ = \alpha + \beta \end{cases}$$

Variational formulation: Find $u \in \tilde{H}_0^{(1)}(x)$

$$\int_0^1 fv dx = - \int_0^1 (au')' v dx \stackrel{?}{=} \int_0^1 au' v' dx - [auv]_{x=0}^{x=1} - \underbrace{[a(x)u'(x)v(x)]}_{g_1} \quad \forall v \in \tilde{H}_0^{(1)}$$

$$= \int_0^1 au' v' dx - \cancel{auu'(1)v(1)} + \cancel{a(x)u'(x)v(x)} \quad \text{by } g_1$$

$$\tilde{H}_0^{(1)} = \{ w''(x) dx < \infty \text{ & } w(0) = 0 \}$$

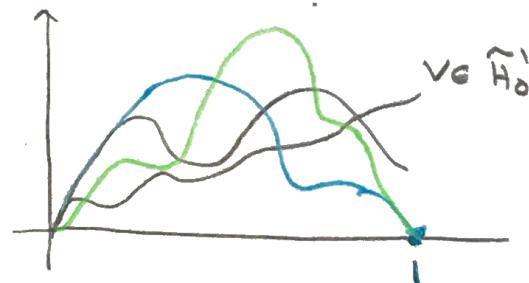
$$H_x^{(1)} = \{ - \quad \text{---} \quad \text{& } w(0) = \alpha \}$$

$$(VF)_2: \int_0^1 au' v' dx = \int_0^1 fv dx + g_1 v(1) \quad \forall v \in \tilde{H}_0^{(1)}$$

$$(BVP)_2 \Rightarrow (VP)_2$$

$$\text{claim } (VF)_2 \Rightarrow (BVP)_2$$

Integration by parts $\xrightarrow{(VF)}$



$$-(\int_0^1 (au')' v dx) + a(1)u'(1)v(1) = \int_0^1 fv dx + g_1 v(1) \quad \forall v \in \tilde{H}_0^{(1)}$$

Step I: choose a $v \in \tilde{H}_0^{(1)}: v(1) = 0$

$$\Rightarrow \int_0^1 (au')' v = \int_0^1 fv$$

This $\xrightarrow{(BVP)} - (au')' = f$ in $(0,1)$ the (PDE) or $(BVP)_2$

$$\text{Step II } (*) \Rightarrow ac(t)u'(t) + v(t) = g, v(t) \\ v(t) \neq 0 \quad \boxed{\Rightarrow ac(t)u'(t) = g}$$

Remark: • Dirichlet boundary conditions (essentially B.C.) is strongly imposed:
Inforced explicitly to the trial & test functions in (VP)
(trial function = solution)

• Newmann & Robin Boundary conditions (natural B.C.s)

Are automatically satisfied in (VP), Hencefore weakly imposed

Scalar Initial value problem (IVP):

$$(DE) \quad \begin{cases} \dot{u}(t) + a(t)u(t) = f(t), \quad 0 \leq t \leq T \\ u(0) = u_0 \end{cases} \quad \begin{cases} a(t) \geq 0 \quad \text{bndd (parabolic case)} \\ a(t) > 0 \quad \rightarrow \text{(dissepativ case)} \end{cases}$$

The analytic solution:

$$\text{Let } A(t) = \int_0^t a(s) ds \Rightarrow A(0) = 0, \text{ then } \stackrel{\text{claim}}{u(t)} = u_0 e^{-A(t)} + \int_0^t e^{-(A(t)-A(s))} f(s) ds$$

$$\text{Pf:} \quad \text{Multip (DE) by } e^{A(t)} \Rightarrow \dot{u}(t) e^{A(t)} + A(t) e^{A(t)} u(t) = e^{A(t)} f(t) \\ \Rightarrow \frac{d}{dt} (u(t) e^{A(t)}) = e^{A(t)} f(t)$$

$$\text{Relabel } t \rightarrow s \quad \int_0^t ds \Rightarrow \int_0^t \frac{d}{ds} (u(s) e^{A(s)}) ds = \int_0^t e^{A(s)} f(s) ds \\ \Rightarrow [u(s) e^{A(s)}]_{s=0}^{s=t} = \int_0^t f(s) e^{A(s)} ds = u(t) e^{A(t)} - u(0) e^{A(0)} = \int_0^t f(s) e^{A(s)} ds \\ \Rightarrow u(t) = u_0 e^{-A(t)} + \int_0^t e^{-(A(t)-A(s))} f(s) ds$$

Stability:

$$1) \quad a(t) \geq \alpha > 0 \Rightarrow |u(t)| \leq e^{-\alpha t} |u_0| + \frac{1}{\alpha} (1 - e^{-\alpha t}) \max_{0 \leq s \leq t} |f(s)|$$

$$2) \quad \text{If } a(t) \geq 0 \quad (\alpha = 0 \text{ in 1}) \Rightarrow |u(t)| \leq |u_0| + \int_0^t |f(s)| ds$$

(continue next time)

Föreläsning 3/2:

Scalar initial value problem (IVP)

- The continuous solution
- The stability
- Present 2 FEMs: CG(1) & DG(0)
- Generalize to CG(q) (DG(q))
- A posteriori (computational) E.E
- A priori (theoretical) error estimation

$$(DE) \quad \begin{cases} \dot{u}(t) + a(t)u(t) = f(t) \\ 0 \leq t \leq T \end{cases}$$

$$(IV) \quad u(0) = u_0$$

Last time \Rightarrow showed the analytic solution: $u(t) = e^{-A(t)} u_0 + \int_0^t e^{-(A(t)-A(s))} f(s) ds$

$$\begin{cases} A(t) = \int_0^t a(r) dr \\ A(0) = 0 \end{cases}$$

stability: (remark) If $f \geq 0$ then $|u(t)| = e^{-\alpha t} |u_0| + \int_0^t (1 - e^{-\alpha(t-s)}) \max_{0 \leq s \leq t} |f(s)| ds \rightarrow 0$ as $t \rightarrow \infty$ (obs: $a(t) \geq 0$)

$$(S1) \quad a(t) \geq \alpha > 0 \Rightarrow |u(t)| \leq e^{-\alpha t} |u_0| + \frac{1}{\alpha} (1 - e^{-\alpha t}) \max_{0 \leq s \leq t} |f(s)|$$

$$(S2) \quad a(t) > 0 \Rightarrow |u(t)| \leq |u_0| + \int_0^t |f(s)| ds$$

$$PF: (S1) \quad a(t) \geq \alpha \Rightarrow \int_s^t a(r) dr \geq \alpha \int_s^t dt \Rightarrow A(t) - A(s) \geq \alpha(t-s) \quad \left. \begin{array}{l} \\ \Rightarrow \{s=0\} \Rightarrow A(t) \geq \alpha t \end{array} \right\}$$

$$\text{Insert in (A.s)} \Rightarrow |u(t)| \leq |u_0| e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} |f(s)| ds$$

$$\Rightarrow |u(t)| \leq |u_0| e^{-\alpha t} + \frac{1}{\alpha} \left(e^{-\alpha(t-s)} \Big|_{s=0}^{s=t} \right) \max_{[0,t]} |f(s)| = |u_0| e^{-\alpha t} + \frac{1}{\alpha} (1 - e^{-\alpha t}) \max_{[0,t]} |f(s)|$$

$$(S2) \quad \alpha = 0 \quad (A(t) - A(s)) \leq 0 \quad \& \quad -A(t) \leq 0 \quad \Rightarrow \quad |u(t)| \leq |u_0| + \int_0^t |f(s)| ds \quad \stackrel{(A.s)}{\Rightarrow} \quad S2$$

Continuous Galerkin of degree 1, CG(1): Relies on piecewise linear and continuous trial functions (solution) and piecewise constant and discontinuous test functions (obss: The space of test & trial functions are different. This is called variational crime)

Discontinuous Galerkin of degree '0' detail:

Piecewise constant & discontinuous test & trial function.

Global Galerkin of degree q:

Find $u \in P^q(0,T)$, with $\underline{u(0)} = u_0$ s.t.

$$\int_0^T (\dot{u} + au) v dt = \int_0^T f v dt, \quad \forall v \in P^q(0,T), \quad \underline{v(0)} = 0$$

$$i.e. V := \{t, t^2, \dots, t^q\}$$

Continuous Galerkin of degree q:

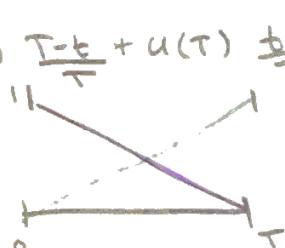
Find $u \in P^q(0,T)$ with $\underline{u(0)} = u_0$ s.t.

$$\int_0^T (\dot{u} + au) v dt = \int_0^T f v dt \quad \underline{\forall v \in P^{q-1}(0,T)}$$

$$V := \{1, t, t^2, \dots, t^{q-1}\}$$

$$\text{Let } q=1 \Rightarrow V=1 \Rightarrow u(T) - u(0) + a \int_0^T u(0) \frac{T-t}{1} + u(t) \frac{t}{1} dt = \int_0^T f dt$$

(assume a constant)



$$\Rightarrow u(T) - u(0) - a u_0 \frac{(T-t)^2}{2T} \Big|_0^T + a u(T) \frac{t^2}{2T} \Big|_0^T = \int_0^T f dt$$

$$(1 + \frac{aT}{2}) u(T) = u_0 (1 - \frac{aT}{2}) = \int_0^T f dt$$

$$\Rightarrow u(T) = \left(\frac{1 - \frac{aT}{2}}{1 + \frac{aT}{2}} \right) u_0 + \left(\frac{1}{1 + \frac{aT}{2}} \right) \int_0^T f dt \quad (***)$$

Obs: f, u_0 known $\Rightarrow u_N$

$u(t)$ linear $\rightarrow u(t) =$

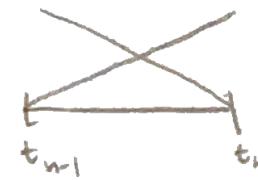
Algorithm for CGCN on a partition $T_k = \{t_0=0, t_1, \dots, t_N=T\}$

I) Compute $u(t_i)$ applying $(***)$ to $[0, t_i]$ with $u(0) = u_0$ given.

II) Assume that $u(t)$ is computed for $t \in (t_{n-2}, t_{n-1}]$

II₂) Compute $u(t_n)$ using $(***)_n$ as follows:

$$\int_{t_{n-1}}^{t_n} (\dot{u} + au) dt = \int_{t_{n-1}}^{t_n} f dt \quad (\Leftrightarrow)$$



$$u(t_n) - u(t_{n-1}) + \int_{t_{n-1}}^{t_n} a(u(t_m)) \left(u(t_{n-1}) \frac{t_n - t}{t_n - t_{n-1}} + u(t_m) \cdot \frac{t - t_{n-1}}{t_n - t_{n-1}} \right) dt = \int_{t_{n-1}}^{t_n} f dt$$

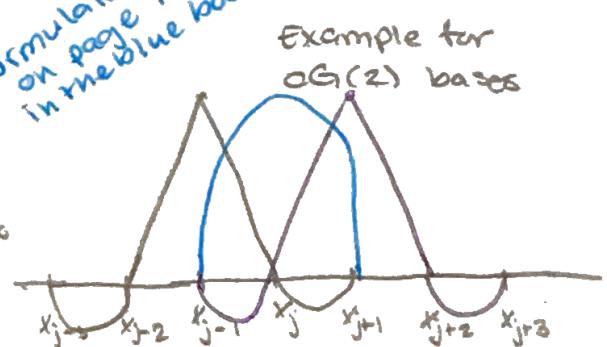
Global forms: $\mathcal{T}_k = \{0 = t_0 < t_1 < \dots < t_N = T\}$

as a partition of the interval: $[0, T]$

formulated 122
on page 122
in the blue book

Continuous Galerkin CG(q):

Find $u \in V_k^{(q)}$ s.t. $u(0) = u_0$ and $\int_0^{t_N} (\dot{u} + au) v dt$
 $= \int_0^{t_N} fv dt, \forall v \in W_k^{q-1}$



$V_k^{(q)} := \{V \mid V \text{ continuous piecewise polynomial of degree } \leq q \text{ on } \mathcal{T}_k\}$

$W_k^{q-1} := \{w \mid w \text{ is discontinuous piecewise polynomial of degree } \leq q-1 \text{ on } \mathcal{T}_k\}$

Discontinuous Galerkin of degree q: dG(q):

Find $u \in P^3(0, T)$ s.t. $\int_0^T (\dot{u} + au) v dt + \alpha(u(0) - u_0) v(0) = \int_0^T fv dt, \forall v \in P^q(0, T)$

[$\alpha=1$ gives best possible stab & accuracy]

Note: $P^q(0, T) \equiv W_k^{(q)}$ & dG(q) gives up the requirement that u satisfies the initial condition

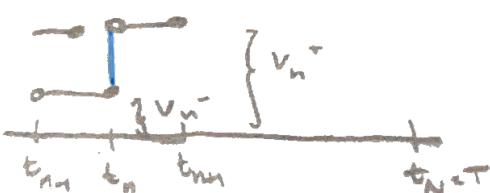
$u_0 =$ _____

$w_0 = u(0)$

Notation: For piecewise constants $u_n = u(t_n), u_{n-1}(t_{n-1}), \dots$

Define $v_n^\pm = \lim_{s \rightarrow 0} v(t_n \pm s)$

$[v_n] = v_n^+ - v_n^-$



$dG(q)$: for $n=1, \dots, N$ find $u \in \mathcal{P}^q([t_{n-1}, t_n])$

such that

$$\int_{t_{n-1}}^{t_n} (\dot{u} + au) v dt + u_{n-1}^+ v_{n-1}^+ = \int_{t_{n-1}}^{t_n} f v dt + u_{n-1}^- v_{n-1}^-$$

$\forall v \in \mathcal{P}^q(I_n)$

For $q=0$ (approx by piecewise constants); $V \equiv 1$, $u = u_n = u_{n-1}^+ = u_n^-$

$\Rightarrow dG(0)$: for $n=1, 2, \dots, N$ find piecewise constant u_n . s.t. on I_n

$$\int_0^t u_n dt + u_n = \int_{t_{n-1}}^{t_n} f dt + u_{n-1} \quad (\text{obs: } u \geq 0 \text{ & } V \equiv 1)$$

Finally summing over n in $dG(q) \Rightarrow$ find $u \in W_k^q$:

$$\sum_{n=1}^N \int_{t_{n-1}}^{t_n} (\dot{u} + au) v dt + \sum_{n=1}^N ([u_{n-1}] v_{n-1}^+) = \int_0^{t_N} f v dt, \quad \forall v \in W_k^q$$

A posterior error estimate for c-G(I) for IVP.

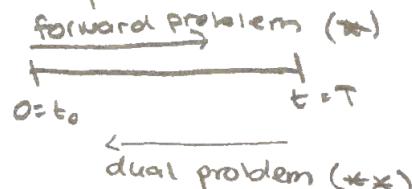
(*) problem: $\dot{u}(t) + a(t)u(t) = f(t), \quad \forall t \in [0, T] \quad \& \quad u(0) = u_0 \quad (\text{DE}) + (\text{V})$

Variational formulation: $\int_0^T (\dot{u} + au) v dt = \int_0^T fv dt, \quad \forall v \stackrel{\text{PE}}{\in}$

$$u(T)v(T) - u(0)v(0) + \int_0^T u(-\dot{v} + av) dt = \int_0^T fv dt$$

Now choose v that satisfies the dual problem: $-\dot{v}(t) + a(t)v(t) = 0 \quad (***)$

$$\Rightarrow u(T)v(T) = u(0)v(0) + \int_0^T fv dt$$



The dual problem for (*):

Find $e(t)$ such that $\begin{cases} -\dot{e}(t) + a(t)e(t) = 0 & \text{for } t_N > t \geq 0 \\ e(t_N) = e_N = u_N \cdot u_N \end{cases}$

Thm: For $n=1, 2, \dots, N$ the c-G(I) solution u for (*) satisfies

$|e_{n-1}| \leq 8(t_N) \max_{[0, t_n]} |\kappa r(u)|$, $r(u) = \dot{u} + au - f$, residual error

$t \in I_n$ on I_n , $t_n = |I_n|$ $I_n = (t_{n-1}, t_n]$ κ $\leq \kappa(t_N)$ the stability factor satisfies.

$$(II) S(t_n) = \int_0^{t_n} \frac{|\dot{e}| dt}{|e|_{\text{ent}}} \leq \begin{cases} e^{t_n} & \text{if } |a(e)| \leq \lambda \quad \forall t \\ 1 & \text{if } a(e) \geq 0, \lambda t \end{cases}$$

PF: $-\dot{e} + ae = 0 \Rightarrow e_N^2 = e_N^2 + 0 = e_N^2 + \int_0^{t_n} e(-\dot{e} + ae) dt = [\text{PS}]$

$$\Rightarrow e_N^2 = e_N^2 - e^2 \Big|_0^{t_n} + \int_0^{t_n} (\dot{e}e + ace) dt = \left\{ \begin{array}{l} e(0) = 0 \\ e(t_n) = e_N \end{array} \right\} \cdot e_N^2 - \overbrace{e^2(t_n)}^{e^2(t_n)} \underbrace{e(t_n)}_{e(t_n)} \cdot \int_0^{t_n} (\dot{e} + ae) dt$$

$$\Rightarrow \{\dot{e} + ae = \dot{u} - \dot{u} + au - au = f - \dot{u} - au = -r(u) = \int_0^{t_n} r(u) e dt$$

Föreläsning 5/2:

(WIP)

Forward problem:

$$\begin{cases} \dot{u}(t) + a(t)u(t) = f(t), & 0 < t \leq t_N \\ u(0) = 0 \end{cases}$$



(Backward) Dual problem:

$$\begin{cases} -\dot{e}(t) + a(t)e(t) = 0 \\ e(t_N) = e_N = u(t_N) - U(t_N) = u_N - U_N \end{cases}$$

Thm: For $N=1, 2, \dots$ the CG(1) solution U_i satisfies

1) $|e_{N,i}| \leq S(t_N) \max_{[0,t_N]} |r(U)| \rightarrow$ Not good, fix it!

This is convergence of order 1: $O(\kappa)$

$$r(U) := \dot{U} + aU - f \quad (\text{is the residual})$$

2) $S(t_N) = \int_0^{t_N} \frac{|\dot{e}| dt}{|e|} \leq \begin{cases} e^{\lambda t_N} & \text{if } |a(t)| \leq \lambda \\ 1 & \text{if } |a(t)| > \lambda \end{cases}$

Requires more refinements (small intervals)

PF: (1)

$$\begin{aligned} -\dot{e} + ae = 0 \Rightarrow e_N^2 = e_N^2 + 0 \Rightarrow e_N^2 = e_N^2 + \int_0^t (-\dot{e} + ae) \epsilon dt \stackrel{P_1}{=} \\ = e_N^2 - [ee]_0^{t_N} + \int_0^{t_N} (\dot{e} + ae) \epsilon(t) dt = e_N^2 - \underbrace{e(t_N) e(t_N)}_{e_N} + \underbrace{e(0) e(0)}_{e_N} + \int_0^{t_N} (\dot{e} + ae) \epsilon dt \end{aligned}$$

$$\Rightarrow |e_N|^2 \cdot \int_0^{t_N} (\dot{e} + ae) \epsilon dt - \{ \dot{e} + ae = \dot{u} - \dot{U} + au - aU = f - \dot{U} - aU \} =$$

$$= \int_0^{t_N} r(U) \epsilon dt \quad \text{def: } \pi_K \epsilon(t) = \frac{1}{k_n} \int_{I_n} \epsilon dt \Rightarrow$$

$$\Rightarrow e_N^2 = - \int_0^{t_N} r(U) (\epsilon - \pi_K \epsilon) dt + \int_0^{t_N} r(U) \pi_K \epsilon(t) dt$$

OBS! $\int_{I_n} |\epsilon - \pi_K \epsilon| dt \stackrel{\text{MVT}}{=} \int_{I_n} |\epsilon(t) - \epsilon(\eta)| dt = \int_{I_n} \left| \int_{\eta}^t \epsilon(s) ds \right| dt \leq \left\{ \left(\int_{I_n} |\dot{\epsilon}| ds \right) \right\}$

$$= k_n \int_{I_n} |\dot{\epsilon}| dt \Rightarrow \left\{ \sum_{n=1}^N \left\{ \dots = \int_0^{t_N} \dots \right\} \right\} \Rightarrow |e_N|^2 \leq \sum_{n=1}^N \underbrace{|r(U)|}_{\max |r(U)|} \int_{I_n} (\epsilon - \pi_K \epsilon) dt$$

$$\Rightarrow \frac{|e_N|^2}{|e_N|} \leq \sum_{n=1}^{\infty} |r(U)| \int_{I_n} |\dot{\epsilon}| dt \leq \max_{\substack{1 \leq n \leq N \\ [0, t_N]}} (k_n |r(U)|_{I_n}) \boxed{\sum_{n=1}^N \frac{\int_{I_n} |\dot{\epsilon}| dt}{k_n}}$$

$\Rightarrow 1) \quad \square$

$S(t_N)$

(2)

To prove (2) we transfer the dual problem to a forward problem:

Let $s = t_N - t$ ($t = t_N - s$) & def: $\Psi(s) := \epsilon(t_N - s) = \epsilon(t)$

then $\frac{d\Psi}{ds} = \frac{d\epsilon}{dt} \cdot \frac{dt}{ds} = -\frac{d\epsilon}{dt} = -\dot{\epsilon}(t_N - s)$

Thus $\dot{\epsilon}$ in the dual problem.

$-\dot{\epsilon}(t_N - s) + a(t_N - s)\epsilon(t_N - s) = 0$ has the corresponding forward problem

$$\begin{cases} \frac{d\Psi}{ds} + a(t_N - s)\Psi(s) = 0, & 0 \leq s \leq t_N \\ \Psi(0) = \epsilon(t_N) = e_N \end{cases}$$

now we use the analytic solution formula:

$$u(t) = u_0 e^{-At} + \int_0^t e^{-(A(t)-A(s))} f(s) ds$$

$$\Rightarrow \{f \equiv 0 \text{ & } u_0 = e_N\} \Rightarrow \Psi(s) = \Psi(0) e^{-\int_0^s a(t_N - r) dr}$$

$$\because \Psi(s) = e_N e^{-\int_0^s a(t_N - r) dr} = \begin{cases} t_N - r = v \\ -dr = dv \end{cases} = e_N e^{\int_{t_N}^{t_N-s} a(v) dv} = e_N e^{A(t_N-s)-A(t_N)}$$

$$\Rightarrow \epsilon(t) = e_N e^{A(t)-A(t_N)} \Rightarrow \dot{\epsilon}(t) = e_N a(t) e^{A(t)-A(t_N)}$$

$$\Rightarrow \int_0^{t_N} |\dot{\epsilon}(t)| dt = |e_N| \int_0^{t_N} a(t) e^{A(t)-A(t_N)} dt = |e_N| \left[e^{A(t)-A(t_N)} \right]_0^{t_N}$$

$$A(0) = 0$$

$$\checkmark = |e_N| (1 - e^{-A(t_N)}) \leq |e_N| \Rightarrow S(t_N) < 1 \text{ if } (a(t) > 0)$$

$$\begin{aligned} \text{The case } |a(t)| \leq \lambda &\Rightarrow |\dot{\epsilon}(t)| \leq \lambda e^{A(t)-A(t_N)} \leq \lambda e^{\int_{t_N}^t a(s) ds} \\ \Rightarrow \int_0^{t_N} |\dot{\epsilon}(t)| dt &\leq \left[-e^{\lambda(t_N-t)} \right]_0^{t_N} |e_N| \leq \lambda e^{\int_{t_N}^t ds} |e_N| = \underline{\lambda e^{\lambda(t_N-t)} |e_N|} = \underline{\left[-1 + e^{\lambda(t_N-t)} \right] |e_N|} \end{aligned}$$

Make a convergence of $a(t)$:

Note that $(g - \pi_K g)$ \perp constants.

$$\text{Because } \pi_K g = \frac{1}{k} \sum_{i=1}^k \{g \rightarrow \int_{I_i} (g - \pi_K g) \cdot c = \int_{I_i} (g - \frac{1}{k} \sum_{i=1}^k g) \cdot c$$

$$= \left\{ \int_{I_n} g - \frac{1}{k_n} (\int_{I_n} g) \cdot C + \left(\int_{I_n} g - \int_{I_n} g \cdot \frac{1}{k_n} \int_{I_n} db \right) \cdot C \right\} = 0$$

Error representation formula:

$$e_N^2 = - \int_0^{t_N} r(t) (e - \pi_K e) dt = \int_0^{t_N} (f - aU - U) (e - \pi_K e) dt$$

$$= \int_0^{t_N} ((f - aU) - \pi_K (f - aU)) (e - \pi_K e) dt \rightarrow "o"$$

$$\text{Thm} \Rightarrow |e_N| \leq S(t_N) \underbrace{|K((f - aU) - \pi_K(f - aU))|}_{[0, t_N]} \leftarrow$$

$$\Rightarrow |e_N| \leq S(t_N) |K^2 \frac{d}{dt} (aU - f)| \stackrel{\text{"interpol."}}{=} \left| \frac{d}{dt} (f - aU) \right|_{[0, t_N]}$$

The draw-back is that to estimate $\frac{d}{dt} (aU - f)$ it is not so easy.

A posterior E.E for dG(0) for (IVP):

Thm: For $N=1, 2, \dots$ the dG(0) solution U satisfies

$$|e_N| = |U(t_N) - U_N| \leq S(t_N) |UR(U)|_{[0, t_N]}$$

$$R(U) = \frac{|U_n - U_{n-1}|}{k_n} + |f - aU| \text{ for } t_{n-1} \leq t \leq t_n$$

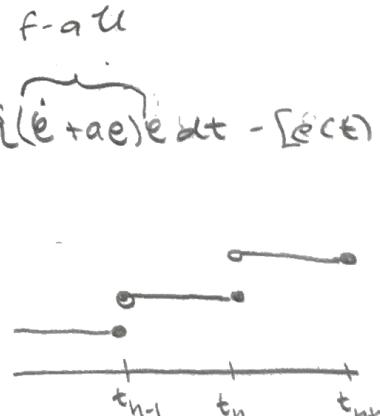
Pf: Use previous thm \Rightarrow

$$e_N^2 = e_N^2 + \sum_{n=1}^N \int_{t_{n-1}}^{t_n} e(-\dot{e} + ae) dt = [\text{PI}] = e_N^2 + \sum_{n=1}^N \left\{ (\dot{e} + ae)e dt - [e(t) e(t)]_{t_{n-1}}^{t_n} \right\}$$

$$\Rightarrow |e_N|^2 = e_N^2 + \sum_{n=1}^N \int_{t_{n-1}}^{t_n} (f - aU) e dt - \sum_{n=1}^N ee \Big|_{t_{n-1}}^{t_n}$$

$$j = \sum_{n=1}^N \{ e(t_n^-) e(t_n^-) - e(t_{n-1}^+) e(t_{n-1}^+) \}$$

$$= \sum_{n=1}^N e_n^- e_n^- - e_{n-1}^+ e_{n-1}^+ = (e_1^- e_1^- - e_0^+ e_0^+) + (e_2^- e_2^- - e_1^+ e_1^+) + \dots + (e_{N-1}^- e_{N-1}^- - e_{N-2}^+ e_{N-2}^+)$$



$$+ (e_N^- e_N^+ - e_{N-1}^+ e_{N-1}^+$$

$$\Rightarrow -j = - \sum_{n=1}^N ee \Big|_{t_{n-1}}^{t_n} = -e_N^- e_N^- + e_0^+ e_0^+ - e_1^- (e_1^- - e_1^+ + e_1^+) + e_1^+ e_1^+ - \dots$$

$$- e_1^- (e_1^- - e_1^+ + e_1^+) + e_1^+ e_1^+ - e_{N-1}^- (e_{N-1}^- - e_{N-1}^+ + e_{N-1}^+) + e_{N-1}^+ e_{N-1}^+$$

(A typical factor: $- e_i^- (e_i^- - e_i^+ + e_i^+) + e_i^+ e_i^+ - e_i^- [e_i^- + e_i^+] [e_i^- + e_i^+]$)

$$= -e_N^2 + e_0^+ e_0^+ + \sum_{n=1}^{N-1} [e_n] e_n^+ + \sum_{n=1}^{N-1} e_n^- \cancel{[e_n]}$$

$$\Rightarrow |e_N|^2 = \cancel{\star}^2 + \sum_{n=1}^N \left\{ \int_{I_n} (f - au) e dt - [u_{n-1}] e_{n-1} \right\} - \cancel{\star}$$

$$= \{as before\} - \sum_{n=1}^N \int_{I_n} (f - au) (e - \pi_n e) dt - \sum_{n=1}^N [u_{n-1}] (e - (\pi_n e)_{n-1})$$

■

Föreläsning 6/2:

A priori error analysis for (IVP):

$x \in$

\mathbb{R}^d
 $d=2,3$

IVBP (heat, wave & convection-diffusion all in 1-space dim)

- Polynom approx [Interpolation in \mathbb{R}^d]
- Poisson equation
- Heat and wave equation

Basically [PI] \leftrightarrow Green's formula

$\Rightarrow a = \text{constant}$ $u + au = f$; find $U = U_n$ such that:

$$\text{dg}(0) \quad \int_{t_{n-1}}^{t_n} u dt + a \int_{t_{n-1}}^{t_n} u dt = \int_{I_n} f dt$$

$$\textcircled{1} \quad \begin{cases} u(t_n) - u(t_{n-1}) + a \kappa_n u_n = \int_{I_n} f dt \\ u_0 = u_0 \end{cases}$$

FOR the exact solution u : $u + au = f$

$$\textcircled{2} \quad u_n - u_{n-1} + a \kappa_n u_n = \int_{I_n} f dt + \boxed{a \kappa_n u_n - a \int_{t_{n-1}}^{t_n} u dt}$$

$$\textcircled{1} \Rightarrow (1 + \kappa_n a) u_n = u_{n-1} + \int_{I_n} f dt, \quad n = 1, 2, \dots \quad \textcircled{I}$$

$$\textcircled{2} \Rightarrow (1 + \kappa_n a) u_n = u_{n-1} + \int_{I_n} f dt + \kappa_n a dt - \underbrace{\int_{I_n} a u dt}_{=: p_n}, \quad n = 1, 2, \dots \quad \textcircled{II}$$

$$\text{Let } e_n = u_n - U_n, \text{ then } \textcircled{II} - \textcircled{I} \Rightarrow (1 + \kappa_n a) e_n = e_{n-1} + \kappa_n a u_n - \underbrace{\int_{I_n} a u dt}_{=: p_n}$$

$$\Rightarrow e_n = (1 - \kappa_n a)^{-1} (e_{n-1} + p_n)$$

Claim: $|P_n| \leq \frac{1}{2} \|a\| \|k_n\|^2 (\max_{I_n} |u|)$

Pf. $|P_n| = |a k_n u_n - \sum_{I_n} a u_n| = \|a\| \|k_n\| \|u_n - \frac{1}{k_n} \sum_{I_n} u_n\| = \|a\| k_n \left| \frac{1}{k_n} \left(\sum_{I_n} u_n - \sum_{I_n} u_{n-1} \right) \right|$

 $\Rightarrow |P_n| \leq \|a\| \left| \sum_{I_n} (b_{n-1})^2 u(I_n) \right| \leq \|a\| \frac{(b_{n-1})^2}{2} \left| \sum_{t=t_{n-1}}^{t_n} \max_{I_n} |u(t)| \right| = \frac{1}{2} \|a\| k_n^2 \max_{I_n} |u|$

Thm: For $k_n \|a\| \leq \frac{1}{2}$, $n \geq 1$ we have the decos approx error e_N satisfying
 $|e_N| \leq \frac{e}{4} (e^{2\|a\| t_N} - 1) (\max_{1 \leq n \leq N} k_n \|u\|_{I_n})$ (No good!)

stability constant (that + with $\|a\| k_N$ large)
the proof relies on the auxiliary result

Lemma: $\|a\| k_n \leq \frac{1}{2}$, $n \geq 1 \Rightarrow (i) (1 - \|a\| k_n)^{-1} \leq e^{2\|a\| k_n}$

$\Rightarrow (ii) |e_N| \leq \frac{1}{2} \left(\sum_{n=1}^N e^{2\|a\| T_n} \|a\| k_n \right) \left(\max_{1 \leq n \leq N} k_n \|u\|_{I_n} \right)$, $T_n = t_n - t_{n-1}$

(iii) $\sum (e^{2\|a\| T_n} \|a\| k_n) \leq e \int_0^{t_N} \|a\| e^{2\|a\| t} dt$

Assume Lemma, then...

Pf of thm: $|e_N| \stackrel{(iii)}{\leq} \frac{1}{2} e \left(\int_0^{t_N} \|a\| e^{2\|a\| t} dt \right) \left(\max_{1 \leq n \leq N} (k_n \|u\|_{I_n}) \right) =$
 $\frac{1}{2} e \cdot \frac{1}{2} [e^{2\|a\| t_N}]_{t=0}^{t=t_N} \left(\max_{1 \leq n \leq N} (k_n \|u\|_{I_n}) \right) = \frac{e}{4} [e^{2\|a\| t_N} - 1] (\max) \quad \square$

Pf of Lemma: (i) $0 \leq x \leq \frac{1}{2} \Rightarrow \frac{1}{1-x} \leq e^{2x}$ (**)

Def: $f(x) = (1-x) e^{2x} - 1$, then (i) $\underline{f(x) \geq 0}$

Obs: $f(0) = 0$ & $f'(x) = -e^{2x} + 2(1-x)e^{2x} - e^{2x}(1-2x) \geq 0$
 $\Rightarrow f(x) \geq 0 \quad \square$

$$(iii) e_n = (1 + \|a\| k_n)^{-1} (e_{n-1} - P_n) \Rightarrow |e_n| \leq \frac{1}{1 + \|a\| k_n} |e_{n-1}| + \frac{1}{\|a\| k_n} |P_n| \stackrel{(i)}{\leq} e^{2\|a\| k_n} (|e_{n-1}| + \|a\|)$$

$$\leq e^{2\|a\| k_n} (e^{2\|a\| k_n} \|a\| |e_{n-1}| + e^{2\|a\| k_n} \|a\| |g_{n-1}| + e^{2\|a\| k_n} \|f_n\|)$$

$$= e^{2\|a\| k_n} e^{2\|a\| k_n} \|a\| \underbrace{(|e_{n-1}| + e^{2\|a\| k_n} \|a\|)}_{|e_{n-1}|} + e^{2\|a\| k_n} \|f_{n-1}\| + \rightarrow$$

$$+ e^{2\lambda|a|t} - e^{-2\lambda|a|t} \frac{1}{2} g_{N+1} + e^{2\lambda|a|t} \frac{1}{2} g_N \quad (\text{so far})$$

$$\Rightarrow \{e_0 = 0\} \Rightarrow \dots \Rightarrow |e_N| \leq \sum_{n=1}^N e^{2|a|t} \frac{1}{m} k_m |g_n|$$

$$\Rightarrow \text{Now } \sum_{m=n}^N k_m = (t_n - t_{n-1}) + (t_{n+1} - t_n) + \dots + (t_N - t_{N-1}) = \underline{\underline{t_N - t_{n-1}}}$$

$$|g_n| < \dots \Rightarrow |e_N| \leq \frac{1}{2} \sum_{n=1}^N e^{2|a|t(t_n - t_{n-1})} |a| t_n^2 (\max_{I_n} |u|)$$

$$\leq \{t_n - t_{n-1}\} \leq \frac{1}{2} \sum_{n=1}^N (e^{2|a|t_n} |a| t_n) (\max_{1 \leq n \leq N} t_n |u|) \\ \Rightarrow (iii)$$

(iii) Note that $\tau_n = t_N - t_n + t_n - t_{n-1} = \tau_{n+1} + \tau_n \Rightarrow$

$$\underline{2|a|\tau_n} = 2|a|\tau_{n+1} + 2|a|k_n \leq \{ |a|k_n \leq \frac{1}{2} \} \leq \underline{2|a|\tau_{n+1} + 1} \quad (***)$$

Further $2|a|\tau_{n+1} \stackrel{(***)}{\leq} 2|a|\tau$ for $\tau_{n+1} < \tau < \tau_n$

$$e^{2|a|\tau_n} |k_n| = \int_{\tau_{n+1}}^{\tau_n} 2|a|\tau \, d\tau \stackrel{(*)}{\leq} \int_{\tau_{n+1}}^{\tau_n} e^{2|a|\tau} \tau_{n+1} \cdot e^{-\tau} \, d\tau \\ \stackrel{(***)}{\leq} e \int_{\tau_{n+1}}^{\tau_n} e^{2|a|\tau} \, d\tau \Rightarrow \sum_{n=1}^N e^{2|a|\tau} |a| k_n \leq e \left(\sum_{n=1}^N \int_{\tau_{n+1}}^{\tau_n} |a| e^{2|a|\tau} \, d\tau \right) = (iii)$$

The heat equation in 1D:

$$\begin{aligned} (\text{PDE}) \quad & \left\{ \begin{array}{l} \dot{u} - u'' = f, \quad 0 < x < 1, \quad t > 0 \\ u(x, 0) = u(x) \\ u(0, t) = u_x(1, t) = 0, \quad t > 0 \end{array} \right. & (\text{BVP}) \end{aligned}$$

Thm: The (BVP) satisfies the following stability estimates

$$(a) \|u(\cdot, t)\| \leq \|u_0\| + \int_0^t \|f(\cdot, s)\| ds \quad \|w(\cdot, t)\| = \left(\int_0^1 w(x, t)^2 dx \right)^{1/2}$$

$$(b) \|u_x(\cdot, t)\| \leq \|u_0'\| + \int_0^t \|f(\cdot, s)\|^2 ds \quad \|w\| t$$

Pf: (a) Multiply the (PDE) by u & $\int_0^1 \dots dx \Rightarrow \int_0^1 \dot{u} u dx - \int_0^1 u'' u dx = \int_0^1 f u dx$
 $\Rightarrow \int_0^1 \frac{1}{2} \frac{d}{dt} (u^2) dx + \int_0^1 u' u' dx - [u(x, t) u'(x, t)]_{x=0}^{x=1} = \int_0^1 f u dx$

$$(b) \|u\| \frac{d}{dt} \|u\| + \frac{1}{2} \|u'\|^2 = \int_0^1 f u dx \leq \|f\| \|u\|$$

$$\Rightarrow \|u\| \frac{d}{dt} (\|u\|) \stackrel{20}{=} \|f\| \|u\| \Rightarrow \int_0^t \frac{d}{ds} \|u(s)\| ds \leq \int_0^t \|f(\cdot, s)\| ds$$

$$\|u(\cdot, t)\| \leq \|u_0\| + \int_0^t \|f(\cdot, s)\| ds \rightarrow (a)$$

(b) Multiply the PDE by \dot{u} and $\int_0^{\cdot} \dots dx \rightarrow \int_0^1 \dot{u}^2 dx - \int_0^1 u'' \cdot \dot{u} dx = \int_0^1 f \dot{u} dx$

$$\Rightarrow \|\dot{u}\|^2 + \int_0^1 u' \dot{u}' dx - [u'(x) u(x, t)]_{x=0} = \int_0^1 f \dot{u} dx$$

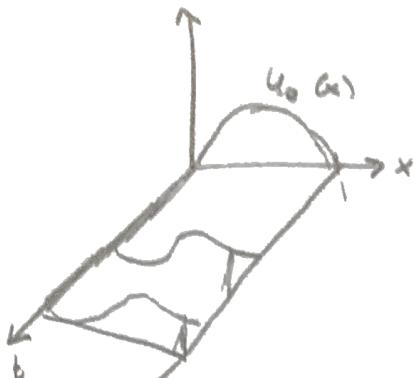
$$\|\dot{u}\|^2 + \frac{1}{2} \frac{d}{dt} \|u'\|^2 = \int_0^1 f \dot{u} \leq \|f\| \|u'\| \leq \frac{1}{2} \|\dot{u}\|^2 + \frac{1}{2} \|f\|^2$$

$$\Rightarrow \frac{1}{2} \int_0^t \|\dot{u}\|^2 + \frac{1}{2} \frac{d}{dt} \|u'\|^2 \leq \frac{1}{2} \|f\|^2$$

$$\Rightarrow \frac{d}{dt} \|u'\|^2 \leq \|f\|^2$$

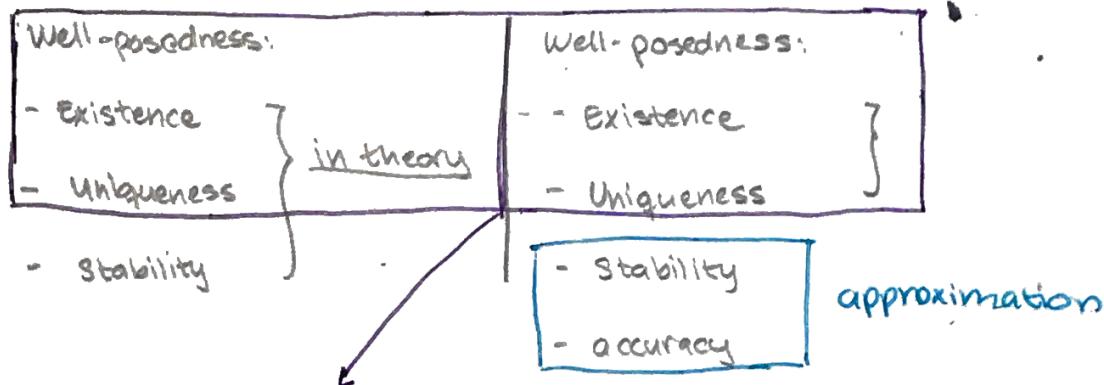
$$\Rightarrow \int_0^t \dots ds \quad \Rightarrow \|u'(0, t)\|^2 - \|u'(x, 0)\|^2 \leq \int_0^t \|f(\cdot, s)\|^2 ds$$

$$\Rightarrow \|u'(0, t)\|^2 \leq \|u'_0\|^2 + \int_0^t \|f(\cdot, s)\|^2 ds \Rightarrow (b)$$



Föreläsning 10/2:

What "V"/"U" expect from a solution of a PDE?



$$a(u, v) = l(v)$$

linear form
bilinear form

$$(PDE) \Leftrightarrow (VF) \ L \vdash (MP)$$

Lax-Milgram (Riesz thm) } (MP) has unique solution

IBVP: (Heat equation in 1-spatial dimension)

$$\begin{aligned} (PDE) \quad & \left\{ \begin{array}{l} \dot{u} - u'' = f \quad \text{on } x \in I \\ u(x, 0) = u_0(x) \end{array} \right. \\ (I.C) \quad & \\ (B.C) \quad & \end{aligned}$$

Thm 1: The solution of IBVP satisfies stability estimate.

$$a) \|u(\cdot, t)\| \leq \|u_0\| + \int_0^t \|f(\cdot, s)\| ds$$

$$b) \|u_x(\cdot, t)\|^2 \leq \|u'_0\|^2 + \int_0^t \|f(\cdot, s)\|^2 ds$$

Thm 2: Stability in homogeneous case ($f = 0$)

$$a) \frac{d}{dt} \|u\|^2 + 2 \|u'\|^2 = 0$$

$$b) \|u(\cdot, t)\| \leq e^{-t} \|u_0\|$$

PF: Multiply (PDE) by u & $\int_0^t \dots dx \Rightarrow$

$$\int_0^t \dot{u} u dx - \int_0^t u'' u dx = 0$$

$$\Leftrightarrow \frac{1}{2} \frac{d}{dt} \int_0^t u^2 dx + \int_0^t u'^2 dx = 0 \quad (\textcircled{A})$$

$$\textcircled{A} \Leftrightarrow \frac{d}{dt} \|u\|^2 \rightarrow 2\|u'\|^2 = 0 \Rightarrow (\textcircled{a})$$

(b): (a) together with Poincaré

$$\Rightarrow \left(\frac{d}{dt} \|u\|^2 + 2\|u'\|^2 \leq 0 \right) \cdot e^{2t}$$

$$\Rightarrow \int_0^t \frac{d}{ds} (\|u(\cdot, s)\|^2 e^{2s}) ds \leq 0$$

$$\|u(\cdot, t)\|^2 e^{2t} - \|u(\cdot, 0)\|^2 \leq 0 \Rightarrow \|u\|^2 \leq e^{-2t} \cdot \|u_0\|^2$$

Thm 3: Consider the homogeneous heat equation & let $\epsilon > 0$ be

"secured". Show that $\int_{-\epsilon}^t \|\dot{u}\|(s) ds \leq \frac{1}{2} \sqrt{\ln \frac{t}{\epsilon}} \|u_0\|$

PF: Multiply (PDE) by $-\dot{u}u''$ & $\int_0^t \dots dx \Rightarrow -t \int_0^t (\dot{u}'u' + t \int_0^s (\dot{u}u'')^2 ds) dx = 0 \quad (\textcircled{B})$

$$\Rightarrow - \int_0^t \dot{u}'u'' = \int_0^t \dot{u}'u' dx - [\dot{u}(x_1, t)u(x_1, t)]_{x=0}^{x=1} = \int_0^t \dot{u}'u' dx \stackrel{\text{in } B}{=} t \cdot \frac{1}{2} \frac{d}{dt} \|u'\|^2 + t\|u''\|^2 = 0$$

$$\Rightarrow \left\{ t \frac{d}{dt} \|u'\|^2 = \frac{d}{dt} (t\|u'\|^2) - \|u'\|^2 \right\} \Rightarrow \frac{d}{dt} (t\|u'\|^2) + 2t\|u''\|^2 = \|u'\|^2$$

$$t \rightarrow s \& \int_0^t \dots ds \Rightarrow \int_0^t \frac{d}{ds} (s\|u'(\cdot, s)\|^2) ds + 2 \int_0^t s\|u''(\cdot, s)\|^2 ds = \int_0^t \|u'(\cdot, s)\|^2 ds$$

$$\leq \underbrace{\left[\text{claim} \right] \leq \frac{1}{2} \|u_0\|^2}_{\textcircled{C}}$$

$$\text{To show } \textcircled{C}: \int_0^t \dots ds \text{ in thm 2 (a)} \Rightarrow \int_0^t \frac{d}{ds} \|u(\cdot, s)\|^2 + 2 \int_0^t \|u'(\cdot, s)\|^2 ds$$

$$\Rightarrow \|u(\cdot, t)\|^2 - \|u(\cdot, 0)\|^2 + 2 \underbrace{\int_0^t \|u'(\cdot, s)\|^2 ds}_{\textcircled{C}} = 0 \Rightarrow \textcircled{C}$$

$$t\|u(\cdot, t)\|^2 + 2 \int_0^t s\|u'(\cdot, s)\|^2 ds \leq \frac{1}{2} \|u_0\|^2 \Rightarrow \|u'(\cdot, t)\|^2 \leq \frac{1}{2t} \|u_0\|^2 \rightarrow$$

$$\Rightarrow \textcircled{1} \|u'(.;t)\| \leq \frac{1}{\sqrt{2\varepsilon}} \|u_0\|$$

$$\textcircled{2} \left(\int_0^t s \|u''(.,s)\|^2 ds \right)^{1/2} \leq \frac{1}{2} \|u_0\|$$

$$\begin{aligned} \|u(t)\| &\leq \int_0^t \|u'(s)\| ds \\ &\leq \frac{1}{\sqrt{2\varepsilon}} \|u_0\| t \\ \|u\| &\leq \frac{1}{\sqrt{2\varepsilon}} \|u_0\| t \quad \text{claim} \\ \|u''\| &\leq \frac{1}{2\varepsilon} \|u_0\| \end{aligned}$$

$$\begin{aligned} \int_{\varepsilon}^t \|u'(s)\| ds &= \int_{\varepsilon}^t \|u''(s)\| ds = \int_{\varepsilon}^t \|u''(s)\| ds = \int_{\varepsilon}^t \frac{1}{\sqrt{s}} \cdot \sqrt{s} \|u''(s)\| ds \\ &\leq \{c-s\} \leq \left(\int_{\varepsilon}^t \left(\frac{1}{\sqrt{s}} \right)^2 ds \right)^{1/2} \cdot \left(\int_{\varepsilon}^t s \|u''(s)\|^2 ds \right)^{1/2} \leq \frac{1}{2} \sqrt{\ln \frac{t}{\varepsilon}} \|u_0\| \quad \blacksquare \end{aligned}$$

FEM for IBVP: CG(1) - CG(1) for heat equation:

$$\begin{cases} u_t - u_{xx} = f, \quad 0 < x < 1, \quad t > 0 & (\text{PDE}) \\ u(0,t) = u(1,t) = 0, \quad t > 0 & (\text{Dirichlet B.C.}) \\ u(x,0) = u_0(x), \quad 0 < x < 1 & (\text{Initial condition}) \end{cases}$$

VF: Variational formulation: for every time interval $I_n = (t_{n-1}, t_n)$,

find u_n ,

$$(\text{VF}) \int_{I_n} \int_0^1 (uv + u'v') dx dt = \int_{I_n} \int_0^1 fv dx dt, \quad \text{for all } v, \text{ such that}$$

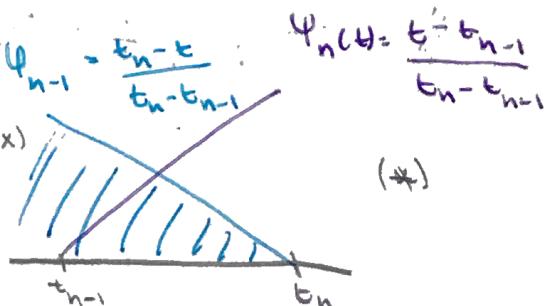
$$v(0,t) = v(1,t) = 0$$

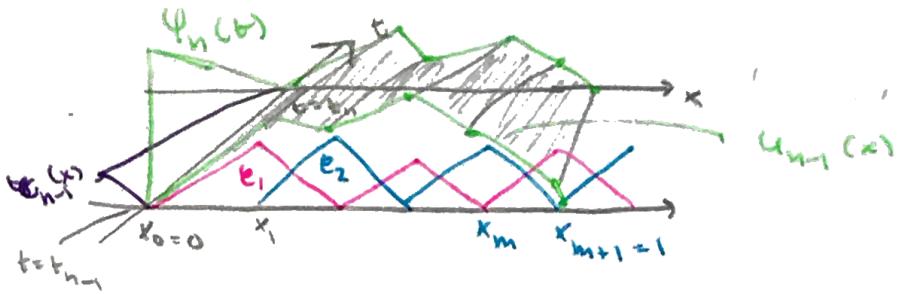
FEM: A piecewise linear (both in space and time) Galerkin approximation

$$(\text{FEM}) \quad u(x,t) := u_{n-1}(x) \Psi_{n-1}(t) + u_n(x) \Psi_n(t)$$

$$\text{where } u_n(x) := u_{n,1} e^{0x} + u_{n,2} e_1(x) + \dots + u_{n,m} e_m(x)$$

$$\text{Obs!} \quad \begin{aligned} \Psi_{n-1}(t) &= -\frac{t-t_{n-1}}{t_n-t_{n-1}}, \quad \Psi_n(t) = \frac{t-t_{n-1}}{t_n-t_{n-1}} \\ \Psi_n'(t) &= \frac{1}{t_n-t_{n-1}} \end{aligned}$$





Here the unknowns are $u_{n,k}$, $k=1, 2, \dots, m$ ($u_{n-1,k}$ are assumed given)

Discrete variational formulation:

$$(DVF) \int_{I_n} \int_0^t (\dot{u}_n e_i + u_n' e_i') dx dt = \int_{I_n} (\int_0^t f e_i dx) dt, \quad i=1, 2, \dots, m$$

In our sample

$$\dot{u}_n = \frac{u_n - u_{n-1}}{\Delta t}$$

$$u_n' = u_{n-1}'(x) \Psi_{n-1}(t) + u_n'(x) \Psi_n(t)$$

$$\Rightarrow \int_{I_n} (\int_0^t \dot{u}_n e_i dx) dt = \frac{1}{\Delta t} \int_{I_n} \int_0^t (u_n(x) - u_{n-1}(x)) e_i(x) dx dt$$

indepth of: t.

$$= \int_0^1 u_n(x) e_i(x) dx - \int_0^1 u_{n-1}(x) e_i(x) dx$$

$$\text{Insert in (DVF)} \Rightarrow \{ u_n(x) = \sum_{j=1}^m u_{n,j} e_j(x) \}$$

$$\Rightarrow \underbrace{\int_0^1 u_n(x) e_i(x) dx}_{MU_n} - \underbrace{\int_0^1 u_{n-1}(x) e_i(x) dx}_{MU_{n-1}} + \underbrace{\int_{I_n} \Psi_{n-1}(t) \int_0^t u_{n-1}'(x) e_i'(x) dx dt}_{\frac{1}{\Delta t} S U_{n-1}}$$

$$+ \underbrace{\int_{I_n} \Psi_n(t) \int_0^t u_n'(x) e_i'(x) dx dt}_{\frac{1}{\Delta t} S U_n} = \int_0^1 (\int_{I_n} f dt) dx$$

\Leftrightarrow a [Crank-Nicolson system]

$$(M + \frac{\Delta t}{2} S) U_n = (M - \frac{\Delta t}{2} S) U_{n-1} = F_n \quad (\text{CNS})$$

$$\Rightarrow U_n = [U_{n,1}, U_{n,2}, \dots, U_{n,m}]^T$$

For each n given
 U_{n-1} & F_n (CNS)
 $\Rightarrow U_n$ \square

Here $S = \{S_{ij}\}_{i,j=1}^m = \left\{ \int_0^1 e_i(x) e_j(x) dx \right\}_{i,j=1}^m$ is the stiffness matrix:

$$S = \frac{1}{4} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

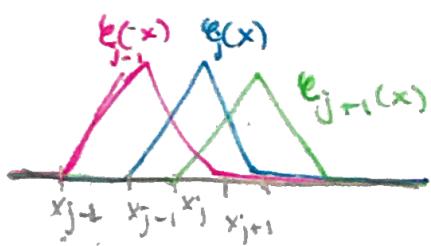
& $M = \{M_{ij}\}_{i,j=1}^m = \left\{ \int_0^1 e_i(x) e_j(x) dx \right\}_{i,j=1}^m$ is the mass matrix

Mass-matrix

To compute M: recall that, for a uniform partition:

$$e_{ij}(x) = \frac{1}{h} \begin{cases} 0 & x \notin [x_{j-1}, x_{j+1}] \\ x_{j+1} - x_i & x_{j-1} \leq x \leq x_j \\ 0 & x_j \leq x \leq x_{j+1} \end{cases}$$

$$x_j \leq x \leq x_{j+1}$$



$$M = \frac{h}{6} \begin{bmatrix} 4 & 1 & 0 & 0 & \dots & 0 \\ 1 & 4 & 1 & 0 & \dots & 0 \\ 0 & 1 & 4 & 1 & 0 & \dots & 0 \\ & & & & \ddots & & \\ & & & & & 4 & \\ & & & & & & \ddots \\ & & & & & & 1 & 1 & 1 \\ & & & & & & & & 4 \end{bmatrix}$$

OBS! $m_{ij} = 0$ if $|i-j| > 1$

$$m_{ij} := \int_0^1 e_j^2(x) dx = \frac{1}{h^2} \left(\int_{x_{j-1}}^{x_j} (x-x_j)^2 dx + \int_{x_j}^{x_{j+1}} (x_{j+1}-x)^2 dx \right) = \frac{1}{h^2} \left[\left[\frac{(x-x_j)^3}{3} \right]_{x_{j-1}}^{x_j} + \left[\frac{(x_{j+1}-x)^3}{3} \right]_{x_j}^{x_{j+1}} \right] = \frac{1}{h^2} \left[\frac{h^3}{3} + \frac{h^3}{3} \right] = \frac{\sqrt{2h}}{3}$$

$$m_{i,i+1} = \int_0^1 e_i(x) e_{i+1}(x) dx = \frac{1}{h^2} \int_{x_i}^{x_{i+1}} (x_{i+1}-x)(x-x_i) dx = \dots = \frac{1}{6} h$$

Föreläsning 12/2:

Wave equation:

- Convection of energy
- CG(1)-cG(1), space time FEM
- Example 7.2

- + Introduce time independent & stationary convection-diffusion (non-linear & linear)
- Introduce the streamline-diffusion method (SDM)

Wave equation:

$$\left\{ \begin{array}{l} (\text{DE}) \quad \ddot{u} - u'' = 0 \quad \text{in } (0,1) \quad (\text{same as: } u_{tt} - u_{xx} = 0) \\ (\text{BC}) \quad u(0,t) = u(1,t) = 0 \\ (\text{IC}) \rightarrow u(x,0) = u_0(x) \\ \quad \quad \quad \dot{u}(x,0) = v_0(x) \end{array} \right.$$

conservation of energy: Multiply the (DE) by \dot{u} & $\int \dots dx$

$$\Rightarrow \int_0^1 \ddot{u} \dot{u} - \int_0^1 u'' \dot{u} = 0 \stackrel{\text{DE}}{\Rightarrow} \int_0^1 \frac{1}{2} \frac{d}{dt} (\dot{u}^2) + \int_0^1 u' \dot{u} - [u(x,t) - \dot{u}(x,t)] \Big|_{x=0}^{x=1} = 0$$

$$\Leftrightarrow \frac{1}{2} \frac{d}{dt} (\|\dot{u}\|^2 + \|u'\|^2) = 0 \quad \int_0^1 \frac{1}{2} \frac{d}{dt} (\dot{u}^2) dx$$

$$\Leftrightarrow \underbrace{\frac{1}{2} \|\dot{u}\|^2}_{\substack{\text{independent of} \\ \text{time}}} + \underbrace{\frac{1}{2} \|u'\|^2}_{\substack{\text{potential} \\ \text{energy}}} = \text{constant} = \frac{1}{2} \|v_0\|^2 + \frac{1}{2} \|u_0\|^2$$

cG(1) - cG(1), FEM for the wave propagation:

$$\left\{ \begin{array}{l} (\text{DE}) \quad \ddot{u} - u'' = \begin{cases} 0 & \text{if } 0 < x < 1, t > 0 \\ f & \text{else} \end{cases} \\ (\text{BC}) \quad u(0,t) = 0, \quad u'(1,t) = g(t) \quad t > 0 \\ (\text{IC}) \quad u(x,0) = u_0(x), \quad \dot{u}(x,0) = v_0(x) \quad 0 < x < 1 \end{array} \right.$$

Reformulate the problem as a system of PDEs:

$$\left\{ \begin{array}{l} \dot{u} - v = 0 \quad (\text{time convective}) \\ \dot{v} - u'' = 0 \quad (\text{diffusion equation}) \end{array} \right.$$

Let $w = \begin{pmatrix} u \\ v \end{pmatrix}$, then $\Leftrightarrow \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -d^2/dx^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\dot{w} + Aw = 0, \quad A = \begin{pmatrix} 0 & -1 \\ -d^2/dx^2 & 0 \end{pmatrix}$$

CGC1 - CGC1: Let for $t \in I_n = [t_{n-1}, t_n]$

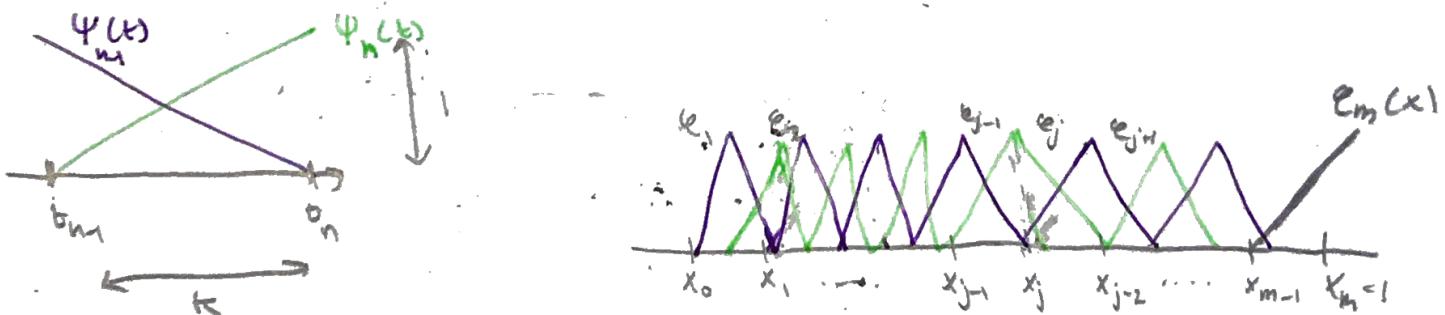
$$\begin{cases} U(x, t) = U_{n-1}(x)\Psi_{n-1}(t) + V_n(x)\Psi_n(t) \\ V(x, t) = V_{n-1}(x)\Psi_{n-1}(t) + V_n(x)\Psi_n(t) \end{cases}$$

where for $\tilde{n} = n-1, n$

$$U_{\tilde{n}}(x) = U_{\tilde{n},1} e_1(x) + U_{\tilde{n},2} e_2(x) + \dots + U_{\tilde{n},m} e_m(x)$$

$$V_{\tilde{n}}(x) = V_{\tilde{n},1} e_1(x) + V_{\tilde{n},2} e_2(x) + \dots + V_{\tilde{n},m} e_m(x)$$

Ques? Just for the sake of "uniformity". Here we choose $m+1$ partition points in x : $0 = x_0 < x_1 < \dots < x_m < x_{m+1} = 1$



Now for $\dot{u} - v = 0$, $t \in I_n$ write the general VF:

$$(I): \int_{I_n} \int_0^1 \dot{u} e dx dt - \int_{I_n} \int_0^1 \dot{v} e dx dt = 0 \quad \text{for all } e(x, t)$$

Similarly for $\dot{v} - u'' = 0$ & $u'(1, t) = g(t)$:

$$\Rightarrow \int_{I_n} \int_0^1 \dot{v} e dx dt + \int_{I_n} \int_0^1 [u' e' - \int_{I_n} [u(x, t) e(x, t)]_{x=0}^{x=1}] dx dt = 0 \quad \text{for all } e(x, t) = 0$$

$$\Leftrightarrow (II) \int_{I_n} \int_0^1 \dot{v} e dx dt + \left(\int_{I_n} \int_0^1 u' e'(x) dx \right) dt = \int_{I_n} g(t) e(1, t) dt$$

We therefore seek $U(x, t)$ & $V(x, t)$ such that:

$$(I) \text{ approx. } \int_{t_n}^t \underbrace{\int_0^x \frac{U_n - U_{n-1}}{k} \psi_j(x)}_{\dot{U}} dt$$

$$- \int_{t_n}^t \int_0^x (V_{n-1}(x) \psi_{n-1}(x) + V_n(x) \psi_n(x)) e_i(x) dt = 0$$

$i=1, 2, \dots, m$

$$\dot{U} = U_n \dot{\psi}_n + U_{n-1} \dot{\psi}_{n-1}$$

$$\left(\frac{t-t_n}{k} \right)$$

$$\left(\frac{t_n-t}{k} \right)$$

$$U_n(x) = \sum_{j=1}^m U_{n,j} \psi_j(x)$$

$$(II) \text{ approx. } \int_{t_n}^t \underbrace{\int_0^x \frac{V_n(x) - V_{n-1}(x)}{k} e_i(x) dt}_{\dot{V}} + \int_{t_n}^t \int_0^x (e'_{n-1}(x) \psi_{n-1}(t) + e'_n(x) \psi_n(t)) e_i'(x) dt = \int_{t_n}^t g(t) e_i(t) dt.$$

Reduced iteration form we have

$$\left\{ \begin{array}{l} \int_0^x \underbrace{U_n(x) e_i(x) dx}_{M U_n} - \frac{k}{2} \int_0^x \underbrace{V_n(x) e_i(x) dx}_{M V_n} = \int_0^x \underbrace{U_{n-1}(x) e_i(x) dx}_{M U_{n-1}} + \frac{k}{2} \int_0^x \underbrace{V_{n-1}(x) e_i(x) dx}_{M V_{n-1}} \\ \int_0^x \underbrace{V_n(x) e_i(x) dx}_{M V_n} + \frac{k}{2} \int_0^x \underbrace{U'_n(x) e'_i(x) dx}_{S U_n} = \int_0^x \underbrace{V_{n-1}(x) e_i(x) dx}_{M V_{n-1}} - \frac{k}{2} \int_0^x \underbrace{U'_{n-1}(x) e'_i(x) dx}_{S U_{n-1}} + g_n \end{array} \right. \quad i=1, 2, \dots, m$$

In matrix form:

$$\begin{bmatrix} M & -\frac{k}{2}M \\ \frac{k}{2}S & M \end{bmatrix} \begin{bmatrix} U_n \\ V_n \end{bmatrix} = \begin{bmatrix} M & \frac{k}{2}M \\ \frac{k}{2}S & M \end{bmatrix} \begin{bmatrix} U_{n-1} \\ V_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ g_n \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{U} \\ \dot{V} \end{bmatrix} = A \begin{bmatrix} U_n \\ V_n \end{bmatrix}$$

$$U_n = w(1:m) \quad V_n = w(m+1:2m)$$

$$S = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots \\ 0 & -1 & 2 & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \dots & \dots & 0 \end{bmatrix} \quad \& \quad M = \frac{h}{6} \begin{bmatrix} 4 & 1 & 0 & \dots & 0 \\ 1 & 4 & 1 & \dots & 0 \\ 0 & 1 & 4 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 4 \end{bmatrix}$$

Because e_m is half hat fcn.

$$u_n = \begin{bmatrix} u_{n,1} \\ \vdots \\ u_{n,m} \end{bmatrix} \quad \& \quad v_n = \begin{bmatrix} v_{n,1} \\ \vdots \\ v_{n,m} \end{bmatrix} \quad \& \quad g_n = \begin{bmatrix} 0 \\ \vdots \\ g_{n,m} \end{bmatrix} \quad \& \quad g_{n,m} = \int_{I_n} g(x) dx$$

Example 7.2 (p. 173)

Consider CGC1 approx for the convection-diffusion equation:

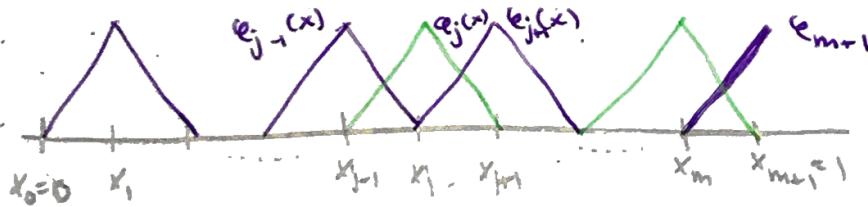
$$\begin{cases} -\varepsilon u''(x) + \rho u'(x) = r, & 0 < x < 1 \\ u(0) = 0 \\ u(1) = \beta + \alpha \end{cases}$$

The function (trial & test) space is $V_h = \{w : \int_0^1 (w^2(x) + w'(x)) dx \leq 200$
 $\& w(0) = 0\}$

Multiply the PDE by a $v \in V_h$ & $\int_0^1 \dots dx$ & (P1) \Rightarrow

$$-\varepsilon u'(x)v' + \varepsilon \int_0^1 u'v' dx + \rho \int_0^1 u'v dx = \int_0^1 r v dx$$

$$\Rightarrow (\text{VF}) \quad \varepsilon \int_0^1 u'v' dx + \rho \int_0^1 u'v dx = r \int_0^1 v(x) dx + \varepsilon \rho v(1)$$



Let $\hat{V}_h := \{w_h : w_h$ is piecewise linear continuous on T_h & $w_h(0) = 0$

Find $u_h \in \hat{V}_h$ s.t.

$$(\text{FEM}) \quad \varepsilon \int_0^1 u_h' v' dx + \rho \int_0^1 u_h' v dx = r \int_0^1 v(x) dx + \varepsilon \rho v(1), \quad \forall v \in \hat{V}_h$$

$$\text{Let now } u_h(x) = \sum_{j=1}^{m+1} \xi_j e_j(x) \quad \& \quad v = e_i(x), \quad i = 1, 2, \dots, m+1$$

$$\Rightarrow \{\xi_j \rightarrow z\}$$

$$\sum_{j=1}^{m+1} \left(\underbrace{\varepsilon \int_0^1 e_j' e_i' dx}_{s_{ij}} + \underbrace{\rho \int_0^1 e_j' e_i dx}_{c_{ij}} \right) \xi_j = r \int_0^1 e_i(x) dx + \varepsilon \rho e_i(1)$$

$$\Leftrightarrow (\varepsilon S + \rho C) \varphi = r \Phi + \varepsilon \beta \Phi,$$

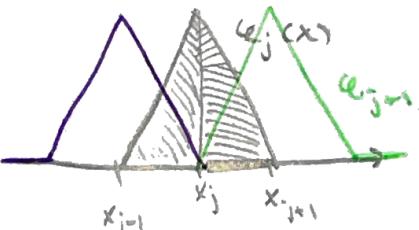
where $S = \frac{1}{n} \begin{bmatrix} 1 & 2 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{bmatrix}$

due to half-hat function ℓ_{m+1}

Note that $c_{ij} = 0$ if $|i-j| > 1$

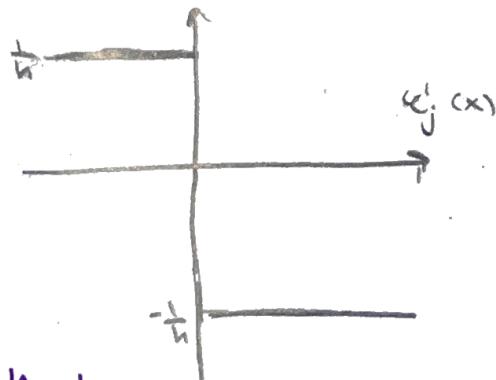
$$c_{ii} = 0$$

$$c_{jj} = \int_0^{x_j} e_j(x) e_j(x) dx$$



$$c_{jj} = \int_{x_j}^{x_{j+1}} e_j(x) e_j(x) dx = \frac{1}{n} \cdot \frac{h}{2} = \frac{1}{2}$$

$$c_{j,j+1} = \int_{x_{j-1}}^{x_j} e_j(x) e_{j+1}(x) dx = -\frac{1}{n} \cdot \frac{h}{2} = -\frac{1}{2}$$



$$c_{m+1,m+1} = \int_{x_m}^{x_{m+1}} \ell_{m+1}(x) \ell_{m+1}(x) dx = \frac{1}{n} \cdot \frac{h}{2} = \frac{1}{2}$$

$$\Rightarrow C = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & \\ 0 & -1 & 0 & 1 & \\ & & -1 & 0 & 1 \\ & & & -1 & 1 \end{bmatrix}$$

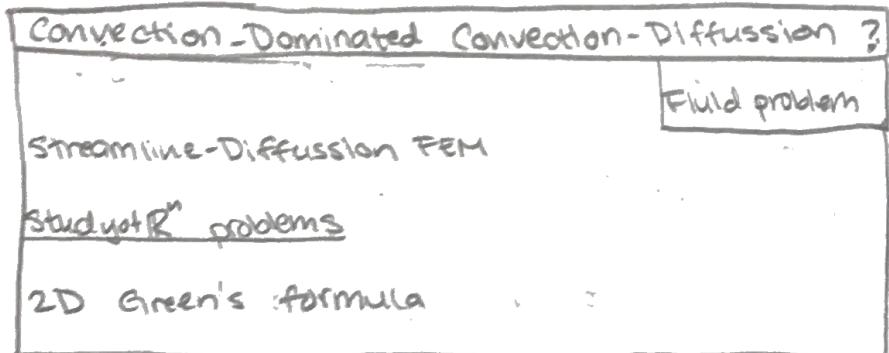
The loadvector $b = (b_k)_{k=1}^{m+1}$
 $\Rightarrow \left\{ b_1 = b_2 = \dots = b_m = r h \right.$
 $\left. b_{m+1} = r \frac{h}{2} + \varepsilon \beta \right.$

Föreläsning 13/2:

Convection Diffusion:

Stationary
time-independent

A traffic flow model:



Example: The traffic flow in a highway;



Let $g \in g(x, t)$ be the density of cars, $0 \leq g \leq 1$ and $u = u(x, t)$ the velocity (speed vector) of cars:

For a highway path (a, b) the difference between the traffic inflow $u(a)g(a)$ at the point $x=a$ and outflow $u(b)g(b)$ at $x=b$ gives the density variation on (a, b) : $\frac{d}{dt} \left(\int_a^b g(x, t) dx \right) = \int_a^b g(x, t) dt = u(a)g(a) - u(b)g(b)$

$$\Rightarrow \int_a^b g'(x, t) dx = - \int_a^b (ug)' dx \Rightarrow \int_a^b \{ \dot{g} + (ug)' \} dx = 0.$$

$$\Rightarrow \{ I = (a, b) \text{ arbitrary} \} \Rightarrow \boxed{\dot{g} + (ug)' = 0} \quad (*)$$

Simple Model: $u = 1 - g \quad (*) \quad \dot{g} + ((1-g)g)' = 0 \Rightarrow \boxed{\dot{g} + (1-2g)g' = 0}$

Assume an alternative model:

$$u = c - \varepsilon \left(\frac{g'}{g} \right) \xrightarrow{c > 0, \varepsilon > 0}$$

$$\Rightarrow \dot{g} + \left(\left(c - \varepsilon \frac{g'}{g} \right) \dot{g} \right)' = 0 \quad \Leftrightarrow \boxed{\dot{g} + c g' - \varepsilon g'' = 0}$$

$\left\{ \begin{array}{l} \text{A non-linear time dependent} \\ \text{convection equation} \end{array} \right.$

time-independent linear convection
-diffusion eqn
 $\varepsilon \gg 1 \Rightarrow$ convection-dominated

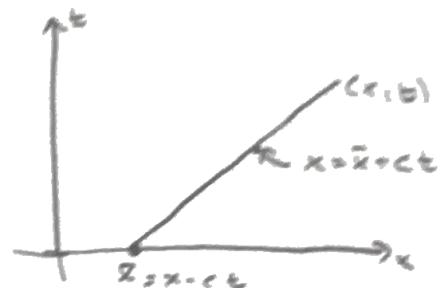
For $\varepsilon \gg 0$, the solution is given by exact transport

$g(x,t) = g_0(x-ct)$ because g is constant on the direction $(C,1)$

$$\Leftrightarrow (C,1) \cdot \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial t} \right) = 0 \quad | \quad g(x,t) = g(x+ct, t)$$

$$\rightarrow \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial g}{\partial t} = 0$$

$$cg' + \dot{g} = 0$$



Change the notation: $g \rightarrow u \xrightarrow[\text{mass}]{} p \quad \dot{u} + \nabla p - \varepsilon u^2 = 0$

Compare with the Navier-Stokes equation for incompressible flow.

$$\underline{\text{N-S}} \quad \begin{cases} (\dot{u} + \nabla p) u - 2\Delta u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases}$$

where $p=u$, $\underline{u}=(u_1, u_2, u_3)$ is the velocity vector
 mass → momentum → energy

p is the pressure and $\varepsilon = \frac{1}{Re}$, \underline{Re} is the Reynolds number

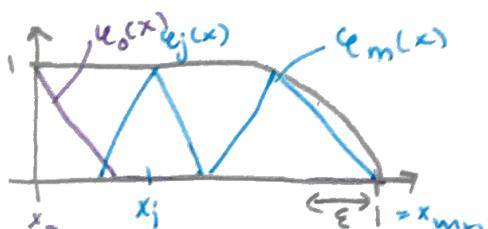
$10^5 < Re < 10^7$ for N-S

↳ Turbulence

Example: Consider the BVP:

$$\begin{cases} \dot{u} - \varepsilon u^2 = 0, \quad 0 < x < 1 \\ u(0) = 1, \quad u(1) = 0 \end{cases}$$

the exact solution is: $u(x) = c(e^{\frac{x}{\varepsilon}} - e^{\frac{-1}{\varepsilon}})$, $c = \frac{1}{(e^{\frac{1}{\varepsilon}} - 1)}$



Note outer boundary larger $\propto \varepsilon$

FEM Set $u(x) = u_0(x) + u_1(x) + \dots + u_m(x)$ & insert in corresponding discrete VF \Rightarrow

$$\rightarrow \int_0^1 (u_i^{(k)} + \varepsilon u_i^{(k+1)}) dx = 0 \quad \text{for } i=1,2,\dots,m.$$

$$\text{at row } j \Rightarrow \frac{1}{2}(u_{j+1} + u_{j-1}) + \sum_{i=1}^n (2u_i - u_{i-1} - u_{i+1}) = 0 \quad (\star\star\star)$$

$$S = \frac{1}{h} \begin{bmatrix} \dots & -1 & 2 & -1 & \dots \end{bmatrix} \quad C = \frac{1}{2} \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & \dots \\ \dots & \dots \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_{j+1} \\ u_j \\ u_{j-1} \\ u_{j-2} \end{bmatrix}$$

Compare with central-difference:

$$\left(\frac{u_{j+1} - u_{j-1}}{2h} - \frac{\epsilon}{h^2} u_{j+1} - 2u_j + u_{j-1} \right) = 0 \quad \Leftarrow \left(\frac{\star\star\star}{h} \right)$$

$$\text{Then if } \epsilon \approx 0 \Rightarrow u_{j+1} \approx u_{j-1}$$

If m even



i.e. oscillations in $u \Rightarrow u$ is a "bad"-approximation

Remedy is to approx $u'(x_j) \approx u'(x_i) \approx \frac{u_j - u_{j-1}}{h}$ \Rightarrow (better approximation)

The streamline-diffusion method (SDM):

Ideal: choose $(v + \frac{1}{2}\beta h v')$ as test function

Let $\beta \equiv 1$ & write the corresponding VF for w

as testfunction.

$$h(u, v) = h(u'', v)$$

$$\int_0^1 [u(v + \frac{1}{2}hv') - \epsilon u''(v + \frac{1}{2}hv')] dx = \int_0^1 f(v + \frac{1}{2}hv') dx \quad (\text{VF})$$

Remark on discrete version:

We intercept the term $\int_0^1 u''v'$ as $\sum_j \int_{x_j}^{x_{j+1}} u''v' \quad (\approx 0 \text{ in case of approx with piecewise linear})$

Let now $v = e_j$ in the discrete version of (VF)

Check each term separately:

$$\text{2nd term in 1st integral: } \int_0^1 u'' \pm h e_j' \stackrel{\substack{\text{row } j \\ \text{on} \\ \text{stiffness matrix}}}{=} u_j - \frac{1}{2}u_{j+1} - \frac{1}{2}u_{j-1}$$

$$\begin{bmatrix} \dots & -1 & 2 & -1 & \dots \end{bmatrix} \begin{bmatrix} u_{j+1} \\ u_j \\ u_{j-1} \\ u_{j-2} \end{bmatrix}$$

1st term in 1st integral: $\int_0^1 u' e_j = \frac{u_{j+1} - u_{j-1}}{2}$

row j in convection matrix

$$\text{minimum} \Rightarrow u_j = u_{j-1} \quad \square$$

Variational formulation in \mathbb{R}^2 :

- Similar to R
- Crucial difference is: [PI] is replaced by Greens formula.

Let $u \in C^2(\Omega)$ & $v \in C^1(\Omega)$, then $\iint_{\Omega} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) v(x, y) dx dy =$

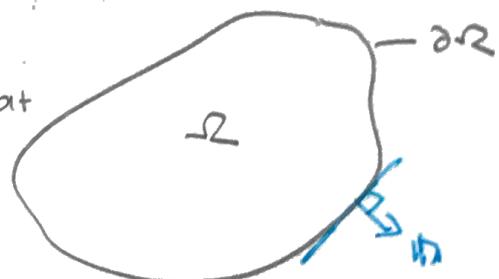
$$\int_{\partial\Omega} \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \cdot \mathbf{n}(x, y) v ds - \iint_{\Omega} \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \cdot \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right) dx dy ,$$

where $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$.

$\mathbf{n} = \mathbf{n}(x, y)$: outward unit normal to $\partial\Omega$ at

$x = (x, y)$

$ds :=$ is the boundary element

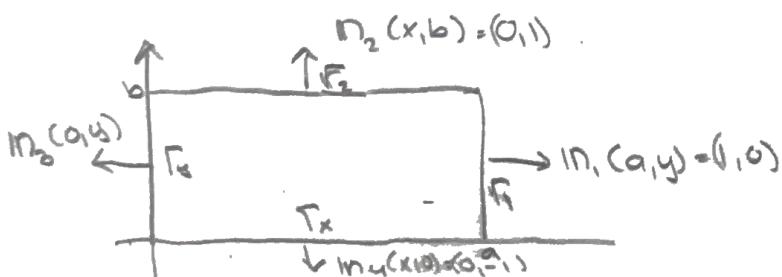


$$\text{In compact form: } \int_{\Omega} (\Delta u) v dx - \int_{\partial\Omega} (\nabla u \cdot \mathbf{n}) v ds - \iint_{\Omega} (\nabla u \cdot \nabla v) dx \quad \textcircled{I}$$

Changing the order of u & v :

$$\Rightarrow \int_{\Omega} (\Delta v) u dx = \int_{\partial\Omega} (\nabla v \cdot \mathbf{n}) u ds - \iint_{\Omega} (\nabla v \cdot \nabla u) dx \quad \textcircled{II}$$

$$\textcircled{I} - \textcircled{II} \Rightarrow \iint_{\Omega} (\Delta uv - \Delta vu) dx = \iint_{\Omega} \{ (\nabla u \cdot \mathbf{n}) v + (\nabla v \cdot \mathbf{n}) u \} ds$$



$$\iint_{\Omega} \frac{\partial^2 u}{\partial x^2} v dx dy = \int_0^b \left(\int_0^a \frac{\partial^2 u}{\partial x^2}(x,y) v(x,y) dx \right) dy = \{ \text{Divergenz} \}$$

$$= \int_0^b \left(\left[\frac{\partial u}{\partial x}(x,y) v(x,y) \right] \Big|_{x=0}^a - \int_0^a \frac{\partial u}{\partial x}(x,y) \frac{\partial v}{\partial x}(x,y) dx \right) dy \quad \textcircled{III}$$

$$= \int_0^b \left[\frac{\partial u}{\partial x}(a,y) v(a,y) - \frac{\partial u}{\partial x}(0,y) v(0,y) \right] dy - \int_0^b \int_0^a \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} dx dy$$

$$\underbrace{\int_0^b \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \right) \cdot \underbrace{\ln(x,y)}_{(-1,0)} v(x,y) dy}_{\textcircled{IV}} + \underbrace{\int_{\Gamma_0} \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \cdot \underbrace{\ln(0,y)}_{(-1,0)} v(0,y) dy}_{\textcircled{V}}$$

$$= \int_{\Gamma_1 \cup \Gamma_3} \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} \cdot \ln(x,y) v(x,y) - \iint_{\Omega} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} dx$$

$$x \rightarrow y \Rightarrow \iint_{\Omega} \frac{\partial^2 u}{\partial y^2} v dx dy - \int_{\Gamma_2 \cup \Gamma_4} \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \cdot \ln(x,y) v(x,y) ds - \iint_{\Omega} \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} dx$$

$$\Rightarrow \textcircled{IV} - \textcircled{V} \Rightarrow \int_{\Omega} (\Delta u) v = \int_{\partial \Omega} (\nabla u \cdot \nabla v) v - \int_{\Omega} \nabla u \cdot \nabla v$$

Föreläsning 17/z:

Finite element in \mathbb{R}^n ($n=2,3$):

$$1D \leftrightarrow n=2$$

PI \hookrightarrow Green's formula (GF)

GF: if $u \in C^2(\Omega)$, $v \in C(\Omega)$, $\Omega \subset \mathbb{R}^2$, then:

$$(GF1) \iint_{\Omega} (\Delta u)v \, dx \, dy = \int_{\partial\Omega} (\nabla u \cdot \mathbf{n})v \, ds - \iint_{\Omega} (\nabla u \cdot \nabla v) \, dx \, dy$$

$$(GF2) \iint_{\Omega} ((\Delta u)v - (\Delta v)u) \, dx = \int_{\partial\Omega} \{(\nabla u \cdot \mathbf{n})v - (\nabla v \cdot \mathbf{n})u\} \, ds$$

An example of a minimization problem (2011-08-24 5a):

Formulate a relevant minimization problem for the poisson equation below:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^2 \\ \mathbf{n} \cdot \nabla u = b(g-u) & \text{on } \Gamma: \partial\Omega \end{cases}$$



Solution: Multiply the eqn by $v \in \{\dots\}_{\Omega}$

$$\Rightarrow \int_{\Omega} fv \, dx = - \iint_{\Omega} (\Delta u)v \, dx - \int_{\Gamma} (\mathbf{n} \cdot \nabla u)v \, ds + \iint_{\Omega} (\nabla u \cdot \nabla v) \, dx = \int_{\Gamma} buv \, ds + \underbrace{\int_{\Omega} \log v \, dx + \int_{\Omega} fu \, v \, dx}_{=: l(v)}$$

$$\Leftrightarrow \underbrace{\iint_{\Omega} (\nabla u \cdot \nabla v) \, dx}_{:= a(u, v)} + \int_{\Gamma} buv \, ds = \underbrace{\int_{\Omega} fv \, dx}_{=: l(v)} \Leftrightarrow a(u, v) = l(v)$$

$$\text{Define: } F(w) = \frac{1}{2} (a(w, w) - l(w)) = \frac{1}{2} \int_{\Omega} \nabla w \cdot \nabla w \, dx + \frac{1}{2} \int_{\Gamma} bw \, ds - \int_{\Omega} fw \, dx - \int_{\Gamma} bgw \, ds$$

Now choose $w = u+v \Rightarrow$

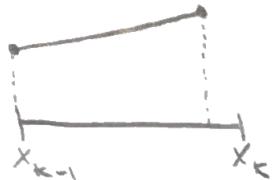
$$\begin{aligned} F(w) &= F(u+v) = \frac{1}{2} \int_{\Omega} \nabla(u+v) \cdot \nabla(u+v) \, dx + \frac{1}{2} \int_{\Gamma} b(u+v)(u+v) \, ds - \int_{\Omega} f(u+v) \, dx - \int_{\Gamma} bg(u+v) \, ds \\ &= \frac{1}{2} \int_{\Omega} \nabla u \cdot \nabla u + \frac{1}{2} \int_{\Omega} \nabla v \cdot \nabla v - \int_{\Omega} fu - \int_{\Omega} \log u \, dx + \int_{\Gamma} buv \, ds - \int_{\Gamma} fv \, ds - \int_{\Gamma} fbgv \, ds \\ &\quad + \frac{1}{2} \int_{\Omega} \nabla v \cdot \nabla v + \frac{1}{2} \int_{\Gamma} bv^2 \, ds = 0 \end{aligned}$$

$$F(u) + \frac{1}{2} \int_{\Omega} \nabla u \cdot \nabla v \geq F(v)$$

$$\Rightarrow F(u) \leq F(v) \quad \forall v \in C^1(\bar{\Omega})$$

Construction of basis functions in 2D:

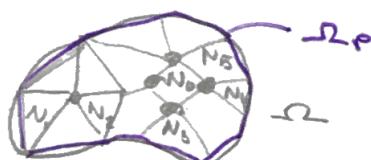
In 1-D case a function which is linear on an interval is uniquely determined by its values at 2-points (ex. endpoints)



Similarly: a plane in \mathbb{R}^3 is uniquely determined by three (non-linear) points. Therefore it is natural to make partitions of 2-dimensional domains using triangular elements and letting the sides of triangles to correspond to the end points of the intervals in 1-D case.

This partitioning is called "Triangulation"

Example: $\Omega \subset \mathbb{R}^2$, $\Omega_p \subset \Omega$
 ↑ polygon



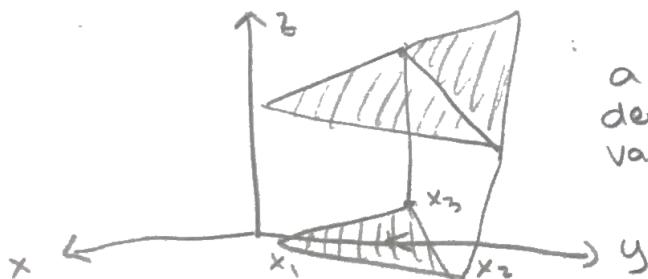
Here we have 6 internal nodes

Triangulations vary 1-principle:

If $k_1 \neq k_2$ tm triangles

then (I1) $k_1 \cap k_2 = \emptyset$

(I2) No vertex of a triangle k_i linear in a side k_j



a piecewise linear fct
determined by its node
values at v-ice of a triangle

Now for every linear function $u: \Omega_p \rightarrow \mathbb{R}$ we have

$$u(x) = u_1 e_1(x) + u_2 e_2(x) + \dots + u_6 e_6(x),$$

where $u_i = u_{(N_i)}$, $i = 1, 2, \dots, 6$ are real numbers

$e_i(N_j) = 1$, $e_i(N_j) = 0$ for $j \neq i$, e_i is card linear (affino)

i.e. $e_i(N_j) = \begin{cases} 1 & \text{if } j=i \\ 0 & \text{if } j \neq i \end{cases} = \delta_{ij}$ & $e_i = 0$ on $\partial\Omega_p$

so, given a (DE) we want to approx. its sol u by a FE sol u

⇒ Finding the numbers (real) u_i , $i = 1, 2, \dots, 6, \dots, m$.

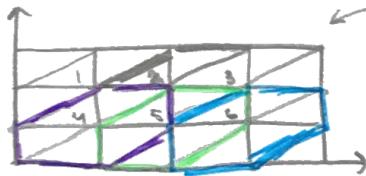
Example: Finite elements for poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega := \{(x,y) : 0 \leq x \leq 4 \text{ & } 0 \leq y \leq 3\} \\ u=0 & \text{on } \partial\Omega \end{cases}$$

FE: Find u vanishing at the boundary $\partial\Omega$:

$$= \{\text{Green}\} \Rightarrow \int_{\Omega} (f u \cdot \nabla v) dx = \int_{\Omega} f v dx \quad \forall v \in C^1(\Omega), v|_{\partial\Omega} = 0$$

we may triangulate Ω as in the figure.



and define the discrete function space

$V_h = \{ \text{space of all cont functions which are linear on each solutions \& 0 at the boundary} \}$

Since such a function (an element of V_h) is uniquely determined by its values at the vertices of the triangles & 0 at the boundary, so in this ex. we have only 6-points (discrete function values) of interest.

Therefore as in the "1D" case we construct basis fun (6 of them here) with values "1" at one of the nodes and "0" at others.

Then we get a two-dimensional mesh with "best" functions as above.

Solution of the equation problem:

④ (2011-08-24)

V is a function space defined by:

$V = \{v : v \text{ is cont in } \Omega, v=0 \text{ on } \partial\Omega\}$

(NF): By Green's formula we have $\langle -\Delta u, v \rangle = \langle f, v \rangle \quad \forall v \in V$

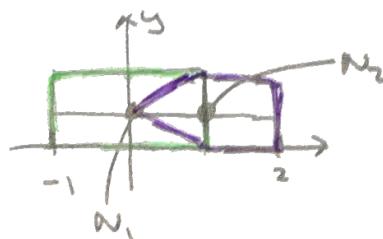
$$\Leftrightarrow (\nabla u, \nabla v)_{\Omega} - \left\langle \nabla u \cdot \frac{\partial v}{\partial \Omega} \right\rangle_{\partial\Omega} = \langle f, v \rangle$$

$$(VP) \quad (\nabla u, \nabla v) = \langle f, v \rangle \quad \forall v \in V$$

The FE-approximation:

Define: $V_h := \{v : v \text{ cont piecewise linear on a triangulation of } \Omega, v=0 \text{ at } \Gamma_h\}$

Then the (CGO) FEM reads on: Find $u \in V_h : (\nabla u, \nabla v) = \langle f, v \rangle \forall v \in V_h$



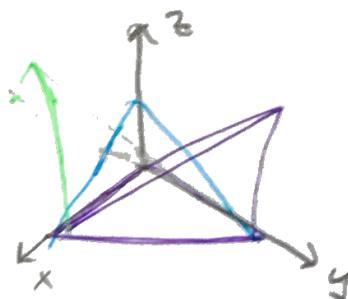
Let $u(x) = \sum_{j=1}^2 \xi_j \varphi_j(x)$, where $\varphi_j, j=1, 2, \dots$

$$\Rightarrow \sum_{j=1}^2 \xi_j \int_{\Omega} (\nabla \varphi_j \cdot \nabla \varphi_i) dx = \int_{\Omega} f \cdot \varphi_i dx \quad \xi_1 = ?, \quad \xi_2 = ?$$

$$\Leftrightarrow S \xi = \underline{F}, \quad S = \{S_{ij}\}_{i,j=1}^2, \quad \underline{F} = (F_i)_{i=1}^2, \quad F_i = \int_{\Omega} f \cdot \varphi_i$$

To proceed we need to compute the stiffness matrix for a standard triangle

a General standard element



The local basis function are:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$



$$\begin{cases} \Phi_1(x, y) = 1 - \frac{x}{h} - \frac{y}{h} \Rightarrow \nabla \Phi_1 = \begin{bmatrix} -1/h \\ -1/h \end{bmatrix} \\ \Phi_2(x, y) = \frac{x}{h} \Rightarrow \nabla \Phi_2 = \begin{bmatrix} 1/h \\ 0 \end{bmatrix} \\ \Phi_3(x, y) = \frac{y}{h} \Rightarrow \nabla \Phi_3 = \begin{bmatrix} 0 \\ 1/h \end{bmatrix} \end{cases}$$

$$\phi_2(x, y) = A + Bx + Cy$$

$$\phi_2(h, 0) \Rightarrow A + Bh = 1$$

$$\phi_2(0, 0) \Rightarrow A = 0$$

$$\phi_2(0, h) \Rightarrow Ch = 0 \Rightarrow C = 0$$

$$|T| = \int_T dx = \frac{h^2}{2}$$

$$S_{11} = (\nabla \phi_1, \nabla \phi_1) = \int_T |\nabla \phi_1|^2 = \frac{2}{h^2} \int_T dx = \frac{2}{h^2} \cdot \frac{h^2}{2} = 1$$

$$S_{22} = -\frac{1}{h^2} \int_T dx = -\frac{1}{h^2} \cdot \frac{h^2}{2} = -\frac{1}{2}$$

$$S_{13} = S_{21} = -\frac{1}{h}$$

$$S_{21} = S_{12}$$

$$S_{23} = 0 = S_{32}$$

$$S_{22} = \frac{1}{h^2} \int_T dx = \frac{1}{h^2} \cdot \frac{h^2}{2} = \frac{1}{2} = S_{33}$$

$$\Rightarrow \text{local stiffness matrix } S = \frac{1}{2} \begin{bmatrix} 2 & -1 & -1 \\ -1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

To compute global S note that

$$S_{11} = 8 S_{22} = (8S_{33}) = 4$$

$$S_{12} = 2 S_{12} = 2 \cdot -\frac{1}{2} = -1$$

$$\Rightarrow S = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$$

$$S_{21} = S_{12} = -1$$

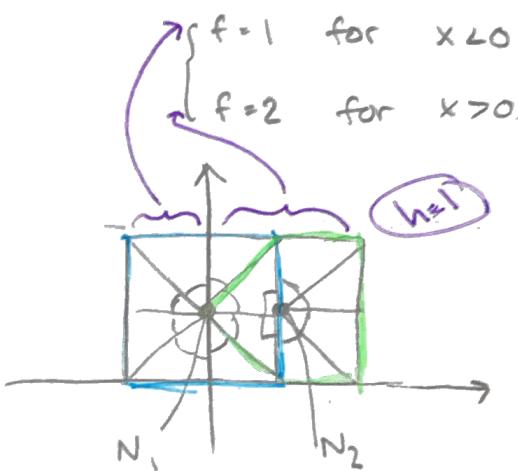
$$S_{22} = 2 S_{11} + 4 S_{22} = 2 + 2 = 4$$

Föreläsning 19/2:

2011-08-24: ④

- Load stiffness & mass matrices in 1 & 2D.
- Compare their generalization to global.
- Poisson equation in \mathbb{R}^d ($d=2$)
- Stability (weak, strong)
- \rightarrow Poincaré in 2D

④ $-\Delta u = f$ in $\Omega = [-1, 2] \times [0, 2]$



Approximate solution in the partition as in the figure: $u_h(x) = \xi_1 e_1(x) + \xi_2 e_2(x)$, $x = (x_1, y)$

$$e_i(N_j) = \begin{cases} 1 & \text{for } i=j \\ 0 & \text{for } i \neq j \end{cases} \quad \text{linear on each element and continuous.}$$

$$(\nabla F) \stackrel{\text{def}}{\Rightarrow} \int_{\Omega} \xi_1 e'_1(x) e'_i(x) dx + \int_{\Omega} \xi_2 e'_2(x) e'_i(x) dx ; \quad i=1, 2, \dots, \quad x = (x_1, y)$$

$$\begin{pmatrix} \int_{\Omega} e'_1(x) e'_1(x) dx & \int_{\Omega} e'_2(x) e'_1(x) dx \\ \int_{\Omega} e'_1(x) e'_2(x) dx & \int_{\Omega} e'_2(x) e'_2(x) dx \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \int_{\Omega} f e_1(x) dx \\ \int_{\Omega} f e_2(x) dx \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} (\nabla e_1, \nabla e_1) & (\nabla e_1, \nabla e_2) \\ (\nabla e_2, \nabla e_1) & (\nabla e_2, \nabla e_2) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

Start with standard element T :

Local basis:

$$\phi_1(x) = 1 - \frac{x}{h} - \frac{y}{h} \Rightarrow \nabla \phi_1 = -\frac{1}{h} [1 \ 1]$$

$$\phi_2(x) = \frac{x}{h} \Rightarrow \nabla \phi_2 = \frac{1}{h} [1 \ 0]$$

$$\phi_3(x) = \frac{y}{h} \Rightarrow \nabla \phi_3 = -\frac{1}{h} [0 \ 1]$$

Elements of local stiffness matrix

$$S_{11} = \int_T \nabla \phi_1 \cdot \nabla \phi_1 dx = \int_T \left(-\frac{1}{h}, -\frac{1}{h}\right) \cdot \left(\frac{1}{h}, -\frac{1}{h}\right) dx \\ = \frac{2}{h^2}$$

$$S_{12} = S_{21} = \int_T \nabla \phi_1 \cdot \nabla \phi_2 dx = -\frac{1}{h^2} \int_T dx = -\frac{1}{h^2} \cdot \frac{h^2}{2} = -\frac{1}{2} \quad \phi_2(0, h) = Bh + 1 = 0 \Rightarrow B = -\frac{1}{h}$$

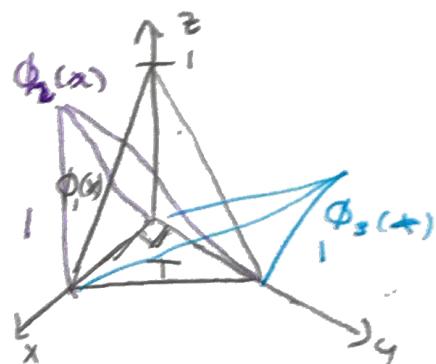
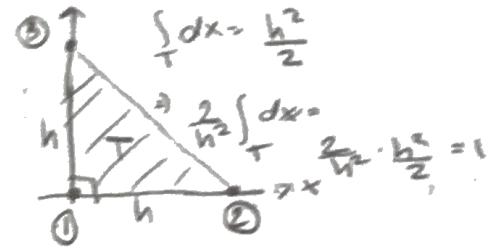
$$S_{13} = S_{31} = \dots = -\frac{1}{2}$$

$$S_{22} = \int_T \nabla \phi_2 \cdot \nabla \phi_2 dx = \int_T \left(\frac{1}{h}, 0\right) \cdot \left(\frac{1}{h}, 0\right) dx = \frac{1}{h^2} \cdot |T| = \frac{1}{h^2} \cdot \frac{h^2}{2} = \frac{1}{2}$$

$$S_{23} = S_{32} = \int_T \nabla \phi_2 \cdot \nabla \phi_3 dx = 0$$

$$S_{33} = \int_T \nabla \phi_3 \cdot \nabla \phi_3 dx = \frac{1}{2}$$

\Rightarrow local stiffness matrix: $S = \frac{1}{2} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$



$$\phi_1(0,0) = C = 1$$

$$\phi_1(h,0) = Ah + 1 = 0 \Rightarrow A = -\frac{1}{h}$$

The discrete problem: $S \xi = F$

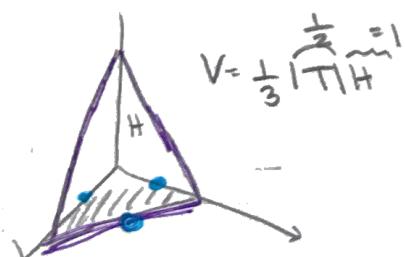
$$S_{11} = 8S_{22} = 8 \cdot \frac{1}{2} = 4$$

$$S_{12} = S_{21} = 2S_{12} = 2 \cdot \left(-\frac{1}{2}\right) = -1$$

$$S_{22} = 2S_{11} + 4S_{22} = 2 \cdot (1) + 4 \cdot \left(\frac{1}{2}\right) = 4$$

$$\Rightarrow S = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$$

$$F_1 = \int f e_1 dx = \int e_1 + \begin{cases} 2e_1 & x \leq 0 \\ 2e_1 + 2e_2 & x > 0 \end{cases} dx = 4 \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot 1 + 2 \cdot 4 \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{4}{6} + \frac{8}{6} = 2$$

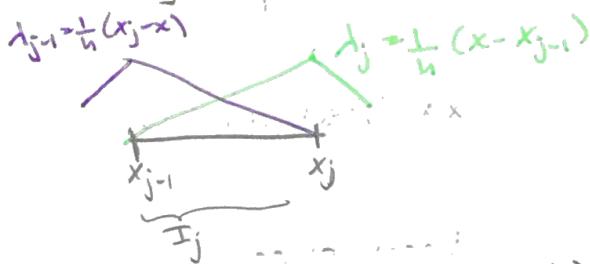


$$F_2 = \int_{-2}^2 f(x) dx = 2 \int_{x=0}^1 6 dx = 2 \cdot 6 \cdot \frac{1}{3} \cdot \frac{1}{2} = 1$$

$$S \xi = \# \Leftrightarrow \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \Rightarrow \xi_1 = \xi_2$$

1D versus 2D:

Starting with 1d local element (subinterval)



a linear u on I_j ($|I_j| = h$)

$$u(x) = u(x_{j-1}) \lambda_{j-1}(x) + u(x_j) \lambda_j(x)$$

$$\begin{cases} u'(x) = u'_{j-1} \lambda'_{j-1}(x) + u'_j \lambda'_j(x) \\ u_k = u(x_k) \quad k = j-1, j \end{cases}$$

For the stiffness matrix:

$$\int_{I_j} u'(x) \lambda'_i(x) dx = \left(\int_{I_j} \lambda'_{j-1}(x) \lambda'_{j-1}(x) \right) u_{j-1} + \left(\int_{I_j} \lambda'_{j-1}(x) \lambda'_j(x) \right) u_j$$

$$\sum_{i=j-1}^j \begin{pmatrix} \int_{I_j} \lambda'^2_{j-1} & \int_{I_j} \lambda'_{j-1} \lambda'_j \\ \int_{I_j} \lambda'_j \lambda'_{j-1} & \int_{I_j} \lambda'^2_j \end{pmatrix} \begin{pmatrix} u_{j-1} \\ u_j \end{pmatrix} = \frac{1}{h^2} \begin{pmatrix} h & -h \\ h & h \end{pmatrix} \frac{1}{h} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

Global stiffness matrix in 1d:

$$S = \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & \dots & & \\ -1 & 2 & -1 & \dots & & \\ 0 & -1 & 2 & -1 & & \\ \vdots & \vdots & \vdots & \ddots & & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}$$

Resonance effect
for the global
(neighboring
elements
contribution)

Similarly for the local massmatrix we have

$$m = \begin{pmatrix} \int_{I_j} \lambda_{j,1}^2(x) dx & \int_{I_j} \lambda_{j,1}(x) \lambda_{j,2}(x) dx \\ \dots & \int_{I_j} \lambda_{j,2}^2(x) dx \end{pmatrix} = \dots = \frac{h}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

\Downarrow

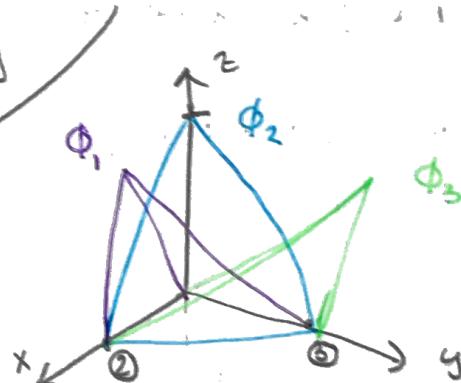
Global matrix in 1d: $M = \frac{h}{6} \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$

In the 2D-case: Standard element has the local basis functions.

$$\begin{cases} \phi_1(x,y) = 1 - \frac{x}{h} - \frac{y}{h} & \nabla \phi_1 = -\frac{1}{h} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \phi_2(x,y) = \frac{x}{h} & \nabla \phi_2 = \frac{1}{h} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \phi_3(x,y) = \frac{y}{h} & \nabla \phi_3 = \frac{1}{h} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$

$$S = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

Local stiffness matrix
in 2D (h-independant)



Local Mass-matrix:

$$m_{11} = (\phi_1, \phi_1) = \int_T \phi_1(x)^2 dx \stackrel{h=1}{=} h^2 \int_0^1 \left(\int_0^{1-x} (1-x-y)^2 dy \right) dx = \frac{h^2}{12}$$

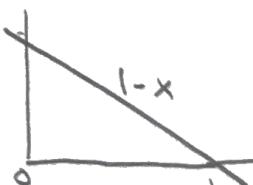
Alt: use mid-point rule (exact for poly & degree 2)

$$m_{11} = \int \phi_1^2 dx = \frac{h}{3} \sum_{j=1}^3 \phi_1(\tilde{x}_j)^2 = \frac{1}{3} \cdot \frac{h^2}{2} (0 + \frac{1}{4} + \frac{1}{4}) = \frac{h^2}{12}$$

$$m_{12} = (\phi_1, \phi_2) = h^2 \int_0^1 \left(\int_0^{1-x} (1-x-y) x dy \right) dx = \dots = \frac{h^2}{24}$$

Other are computed similarly $\Rightarrow \dots \Rightarrow$

$$\Rightarrow m = \frac{h^2}{24} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$



The poisson equation in $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^2 \\ u=0 & \text{on } \partial\Omega \end{cases}$$

Ω : bdd, polynomial (convex)

The Green's formula $\Rightarrow \{ \text{if } u=0 \text{ on } \Gamma \}$

$$(1) \|\nabla u\|^2 = \int_{\Omega} fu \leq \{C-S\} \leq \|f\| \|u\|$$

$$\|u\|^2 = \int_{\Omega} |u(x)|^2 dx$$

Poincaré inequality in 2D: $u \in C^1(\bar{\Omega})$

$$\|u\| \leq C_{\Omega} \|\nabla u\|$$

Proof: Let e be a function such that $\begin{cases} \Delta e = 1 & \text{in } \Omega \\ 2|\nabla e| \leq C_{\Omega} & (\text{size of } \Omega) \end{cases}$

$$\text{Then } \|u\|^* = \int_{\Omega} u^2 (\Delta e) \stackrel{\text{Green's}}{=} - \int_{\Omega} 2u (\nabla u \cdot \nabla e) dx \leq C_{\Omega} \|u\| \|\nabla u\|. \Rightarrow \text{pf } \square$$

It remains to find such a e . Define $e(x,y) = \frac{1}{4}(x^2+y^2)$

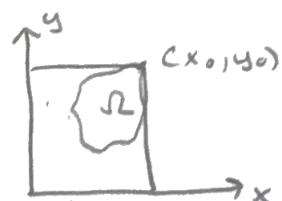
$$\Rightarrow \nabla e = (e_x, e_y) = \frac{1}{4}(2x, 2y) = \frac{1}{2}(x, y)$$

$$\Rightarrow 2|ue| = |(x,y)| = \sqrt{x^2+y^2} = \text{diam } C_{\Omega} = C_{\Omega}$$

$$\text{Also } \Delta e = e_{xx} + e_{yy} = \frac{1}{2} + \frac{1}{2} = 1.$$

$$\textcircled{1} + \text{Poincaré} \Rightarrow \|u\|^* \leq \|f\| \|u\| \leq \|f\| C_{\Omega} \|\nabla u\|$$

$$\Rightarrow \textcircled{2} \|\nabla u\| \leq C_{\Omega} \|f\| \quad (\text{weak-stability.})$$



Föreläsning 20/2:

[2011-08-24]

(GCI) for the poisson equation:

The equation: $\begin{cases} -\Delta u = f \text{ in } \Omega \subset \mathbb{R}^2 \\ u=0 \text{ on } \Gamma := \partial\Omega \end{cases}$

Green's $\Rightarrow \| \nabla u \|^2 = \int_{\Omega} fu$

$$(\Delta u, u) := -\langle \nabla u \cdot \nu, u \rangle_{\Gamma} + (\nabla u, \nabla u) \quad \text{(Lebesgue)}$$

Poincaré: $u \in C^1(\Omega)$ & $u=0$ on $\Gamma_c \subset \Gamma$ with $|\Gamma_c| \neq 0$

Then $\|u\|_1 \leq C_{\Omega} \|\nabla u\|$

\Rightarrow weak-stability Poincaré

$$\|\nabla u\| \stackrel{\textcircled{1}}{\leq} \|f\| \|u\| \leq C_{\Omega} \|\nabla u\|$$

$\boxed{\|\nabla u\| \leq C_0 \|f\|} \quad \textcircled{2}$

I Gradient estimate:

Assume that Ω is a polygonal domain. Then we have:

$$(NF) \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} fv, \forall v: v \in H_0^1(\Omega)$$

Finite element method: Let $T_h := \{K: \cup K = \Omega\}$ such that

i) $x_i \cap x_j$ for $i \neq j$



Not OK!

ii) No vertex of a triangle (an element) lies on a side of another



OK!



Not OK!

Let now e_j , $j=1, 2, \dots, m$ be a corresponding basis function for the m interior nodes of the triangulation. i.e. $e_j(x)$ is continuous piecewise linear (linear on each triangle) and $e_j(N_i) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

To proceed set the approx solution:

$$u(x) = u_1 \varphi_1(x) + u_2 \varphi_2(x) + \dots + u_m \varphi_m(x)$$

where $u_i = u(N_i)$ $i = 1, 2, \dots, m$

Then (FEM): $\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f v dx \quad \forall v \in \tilde{V}_h$

where $\tilde{V}_h = \{v: v \text{ is continuous piecewise linear on the partition } \mathcal{T}_h \text{ & } v=0 \text{ on } \partial\Omega\}$

Define the error: $e := u - u_h$

then (VF) - (FEM) $\Rightarrow \boxed{\int_{\Omega} \nabla e \cdot \nabla v = 0, \forall v \in \tilde{V}_h} \quad (G^+)$

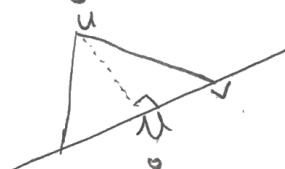
Thus for $v \in \tilde{V}_h$ we can write

$$\begin{aligned} \|e\| &= \sqrt{\int_{\Omega} \nabla e \cdot \nabla e} = \int_{\Omega} \nabla e \cdot \nabla u - \int_{\Omega} \nabla e \cdot \nabla u_h = \int_{\Omega} \nabla e \cdot \nabla u - \int_{\Omega} \nabla e \cdot \nabla v \\ &= \int_{\Omega} \nabla e \cdot \nabla (u-v) \leq \{c-s\} \leq \|\nabla e\| \|\nabla (u-v)\| \end{aligned}$$

$$\|e\| \leq \|\nabla (u-v)\| \quad \forall v \in \tilde{V}_h \quad (5)$$

$$\|\nabla (u-v)\| \leq \|\nabla (u-v)\|$$

i.e. u_h is closest to u among all other $v \in \tilde{V}_h$



Now again as in 1-D case,

it is possible to show that $\exists v \in \tilde{V}_h$ s.t.

$$\|\nabla (u-v)\| \leq c \|h D^2 u\|$$

a typical choice of such "v" is $v = \pi_h u$ (9)

h : mesh parameter

$$\Rightarrow \|\nabla e\| \leq c \|h D^2 u\| \quad (5)$$

$$D^2 u = (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)^{1/2}, \text{ while } \|D^2 u\|^2 = u_{xx}^2 + u_{yy}^2 + 2u_{xx} u_{yy}$$

$h = \max_{\Omega} \text{diam}(x_i)$

To proceed set the approx solution:

$$u(x) = u_1 e_1(x) + u_2 e_2(x) + \dots + u_m e_m(x)$$

where $u_i \in V_h$, $i = 1, 2, \dots, m$

Then (FEM): $\int_{\Omega} \nabla u \cdot \nabla v - \int_{\Omega} f v dx \quad \forall v \in V_h$:

where $V_h = \{v: v \text{ is continuous piecewise linear on the partition } T_h \text{ & } v=0 \text{ on } \partial\Omega\}$

Define the error, $e := u - u_h$

then (F) - (FEM) $\rightarrow \boxed{\int_{\Omega} \nabla e \cdot \nabla v = 0, \forall v \in V_h}$ (G⁺)

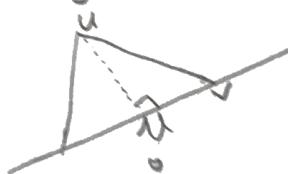
Thus for $v \in V_h$ we can write

$$\begin{aligned} \|e\|_h^2 &= \int_{\Omega} \nabla e \cdot \nabla (u - u_h) = \int_{\Omega} \nabla e \cdot \nabla u - \int_{\Omega} \nabla e \cdot \nabla u_h = \int_{\Omega} \nabla e \cdot \nabla u - \int_{\Omega} \nabla e \cdot \nabla v \\ &= \int_{\Omega} \nabla e \cdot \nabla (u - v) \leq \|e\|_h \| \nabla (u - v) \| \end{aligned}$$

$$\|e\|_h \leq \| \nabla (u - v) \| \quad \forall v \in V_h \quad \textcircled{5}$$

$$\| \nabla (u - v) \| \leq \| \nabla (u - v) \|$$

i.e. u_h is closest to u among all other $v \in V_h$



Now again as in 1-D case,

it is possible to show that $\exists v \in V_h$ s.t.

$$\| \nabla (u - v) \| \leq C h^2 \| u \|$$

a typical choice of such "v" is $v = \pi_h u$ $\textcircled{6}$

$$\Rightarrow \|e\|_h \leq C h^2 \|u\| \quad \textcircled{5}$$

$$D^2 u = (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)^{1/2}, \text{ while } \|Du\|^2 = u_{xx}^2 + u_{yy}^2 + 2u_{xx}u_{yy}$$

h : mesh parameter

$$h = \max_{\Omega} \text{diam}(\omega)$$

Estimate of the solution error:

Let e be the solution of the dual problem:

$$\begin{cases} -\Delta e = e & \text{in } \Omega, e = u - u \\ e = 0 & \text{on } \partial\Omega \end{cases}$$

Then, $\forall v \in V_h$: Green's

(G+)

$$\|e\|_h^2 = \int_{\Omega} e(-\Delta e) = \int_{\Omega} \nabla e \cdot \nabla e = \int_{\Omega} \nabla e \cdot \nabla(e-v) \stackrel{\text{c.s.}}{\leq} \|\nabla e\| \|\nabla(e-v)\| \leq \left\{ \begin{array}{l} \text{choose} \\ \text{with } v \end{array} \right\}$$

$$\leq C \|\nabla e\| (\max \alpha \|D^2 e\|) \stackrel{\text{claim}}{\leq} C C_{\Omega} \|h D^2 u\| \max(h) C_{\Omega} \|\Delta u\|$$

$$\Rightarrow \{-\Delta e = e\} \Rightarrow \|e\|_h \leq C C_{\Omega} \|h D^2 u\| \max(h) \quad \cancel{\|\Delta u\|}$$

$$\Rightarrow \|e\|_h \leq C C_{\Omega} (\max(h)) \|h D^2 u\|$$

$$= (\max(h)) \|D^2 u\| = (\max(h)) C_{\Omega} \|\Delta u\|$$

$$\Rightarrow \|e\|_h \leq C C_{\Omega}^2 (\max(h))^2 \|f\|$$

Strong stability

Lemma: For Ω with no re-entrants $u \in H^2(\Omega)$ (actually you used more)

$$u=0 \quad (\text{or} \quad \frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n} = 0) \quad \text{on } \partial\Omega$$

$$\|D^2 u\| \leq C_{\Omega} \|\Delta u\|$$

$$(D^2 u)^2 = (u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2)$$

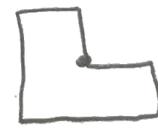
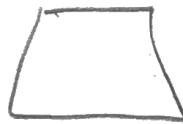
Remark:

$$C_{\Omega} \leq 1$$

$$C_{\Omega} \geq 1$$

$$C_{\Omega} \rightarrow \infty$$

convex
 Ω



Proof: Let Ω be a rectangle

$$\|u\|_h^2 = \int_{\Omega} (u_{xx} + u_{yy})^2 dx dy = \int_{\Omega} u_{xx}^2 + u_{yy}^2 + 2u_{xx}u_{yy}$$

$$n_x = 0 \quad u_x = 0$$

$$n_x = 1$$

$$n_y = 1$$

$$u_{yy} = 0$$

$$n_y = 0 \quad n_y = -1$$

$$u_x = 0$$

By Green's formula:

$$\begin{aligned} \int_{\Omega} u_{xx} u_{yy} &= \int_{\partial\Omega} u_{yy} (u_x \cdot n_x) - \int_{\Omega} u_x \cdot \underbrace{u_{yyx}}_{=u_{xy}} = \{ \text{another Green's} \} = \\ &\quad \text{on horizontal part} \quad \text{on vertical part} \\ &= \int_{\partial\Omega} (u_x n_x u_{yy} - u_x u_{xy} n_y) + \int_{\Omega} u_{xy} u_{yy} \\ &\quad \text{= 0 everywhere} \end{aligned}$$

$$\Rightarrow \{ \text{for } \Omega = \square \} \quad \boxed{\int_{\Omega} u_{xx} u_{yy} = \int_{\Omega} u_{xy}^2} \quad (*)$$

$$(*) \Rightarrow \|\nabla^2 u\| = \|A u\| \quad (\text{in this geometry})$$

An posterior error for estimate for the poisson equation that also yields strong stability:

[For some simplicity just consider the 1D-case]

Dual problem: $\begin{cases} -e'' = e, \quad 0 < x < 1 \\ e(0) = e(1) = 0 \end{cases}$

$$\text{Let } e = u - u_1, \text{ then } r(u_1) = f + u'' - u'' + e'' = -(u - u_1)'' = -e''$$

$$\text{Then } \|e\|^2 = \int_{\Omega} e(-e'') = \int_{\Omega} e'e' \stackrel{\text{PI}}{=} \int_{\Omega} e'(u-v)' = \int_{\Omega} (-e'')(u-v) =$$

$$\int_{\Omega} \frac{h^2}{h^2} r(u-v) \leq \|e\|_3 \leq \|h^2 r\| \|h^2(u-v)\| \leq \|h^2 u\| \underbrace{\|h^{-2}(e - \Pi_h e)\|}_{\leq c_i \|e''\|} =$$

\uparrow
 { if v is an
interpolant of e }

$$\Rightarrow \|e\| \leq c_i \|h^2 r\| = \begin{cases} \text{if } u \text{ approx w} \\ \text{is piecewise linear,} \\ \text{then } v = f \end{cases} = \underline{c_i \|h^2 f\|}$$

$$\underline{\|e\| \leq c_i \|h^2 f\|}$$

A better strong stability

Both a prior & a posterior E.E. in here are asked in problem ⑤

2011-08-24

Föreläsning 24/2:

problem 27: if $u(x) = \log \frac{1}{|x|}$, $x = (x, y)$, $x \neq 0$, then $\Delta u = 0$

(problem 26: $-\Delta u = \text{rot}(\text{rot } u)$)

$$\text{Alt 1: } \Delta u = u_{xx} + u_{yy} = \frac{\partial^2}{\partial x^2} \left(\log \frac{1}{|x|} \right) + \frac{\partial^2}{\partial y^2} \left(\log \frac{1}{|x|} \right) \stackrel{\text{last time}}{=} 0$$

Alt 2: (via problem 26)

If $u(x, y) = (u_1, u_2)^T$ a vector then $\text{rot } u := \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}$ is a scalar

If $u(x, y)$ is scalar, then $\text{rot } u := \left(\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x} \right)$ is a vector.

$$\text{Now } u(x, y) = \log \frac{1}{\sqrt{x^2+y^2}} = \log w, \quad w = (x^2+y^2)^{-1/2}$$

$$(\text{obs! } u: \mathbb{R}^2 \rightarrow \mathbb{R}) \Rightarrow \begin{cases} u_x = \frac{w_x}{w} \\ u_y = \frac{w_y}{w} \end{cases}$$

$$w = (x^2+y^2)^{-1/2} \Rightarrow \begin{cases} w_x = -\frac{1}{2} (2x) (x^2+y^2)^{-3/2} = x(x^2+y^2)^{-3/2} \Rightarrow u_x = \frac{w_x}{w} = \frac{-x(x^2+y^2)^{-3/2}}{(x^2+y^2)^{-1/2}} \\ w_y = -y (x^2+y^2)^{-3/2} \Rightarrow u_y = \frac{w_y}{w} = \frac{-y(x^2+y^2)^{-3/2}}{(x^2+y^2)^{-1/2}} \end{cases}$$

$$u = \log \frac{1}{|x|} \Rightarrow \underbrace{\text{rot } u}_{\text{a scalar}} = \left(\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x} \right) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) \text{ a vector}$$

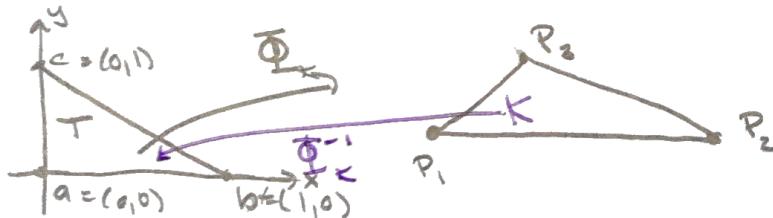
$$-\Delta u = \text{rot}(\text{rot } u) = \text{rot} \tilde{u} = \frac{\partial}{\partial x} \left(\frac{x}{x^2+y^2} \right) + \frac{\partial}{\partial y} \left(\frac{y}{x^2+y^2} \right)$$

$$\Rightarrow \left\{ \left(\frac{f}{g} \right)' = \frac{f'g - fg'}{g^2} \right\} \Rightarrow -\Delta u = \text{rot}(\text{rot } u) = \text{rot}(\tilde{u}) = \frac{1 \cdot (x^2+y^2) - 2x \cdot x + 1 \cdot (x^2+y^2) - 2y \cdot y}{(x^2+y^2)^2} = 0$$

Remark: $-\Delta (\log \frac{1}{|x|}) = 0 \Rightarrow \log \frac{1}{|x|}$ is a harmonic function. \square

10.8 Express the basis functions $\lambda_1, \lambda_2, \lambda_3$ on a triangle K with nodes P_1, P_2, P_3 .

Solution: Consider the "unit" standard triangle T :



The standard basis are:

$$\begin{cases} \phi_1 = 1 - x - y \\ \phi_2 = x \\ \phi_3 = y \end{cases} \quad / \text{basis functions for } T$$

OBS! $\Phi_K: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$\Rightarrow \begin{cases} \Phi_K(a) = P_1 \\ \Phi_K(b) = P_2 \\ \Phi_K(c) = P_3 \end{cases} \xrightarrow{\Phi_K} (\underbrace{\vec{P}_2 - \vec{P}_1, \vec{P}_3 - \vec{P}_1}_{:= A}) \mathbf{x} + \vec{P}_1$$

Let now $B = A^{-1}$, then $\Phi_K^{-1}(x) = A^{-1}(x - \vec{P}_1) = B(x - \vec{P}_1)$ and the basis functions $\lambda_1, \lambda_2, \lambda_3$ are given by $\lambda_i(x) = P_i(\Phi_K^{-1}(x))$, $i=1,2,3$.

10.9 Basic properties of the basis functions: Let $P_i = \begin{pmatrix} (P_i)_1 \\ (P_i)_2 \end{pmatrix}$.

$$(a) \sum_{i=1}^3 \lambda_i(x) = 1$$

$$(b) \sum_{i=1}^3 ((P_i)_j - x_j) \lambda_i(x) = 0, \quad j=1,2, \dots \quad x = (x_1, x_2)$$

$$(c) \sum_{i=1}^3 \frac{\partial}{\partial x_k} \lambda_i(x) = 0$$

$$(d) \sum_{i=1}^3 ((P_i)_j - x_j) \frac{\partial \lambda_i}{\partial x_k}(x) = \delta_{jk}, \quad j,k = 1,2, \dots$$

Solution: 1st observe that $\Phi_K^{-1}(x) = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} x_1 - (P_1)_1 \\ x_2 - (P_1)_2 \end{pmatrix} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$

$$\Rightarrow \frac{\partial(\Phi_K^{-1})_j}{\partial x_k} = P_{jk}, \quad j,k = 1,2, \dots$$

$$\Rightarrow (a): \sum_{i=1}^3 \lambda_i(x) = \sum_{i=1}^3 \Phi_i(\tilde{x}) = 1 - \cancel{\tilde{x}_1} - \cancel{\tilde{x}_2} + \cancel{\tilde{x}_1} + \cancel{\tilde{x}_2} = 1$$

(b): Consider the function: $f_j(\tilde{x}) = x_j$, $j=1,2, \dots$ (component function)

Taylor's formulation in 2D: $f_j(P_i) = f_j(x) + \nabla f_j(x) \begin{pmatrix} (P_i)_1 - x_1 \\ (P_i)_2 - x_2 \end{pmatrix}$

OBS! (No remainders due to piecewise linear approx)

$$\begin{aligned} \text{Linear interpolation of } f_j: \text{trif}_j(x) &= \sum_{i=1}^3 f_j(P_i) \lambda_i(x) = \sum_{i=1}^3 [f_j(x) + (P_i)_j - x_j] \lambda_i(x) \\ &= \sum_{i=1}^3 f_j(x) \lambda_i(x) + \sum_{i=1}^3 ((P_i)_j - x_j) \lambda_i(x) = f_j(x) \underbrace{\sum_{i=1}^3 \lambda_i(x)}_{=1 \text{ by (a)}} + \sum_{i=1}^3 (P_i)_j \lambda_i(x) \end{aligned}$$

$$\Rightarrow \sum_{i=1}^3 ((P_i)_j - x_j) \lambda_i(x) = 0 \rightarrow (b)$$

$$(c) \sum_{i=1}^3 \frac{\partial \lambda_i}{\partial x_k}(x) = \sum_{i=1}^3 \left(\underbrace{\partial_1 \phi_i(\bar{\Phi}_k^{-1}(x))}_{\partial_1 \phi_1 = 1} \underbrace{\frac{\partial (\bar{\Phi}_k^{-1}(x))}{\partial x_k}}_{\beta_{1k}} + \underbrace{\partial_2 \phi_i(\bar{\Phi}_k^{-1}(x))}_{\partial_2 \phi_2 = 0} \underbrace{\frac{\partial (\bar{\Phi}_k^{-1}(x))}{\partial x_k}}_{\beta_{2k}} \right)$$

$\phi_1 = 1 - x_1 - x_2$
 $\phi_2 = x_1$
 $\phi_3 = 0$
 $\partial_1 \phi_3 = 0$
 $\partial_2 \phi_3 = 1$

$$\partial \Phi_3 = x_2 = \dots = 0$$

(d) Once again we consider $f_j(x) = x_j$, $j=1, 2, \dots$

$$f_j(x) = \{ f_j : \text{linear} \} = (\Pi f_j)(x) = \sum_{i=1}^3 f_j(P_i) \lambda_i(x)$$

$$\Rightarrow \frac{\partial f_j(x)}{\partial x_k} = \sum_{i=1}^3 \boxed{f_j(P_i)} \frac{\partial \lambda_i}{\partial x_k}(x) = \sum_{i=1}^3 f_j(x) \frac{\partial \lambda_i}{\partial x_k}(x) + \sum_{i=1}^3 ((P_i)_j - x_j) \frac{\partial \lambda_i}{\partial x_k}(x) =$$

$\boxed{f_j(x) + \nabla f_j(x) \begin{pmatrix} (P_i)_1 - x_1 \\ (P_i)_2 - x_2 \end{pmatrix} = (P_i)_j - x_j}$

$$= f_j(x) \sum_{i=1}^3 \cancel{\frac{\partial x_j}{\partial x_k}}(x) + \sum_{i=1}^3 ((P_i)_j - x_j) \frac{\partial \lambda_i}{\partial x_j}(x) \stackrel{=0}{\cancel{\rightarrow}} (d)$$

Chapter 10: Consider the interpolation error in 2D.

$$f(x) - \Pi f(x) = - \sum_{i=1}^3 r_i(x) \lambda_i(x)$$

Here λ_i :s are the basis function & $r_i(x)$ is the remainder?

Show that

$$|r_i(x)| \leq \frac{1}{2} h_k \|D^2 f\|_{L_\infty(\mathbb{R}^2)}, \quad i=1, 2, 3, \dots$$

Solution: for $f \in C^2(\mathbb{R}^2)$, By Taylor's expansion we have

$$f(P_i) = f(x) + \nabla f(x) \cdot \begin{pmatrix} (P_i)_1 - x_1 \\ (P_i)_2 - x_2 \end{pmatrix} + \sum_{j,k=1}^2 \frac{1}{2} \partial_j \partial_k f(x) ((P_i)_j - \eta_{jk}) ((P_i)_k - \xi_k)$$

$\stackrel{=: L_i(x)}{\underline{=}}$ $\stackrel{=: r_i(x)}{\underline{=}}$

η_{jk}, ξ_k , $j, k = 1, 2$, are segments connecting P_i & x .

We plug this term in the linear interpolant of f :

$$(\pi f)(x) = \sum_{i=1}^3 f(P_i) \lambda_i(x) = \sum_{i=1}^3 (f(x) + \lambda_i(x) r_i(x)) \lambda_i(x)$$

We have that

$$\cdot \sum_{i=1}^3 f(x) \lambda_i(x) = f(x) \sum_{i=1}^3 \lambda_i(x) = f(x) \quad \text{by 10.9 (b)}$$

$$\cdot \sum_{i=1}^3 \lambda_i(x) \lambda_i(x) = \sum_{i=1}^3 (\partial_1 f)(x) (P_i)_1 - x_1 \lambda_i(x) + (\partial_2 f)(x) (P_i)_2 - x_2 \lambda_i(x) \equiv 0$$

$$\Rightarrow f(x) - (\pi f)(x) = - \sum_{i=1}^3 r_i(x) \lambda_i(x)$$

$$\Rightarrow \{ \text{by def of } r_i(x) \} \Rightarrow |r_i(x)| \leq \frac{1}{2} \sum_{j,k=1}^2 \underbrace{|\partial_j \partial_k f(x)|}_{\leq \|D^2 f\|_{L_\infty}} \underbrace{|(P_i)_j - x_j|}_{\leq h} \underbrace{|(P_i)_k - x_k|}_{\leq h}$$

$$\Rightarrow |f(x) - \pi_h f(x)| \leq \frac{1}{2} h^2 \|D^2 f\|_{L_\infty}$$

$$\text{Gradient estimate: } \textcircled{I} \quad (\nabla(\pi f))(x) = \sum_{i=1}^3 \boxed{f(P_i)} \nabla \lambda_i(x) = \sum_{i=1}^3 (f(x) + \lambda_i(x) r_i(x)) \nabla \lambda_i(x)$$

$$\Rightarrow \sum_{i=1}^3 \boxed{f(x)} \nabla \lambda_i(x) = 0 \quad \textcircled{II}$$

~~$$\sum_{i=1}^3 \lambda_i(x) \nabla \lambda_i(x) = \sum_{i=1}^3 [(\partial_1 f)(x) (P_i)_1 - x_1 + (\partial_2 f)(x) (P_i)_2 - x_2] \begin{pmatrix} \nabla \lambda_i(x) \\ \nabla \lambda_i(x) \end{pmatrix}$$~~

$$= \begin{pmatrix} \partial_1 f(x) \\ \partial_2 f(x) \end{pmatrix} = \nabla f(x) \quad \text{(III)}$$

Plug \textcircled{II} & \textcircled{III} into $\textcircled{I} \Rightarrow$

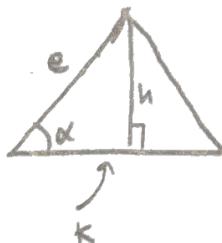
$$\nabla(f(x) - \pi f(x)) = - \sum_{i=1}^3 r_i(x) \nabla \lambda_i(x)$$

Föreläsning 26/2:

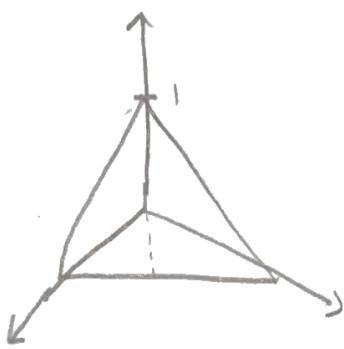
10.11 Estimate $|\nabla \lambda_1(x)|$



$$|\nabla \lambda_1(x)| = \frac{1}{h}$$



$$\Rightarrow \frac{\sin \alpha}{l} = \frac{1}{h} \Rightarrow |\nabla \lambda_1| = \frac{1}{h} = \frac{1}{\text{Graude}}$$



$$e \geq \frac{h_k}{2}$$

$$\Rightarrow |\nabla \lambda_1| \leq \frac{2}{h_k} \text{tsin}\alpha \quad \alpha: \text{smallest angle of element } k$$

The heat equation in \mathbb{R}^n ($n=2$):

- Stability
- Semi-discrete: error estimate
↑ discretization in special domain.
- Error estimate

P {L₂-projection
- Ritz-projection}

$$(DE) \quad \begin{cases} \dot{u} - \Delta u = f & \text{in } \Omega \subseteq \mathbb{R}^d \\ u = 0 & \text{on } \Gamma: \partial\Omega \end{cases} \quad (d=1, 2, 3)$$

$$(BV) \quad u(x, 0) = u_0(x) \quad \text{for } x \in \Omega$$

$$(I) \quad u(x, t) = u_0(x) + \int_0^t \dot{u}(x, s) ds$$

$$\dot{u} = \frac{du}{dt}$$

Stability: Multiply by u and $\int_{\Omega} \dots dx$ & use Green's

$$\Rightarrow (II) \quad \int_{\Omega} \dot{u} u dx - \int_{\Omega} (\Delta u) u dx = 0 \Rightarrow \left\{ \dot{u} u = \frac{1}{2} \frac{d}{dt} (u^2) \right\}$$

$$\Rightarrow (I) \Leftrightarrow$$

$$(2) \frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla u|^2 = 0 \Leftrightarrow \frac{1}{2} \frac{d}{dt} \|u\|^2 + \|\nabla u\|^2 = 0$$

$$\stackrel{t \rightarrow s}{\Rightarrow} \frac{1}{2} \|u\|^2(t) - \frac{1}{2} \|u\|^2(0) + \int_0^t \|\nabla u\|^2 ds \leq 0$$

$$(\text{obs: } \|u\|(t) := \left(\int_{\Omega} |u(x,t)|^2 dx \right)^{\frac{1}{2}})$$

$$\|u\|^2(t) + 2 \int_0^t \|\nabla u\|^2 ds = \|u_0\|^2$$

$$(3a) \boxed{\|u(t)\| \leq \|u_0\|}$$

$$(3b) \boxed{\int_0^t \|\nabla u\|^2 ds \leq \frac{1}{2} \|u_0\|^2}$$

To obtain further stabilities multiply (DE) by $-t \Delta u$

$$\Rightarrow (4) -t \int_{\Omega} \tilde{u} \Delta u + t \int_{\Omega} (\Delta u)^2 dx = 0$$

$$\text{Greens on 1st term} \Rightarrow \int_{\Omega} \tilde{u} \Delta u = - \int_{\Omega} \nabla \tilde{u} \cdot \nabla u = \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u|^2 dx$$

$$\Rightarrow (4) \Leftrightarrow t \frac{d}{dt} \frac{1}{2} \|\nabla u\|^2 + t^2 \|\Delta u\|^2 = 0$$

$$\Leftrightarrow \frac{d}{dt} (t \|\nabla u\|^2) + 2t \|\Delta u\|^2 = \|\nabla u\|^2$$

$$\stackrel{t \rightarrow s}{\Rightarrow} \int_0^t \frac{d}{ds} (s \|\nabla u\|^2) ds + 2 \int_0^t s \|\Delta u\|^2 ds = \int_0^t \|\nabla u\|^2 ds \stackrel{(3b)}{\leq} \frac{1}{2} \|u_0\|^2$$

$$t \|\nabla u\|^2(t) + 2 \int_0^t s \|\Delta u\|^2 ds \leq \frac{1}{2} \|u_0\|^2$$

$$\Rightarrow (5a) \boxed{\|\nabla u\|(t) \leq \frac{1}{\sqrt{2t}} \|u_0\|}$$

$$\Leftrightarrow (5b) \boxed{\left(\int_0^t s \|\Delta u\|^2 ds \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{2t}} \|u_0\|}$$

$$\text{Analogously: (6) } \|\Delta u\| \leq \frac{1}{\sqrt{2t}} \|u_0\|$$

Evidently (5a), & (6) below up as $t \rightarrow 0$

Following its estimate for t avoiding '0'.

$$\begin{aligned}
 \int_{\varepsilon}^t \|u\| ds &= \int_{\varepsilon}^t \|\Delta u\| ds = \int_{\varepsilon}^t \frac{1}{\sqrt{s}} \sqrt{s} \|\Delta u\|(s) ds \\
 &\leq \{C \cdot s\} \leq \left(\int_{\varepsilon}^t \frac{1}{s} ds \right)^{1/2} \cdot \underbrace{\left(\int_{\varepsilon}^t s \|\Delta u\|^2(s) ds \right)^{1/2}}_{\leq \frac{1}{2} \|u_0\|} \\
 &\leq \frac{1}{2} \sqrt{\ln\left(\frac{t}{\varepsilon}\right)} \|u_0\|
 \end{aligned}$$

Weak formulation using Green's:

$$(Vf) (u_t, v) + (\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$$

Let \mathcal{T}_h be a partition (triangulation) of Ω with interior nodes $\{x_j\}_{j=1}^M$

Def. $\overset{\circ}{V_h} := \{x : x \text{ is continuous piecewise linear on } \mathcal{T}_h \text{ & } x=0 \text{ on } \partial\Omega\}$
 Let $\{\psi_j\}_{j=1}^M$ be the standard basis for $\overset{\circ}{V_h}$

The p-L. FE approx.: For each t , find $u_h(\cdot, t) \in \overset{\circ}{V_h}$ s.t.

$$\begin{cases} (u_{h,t}, x) + \nabla(u_h, \nabla x) = (f, x) \quad \forall x \in \overset{\circ}{V_h} \\ u_h(x, 0) = u_{0,h}(x) \end{cases}$$

(FEM) \Rightarrow To find the time-dependent coefficients $\zeta_j(t)$ in

$$u(x, t) = \sum_{j=1}^M \zeta_j(t) \psi_j(x) \text{ such that } \sum_{j=1}^M \zeta_j(t) (\psi_j, \psi_i) + \sum_{j=1}^M \zeta_j(t) (\nabla \psi_j, \nabla \psi_i)$$

$$= (f(t), \psi_i) \quad i=1, 2, \dots, M$$

Matrix form:

$$\Leftrightarrow A \zeta' + S \zeta = b(t) \quad \zeta_j(0) = \eta_j \quad (\eta_j \text{ are model-values of } u_{0,h})$$

where $\begin{cases} A = (a_{ij})_{i,j=1}^M = ((\psi_i, \psi_j))_{i,j=1}^M = \left(\int_{\Omega} \psi_i(x) \psi_j(x) dx \right)_{i,j=1}^M \text{ is the mass matrix} \\ S = (s_{ij})_{i,j=1}^M = ((\nabla \psi_i, \nabla \psi_j))_{i,j=1}^M \text{ is the stiffness matrix} \\ b = (b_i)_{i=1}^M, b_i = (f, \psi_i), i=1, \dots, M. \end{cases}$

A and S are symmetric positive definite & in particular
 A is invertible

$$\begin{array}{l} \text{WP} \\ \bar{A}^{-1} \rightarrow \left\{ \begin{array}{l} \dot{\xi}(t) + (A^{-1}S)\xi(t) = A^{-1}b(t) \\ \text{a system of ODE, } \xi(0) = \eta_0 \end{array} \right. \end{array} \quad \text{Has unique sol for } t > 0$$

Semi-discrete stability:

Let $x = u_n$ in the (FEM) $\Rightarrow (u_{n,t}, u_n) + (\nabla u_n, \nabla u_n) = (f, u_n)$, $u_n(0) = u_{0,n}$

A studied procedure $\xrightarrow{\text{(VP)}} \{ \text{stability II} \}$

$$\textcircled{D} \|u_n(t)\| \leq \|u_{0,n}\| + \int_0^t \|f\| ds$$

To proceed we define discrete Laplacian: $\Delta_n: \overset{\circ}{V}_n \rightarrow \overset{\circ}{V}_n$
 $(-\Delta_n \Psi, x) = (\nabla \Psi, \nabla x) \quad \forall x \in \overset{\circ}{V}_n$

Δ_n is self-adjoint & $-\Delta_n$ is positive definite

Let P_n be the L_2 projection on $\overset{\circ}{V}_n$:

$P_n g \in V_n$ s.t.

$$(g, x) - P_n(g, x) \Leftrightarrow (g - P_n g, x) = 0 \quad \forall x \in \overset{\circ}{V}_n$$

Then (VF) is written as

$$\underbrace{(u_{n,t} - \Delta_n u_n - P_n f, x)}_{\in \overset{\circ}{V}_n} = 0 \quad \forall x \in \overset{\circ}{V}_n$$

$$\text{Take } x = u_{n,t} - \Delta_n u_n - P_n f \Rightarrow \|u_{n,t} - \Delta_n u_n - P_n f\|^2 = 0$$

$$\Leftrightarrow \boxed{u_{n,t} - \Delta_n u_n = P_n f} \quad \text{X} \quad \text{- semi-discrete problem}$$

Consider $E_n(t)$ the solution operator of the homogenous ($f = 0$) problem

Then $u_n(t) = E_n(t) u_{0,n}$ & Duhamel's principle

$$u_n(t) = E_n(t) u_{0,n} + \int_0^t E_n(t-s) P_n f(s) ds$$

Thm: u_h, u_n are solutions for continuous & semi-discrete heat eqn resp.

Semi-discrete
error estimate
for heat equation

$$\text{Then: } \|u_{n,h}(t) - u(t)\| \leq \|u_{0,h} - u_0\| + ch^2 (\|u_0\|_2 + \int_0^t \|u_s\|_2 ds)$$

Proof: Define the Ritz projection:

$$R_h: \underbrace{a(R_h v - v_h, \epsilon)}_0 = 0, \forall \epsilon \in V_h \\ \nabla(R_h v - v) \cdot \nabla \epsilon \quad \forall \epsilon \in H_0'$$

Note that $\|v \cdot v_h\| \leq ch^2 \|v\|_2 \quad \forall v \in V_h \leftarrow$ interpolant of v

We shall use the Ritz projection:

$$(I) \quad u - u_h = (u - R_h u + R_h u - u_h) := f + \epsilon \quad \begin{cases} v_h = P_h v \leftarrow L_2 \text{ proj of } v \\ v_h = R_h v \leftarrow \text{Ritz proj of } v \end{cases}$$

$$(II): \|u - R_h u\| + h \|u - R_h u\|_1 \leq ch^s \|u\|_s, \quad s=1,2, \quad \forall u \in H^s(\Omega) \cap H_0'(\omega)$$

we have:

$$(III): \|g(t)\| \leq ch^2 \|u(t)\|_2 = ch^2 \|u_0 + \int_0^t u_s ds\|_2 \leq ch \|u_0\|_2 + \underbrace{\int_0^t \|u_s\|_2 ds}_{\text{ch}^2 \|u(t)\|_2}$$

$$\text{OBS! } a(u, \epsilon) = (\nabla u, \nabla \epsilon)$$

$$\begin{aligned} (u_t, \epsilon) + a(u, \epsilon) &= - \underbrace{(u_{n,t}, \epsilon)}_{= - (f, \epsilon)} - a(u_n, \epsilon) + \underbrace{(R_h u, \epsilon)}_{= a(R_h u, \epsilon)} + a(R_h u, \epsilon) \\ &= - (f, \epsilon) + (R_h u, \epsilon) + a(u, \epsilon) = \underbrace{(-u_t + R_h u, \epsilon)}_{(-u_t + (R_h u)_t, \epsilon)} \end{aligned}$$

$$(u_t, \epsilon) + a(u, \epsilon) = (f, \epsilon)$$

$$\Rightarrow (IV) ((u, \epsilon) + a(u, \epsilon)) = (f, \epsilon) \Rightarrow \boxed{\|u(t)\| \leq \|u(0)\| + \int_0^t \|g_s\| ds}$$

Föreläsning 27/2:

Heat equation Duhamel's principle:

1st Homogeneous case:

$$\left\{ \begin{array}{l} u_0 - \Delta u = 0 \quad (\text{DE}) \\ u(x,0) = u_0(x) \quad (\text{IV}) \end{array} \right.$$

$$\text{Fourier transform } \Rightarrow \hat{u}(z,t) = \int_{\mathbb{R}^d} u(x,t) e^{-izx} dx, z \in \mathbb{R}^d$$

$$F(u(\cdot, t))(q) = \frac{d\hat{u}}{dt}(q, t)$$

$$F(\Delta u(\cdot, t))(q) = \int_{\mathbb{R}^d} \Delta u(x, t) e^{ixq} dx = \dots = -|q|^2 \hat{u}(q, t)$$

$$F(\text{DE}) \Rightarrow \left\{ \begin{array}{l} \frac{d\hat{u}}{dt} = -|q|^2 \hat{u} \quad (\text{ODE}) \\ \hat{u}(q, 0) = \hat{u}_0(q) \end{array} \right.$$

$$\Rightarrow \frac{\hat{u}_t}{\hat{u}} = -|q|^2 \Rightarrow \ln(\hat{u}) = -|q|^2 t$$

$$\hat{u}(q, t) = c e^{-|q|^2 t}$$

$$\hat{u}(q, t) = \hat{u}_0(q) e^{-|q|^2 t}$$

By a fixed-point thm for F -transf.:

$$F^{-1}(e^{-|q|^2 t}) = \frac{1}{(4\pi t)^{d/2}} e^{\frac{-|x|^2}{4\pi t}} =: u(x, t)$$

See Folland in \mathbb{R} :

$$\frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4\pi t}} \stackrel{f}{\rightarrow} e^{-|q|^2 t}$$

scaling

$$F(f(ax)) = \frac{1}{a} F(f(\frac{x}{a}))$$

$$\Rightarrow u(x, t) = (u(\cdot, t) * u_0)(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} u_0(y) e^{-\frac{|x-y|^2}{4\pi t}} dy =: (G(t)u_0)(x)$$

claim: $\|G(t)w\| \leq \|w\|$

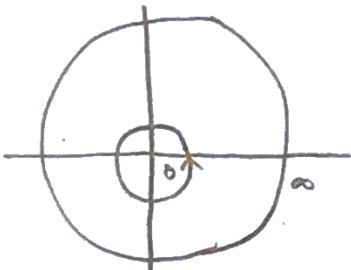
Hint: Let $\eta = \frac{y-x}{\sqrt{4t}}$ $\Rightarrow \frac{1}{(4\pi t)^{d/2}} \int_{R^d} e^{-\frac{|x-y|^2}{4t}} dy$

$$= \frac{1}{(4\pi t)^{d/2}} \cdot \frac{1}{(4\pi t)^{d/2}} \int_{R^d} e^{-|\eta|^2} (4\pi t)^{d/2} d\eta = 1$$

Check it (Ex $d=1$)

$$J = \int_R e^{-x^2} dx = \sqrt{\pi} \leftarrow J^2 = \left(\int_R e^{-x^2} dx \right)^2 = \left(\int_R e^{-x^2} dx \right) \left(\int_R e^{-y^2} dy \right) = \iint_{R^2} e^{-(x^2+y^2)} dx dy$$

$$= \iint_{R^2} e^{-r^2} r dr d\theta = \int_0^{2\pi} \int_0^\infty e^{-r^2} r dr d\theta = 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^\infty = \pi$$



$$J = \sqrt{\pi} \Rightarrow \|u(t)\| \leq \|u_0\| + \int_0^t \|f(s)\| ds$$

Inhomogeneous Case: $\begin{cases} u_t - \Delta u = f \\ u(\cdot, 0) = u_0 \end{cases}$

$$\Rightarrow \dots \Rightarrow u(x, t) = E(t)u_0 + \underbrace{\int_0^t E(t-s)f(\cdot, s) ds}_{\text{some procedure}}$$

Thm: ① $\|u(t)\|^2 + \int_0^t \|u(s)\|_1^2 ds \leq \|u_0\|^2 + C \int_0^t \|f(s)\|^2 ds$

② $\|u(t)\|_1^2 + \int_0^t \|u_x(s)\|^2 ds \leq \|u_0\|_1^2 + \int_0^t \|f(s)\|^2 ds$

Proof: (VF): $\begin{cases} (u_t, v) + a(u, v) = (f, v) \quad \forall v \in H_0 \\ a(u, v) = \int_{\mathbb{R}^d} \nabla u \cdot \nabla v dx, \quad (u, w) = \int_{\mathbb{R}^d} uw dx \end{cases}$

Take $v = u$ in (VF) $\Rightarrow (u_t, u) + a(u, u) = (f, u)$ $\quad \text{XXXX}$

$$(u_t, u) = \int_{\mathbb{R}^d} u_t u dx - \int_{\mathbb{R}^d} \frac{1}{2}(u^2)_t dx = \frac{1}{2} \frac{d}{dt} \|u\|^2 \quad \text{④}$$

$$\text{Poincaré} \Rightarrow \|u(t)\| \stackrel{\textcircled{I}}{\leq} C \|v\| = c \|\nabla u(t)\|^{1/2} \quad \forall v \in H_0^1$$

$$\left(\int_{\Omega} |\nabla u|^2 dx \right)^{1/2}$$

Further



$$\text{Insert } \textcircled{I}, \textcircled{II} \text{ & } \textcircled{III} \text{ in } \textcircled{**} \Rightarrow \frac{1}{2} \frac{d}{dt} \|u\|^2 + \frac{1}{2} \|u_t\|^2 \leq \frac{1}{2} c^2 \|f\|^2 \int_0^t \|f(s)\|^2 ds$$

$$\|u(t)\|^2 + \int_0^t \|u(s)\|^2 ds \leq \|u_0\|^2 + C \int_0^t \|f(s)\|^2 ds \Rightarrow \textcircled{T}$$

Proof of T₂: Sudden direct.

Wave equation in \mathbb{R}^N :

$$\begin{cases} \ddot{u} - \Delta u = \begin{cases} p \neq 0 & \text{Inhomogeneous} \\ 0 & \text{Homogeneous} \end{cases} & (\text{DE}) \\ u = 0 \quad \text{on } \Gamma := \partial \Omega & (\text{B.C.}) \\ (u=u_0), (\dot{u}=v_0) \quad \text{in } \Omega, \text{ for } t=0 & (\text{IC}) \end{cases}$$

Conservation of energy:

$$\text{Multiply (DE) by } \dot{u} \text{ & } \int_{\Omega} \dots dx \Rightarrow \int_{\Omega} \ddot{u} \dot{u} - \int_{\Omega} (\Delta u) \dot{u} = 0 \quad \leftarrow \underbrace{\text{Green's + B.C.}}$$

$$\int_{\Omega} \frac{1}{2} (\dot{u})_t + \int_{\Omega} \frac{1}{2} (|\nabla u|^2)_t = 0$$

$$\text{i.e. } \underbrace{\frac{1}{2} \frac{d}{dt} (\|u\|^2 + \|\nabla u\|^2)}_{\text{kinetic energy}} = 0 \Rightarrow \frac{1}{2} \|\dot{u}\|^2 + \frac{1}{2} \|\nabla u\|^2 = \text{constant} \quad [\text{independent of } t]$$

$$\text{potential energy} = \frac{1}{2} \|v_0\|^2 + \frac{1}{2} \|\nabla u_0\|^2$$

Therefore the total energy is conserved

$$V \in H_0^1(\Omega)$$

Weak formulation: Multiply (DE) by v & $\int_{\Omega} \Rightarrow$

$$\int_{\Omega} fv = \int_{\Omega} \dot{u} v dx - \int_{\Omega} (\Delta u) v dx$$

$$\Rightarrow \{ \text{Green's} \} \Rightarrow \int_{\Omega} f v d\mathbf{x} = \int_{\Omega} \nabla u \cdot \nabla v d\mathbf{x} - \int_{\partial\Omega} (\nabla u \cdot \mathbf{n}) v ds + \int_{\partial\Omega} v ds \quad (\text{F})$$

The semi-discrete problem: (SDF):

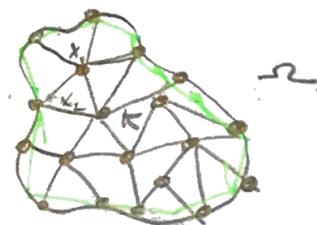
Let \mathcal{T}_h be a partition of Ω into elements K and $\{x_j\}_{j=1}^M$ internal nodes in the partition (triangulation) & $\{\psi_j\}_{j=1}^M$ basis functions for the discrete function space \tilde{V}_h

$$V_h := \{v: v \text{ cont piecewise linear on } \mathcal{T}_h, v=0 \text{ on } \partial\Omega\}$$

(SOP) \Rightarrow Find $u_h \in V_h$ such that

$$(\text{FEM}) \int_{\Omega} u_h v d\mathbf{x} + \int_{\Omega} \nabla u_h \cdot \nabla v d\mathbf{x} = \int_{\Omega} f v d\mathbf{x}$$

$$\Rightarrow u_h(x, t) = \sum_{j=1}^M \xi_j(t) \psi_j(x) \quad \stackrel{\text{in}}{\Rightarrow} \quad \forall v \in V_h \subset H,$$



$$\sum_{j=1}^M \xi_j(t) \int_{\Omega} \psi_i \psi_j d\mathbf{x} + \sum_{j=1}^M \xi_j(t) \int_{\Omega} \nabla \psi_i \cdot \nabla \psi_j d\mathbf{x} = \int_{\Omega} f \psi_i d\mathbf{x}, \quad i=1, 2, \dots, M$$

$$\Leftrightarrow A \xi(t) + S \xi(t) = b(t) \quad \Rightarrow: \quad \xi(t) + B \xi(t) = A^{-1} b(t)$$

Massmatrix $\xrightarrow{\text{Stiffness matrix}}$

$$\boxed{B = A^{-1} S}$$

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_M \end{pmatrix}$$

$$\text{Thm: } \exists c(t): \|u(t) - u_h(t)\| + h \|u(t) - u_h(t)\|_1 + \|u_t(t) - u_{h,t}(t)\|_1 \leq C(\|u_h - R_h u\|_1 + \|v_h - R_h v\|) + c(t) h^2 [\|u(t)\|_2 + \|u_t(t)\|_1 + (\|u_{tt}\|_2^2 ds)^{1/2}]$$

$$\left\{ \begin{array}{l} u_{tt} - \Delta u = 0 \\ u = 0 \text{ on } \partial\Omega \\ u = u_0, \dot{u} = \dot{u}_0 \text{ for } t=0 \end{array} \right.$$

$$\left\{ \begin{array}{l} u_0(x) = u(x, 0) \\ v_0(x) = \dot{u}(x, 0) \end{array} \right.$$

Q8: Write $u - u_h = (u - R_h u) + (R_h u - u_h) = g + \theta$

As in the heat equation we estimate g :

$$\begin{cases} \|g(t)\| + h\|g'(t)\|_1 \leq Ch^2 \|u(t)\|_2 \\ \|g_t(t)\| \leq ch^2 \|u_{tt}(t)\|_2 \end{cases}$$

As for θ : $(\theta_{tt}, \psi) + a(\theta, \psi) = (\beta_{tt}, \psi), \forall \psi \in V_h$

(Separate the effect of initial & boundary data)

$\theta = \eta + \xi$, where

$$\begin{cases} (\eta_{tt}, \psi) + a(\eta, \psi) = 0 \quad \forall \psi \in V_h \\ \eta(0) = \theta(0), \quad \eta_{tt}(0) = \theta_{tt}(0) \end{cases}$$

$$\Rightarrow S(0) = S_t(0) = 0$$

$$(S_{tt}, \psi) + a(\xi, \psi) = -(\beta_{tt}, \psi)$$

$$\text{Now let } \psi = S_t \Rightarrow \frac{1}{2} \frac{d}{dt} (\|S_t\|^2 + \|\xi\|^2) = -(\beta_{tt}, \xi_t) \leq \frac{1}{2} \|\beta_{tt}\|^2 + \frac{1}{2} \|\xi_t\|^2$$

$$\stackrel{\int_0^t}{\Rightarrow} \|\xi_t(t)\|^2 + \|\xi(t)\|^2 \leq \|\xi(0)\|^2 + \|\xi(s)\|^2 + \int_0^t \|\beta_{tt}\|^2 ds + \int_0^t \|S_t\|^2 ds$$

$$\Rightarrow \{\text{Gronwall}\} \Rightarrow \|S_t(t)\|^2 + \|\xi(t)\|^2 \leq c(t) \int_0^t \|g_{tt}\|^2 ds \leq c(t) h^4 \int_0^t \|u_{ttt}\|^2 ds$$

Fürstentum 2/3

2004-04-13

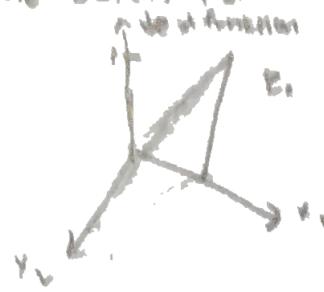
- ② Determine the stiffness matrix & load vector if the $\text{e}(1)$ FEM is applied for the Poisson equation:

$$\int -\Delta u = 1 \quad \text{on} \quad \partial B(0,1)^2$$

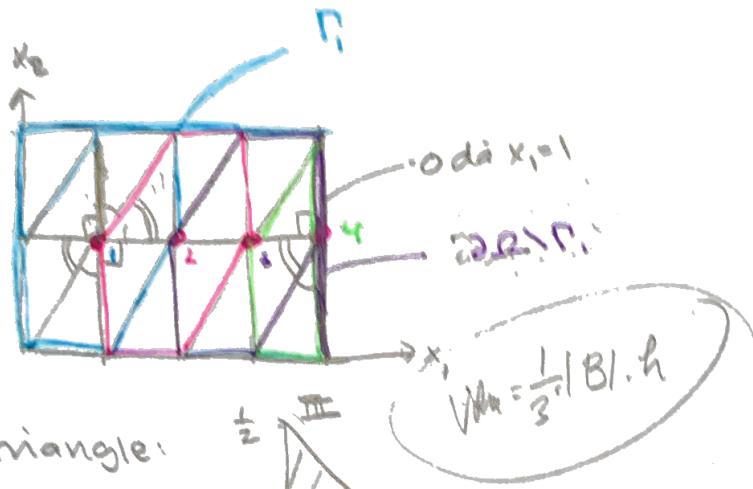
$$\frac{\partial L}{\partial x_1} = 0 \quad \text{for } x_1 = 1$$

$u=0$ for $x \in \partial D \setminus \{x_1\}$

On a triangulation with sidelength $\frac{1}{4}$ in x_1 -direction & $\frac{1}{2}$ in the x_2 -direction



Solution:



Standard triangle:

$$\phi_1(x,y) = Ax + By + C$$

$$\Phi(c_0, 0) = 1 \rightarrow c = 1$$

$$\phi_1(\frac{1}{4}, 0) \Rightarrow A \cdot \frac{1}{4} + 1 = 0 \Rightarrow A = -4$$

$$\Phi_1(q_{\frac{1}{2}}) = B \cdot \frac{1}{2} + 1 = 0 \Rightarrow B = -2$$

Similarly $\phi_2(x, y) = 4x$

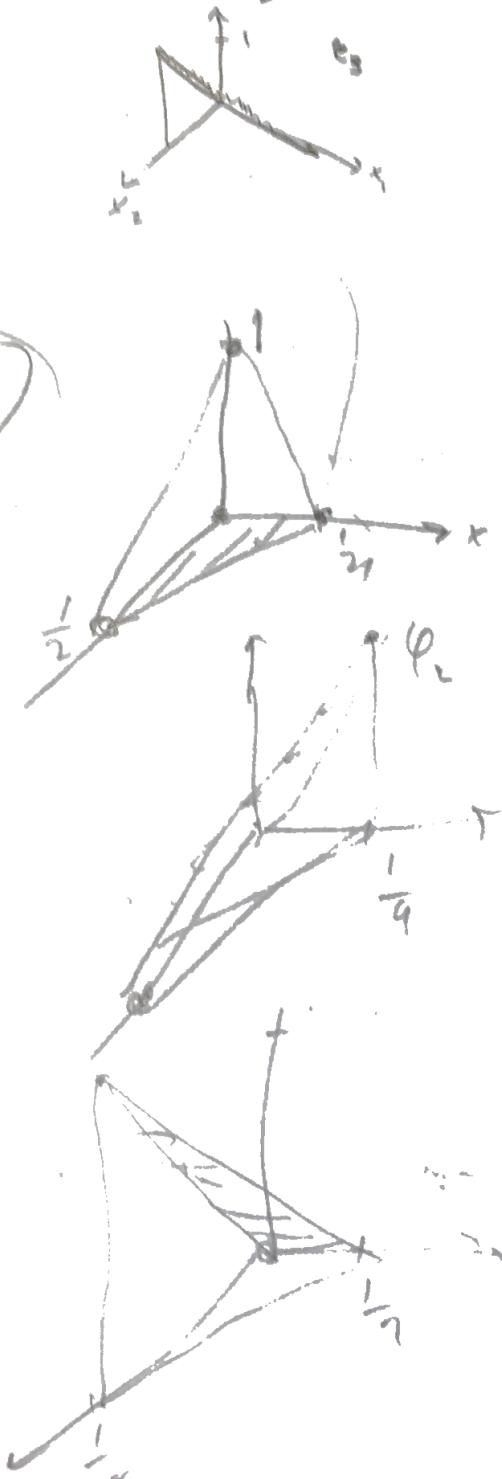
$$\Phi_3(x,y) = 2y$$

$$\Rightarrow \Phi_1(x,y) = -4x - 2y + 1$$

$$\{ \phi_2(x,y) = yx \}$$

$$\phi_2(x,y) = 2y$$

$$\rightarrow \begin{cases} \nabla \Phi_1 = -\begin{pmatrix} 4 \\ 2 \end{pmatrix} \\ \nabla \Phi_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \\ \nabla \Phi_3 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \end{cases}$$



Stiffness matrix for the Standard element

$$|T| = \int_T dx = |T| = \frac{1}{16} \quad (\text{area of standard triangle})$$

$$S_{11} = \int_T \nabla \Phi_1 \cdot \nabla \Phi_1 = \int_T (4^2 + 0^2) dx = \frac{16}{16} = \frac{5}{4}$$

$$S_{22} = \int_T \nabla \Phi_2 \cdot \nabla \Phi_2 = \int_T 4^2 dx = \frac{16}{16} = 1$$

$$S_{33} = \int_T \nabla \Phi_3 \cdot \nabla \Phi_3 = \int_T 2^2 dx = \frac{4}{16} = \frac{1}{4}$$

$$S_{12} = \int_T \nabla \Phi_1 \cdot \nabla \Phi_2 = \int_T -4 \cdot 4 dx = -\frac{16}{16} = -1$$

$$S_{13} = \int_T \nabla \Phi_1 \cdot \nabla \Phi_3 = \int_T -2 \cdot 2 dx = -\frac{4}{16} = -\frac{1}{4}$$

$$S_{23} = 0$$

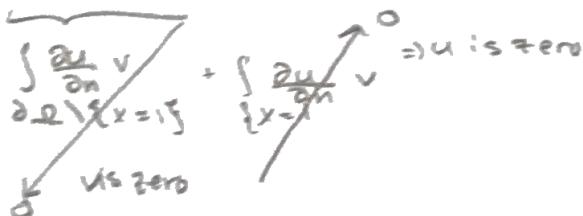
$$\Rightarrow S = \begin{pmatrix} \frac{5}{4} & -1 & -\frac{1}{4} \\ -1 & 1 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix} \leftarrow \text{Stiffness matrix for the standard element}$$

Weak formulation:

$$(f u, v) = (1, v) \quad \forall v \in \tilde{H}_0 \quad v \in H^1 \text{ & } v=0 \text{ on } \partial\Omega \setminus \{x=0\}$$

Green's
=)

$$(f u, v) = \int_{\Omega} f u \cdot v ds = \int_{\Omega} v dx$$



$\Rightarrow (V=)$: Find $u \in V$ such that $u=0$ on $\partial\Omega \setminus \{x=0\}$ & $\frac{\partial u}{\partial n} = 0$ for $x=1$
and $(f u, v) = (1, v) \quad \forall v \in \tilde{H}_0$

EFM: Find $u \in V_h$ such that $(\nabla u, \nabla v) = (1, 1), \quad \forall v \in V_h$

$$\int_{\Omega} (\nabla u \cdot \nabla v) dx = \int_{\Omega} v dx$$

$V_h = \{V \in V \mid V \text{ is piecewise linear on } T_h \text{ & cont in } \omega_2, V=0 \text{ on } \Gamma_1\}$

A set of test functions (basis functions) for the V_h on ω_2 : $\{e_i\}_{i=1}^4$

$$\{e_i \in V_h \mid i=1,2,3,4\}$$

$$\{a_{ij}(x_j) = S_{ij} \mid j=1,2,3,4\}$$

→ (FEM) To find $\{u_j\}_{j=1}^4$: $e_i(x) = \sum_{j=1}^4 u_j e_j(x)$ and $\sum_{j=1}^4 \left(\int_{\omega_2} \nabla e_i \cdot \nabla e_j u_j dx \right)$

$$\Leftrightarrow [S^{-1} u = b]$$

$$S^{-1} = (S_{ij})_{i,j=1}^4 = (\nabla e_i, \nabla e_j)_{i,j=1}^4$$

$$b_i = (b_i)_{i=1}^4, b_i = \int_{\omega_2} e_i(x) dx$$

2 av varje
vinkel

samma antal vinklar

$$S_{11} = 2(S_{11} + S_{22} + S_{33}) - 2\left(\frac{1}{2} + 1 + \frac{1}{2}\right) = 5 = S_{22} = S_{33}$$

$$S_{44} = S_{11} + S_{22} + S_{33} = \frac{5}{4} + 1 + \frac{1}{4} = \frac{5}{2}$$

central trianglar volymar pyramid kring 1 punkt
med sig

$$S_{12} = S_{23} = S_{34} = 2S_{12} = 2 \cdot -1 = -2$$

Alla utom punkt 4 har
6 trianglar runt
sig

$$6 \cdot \left(\frac{1}{3} \cdot \frac{1}{6}\right) = \frac{1}{8}$$

$$V = h \cdot 3 \cdot \frac{1}{3}$$

$$S = \begin{pmatrix} 5 & -2 & 0 & 0 \\ -1 & 5 & 2 & -2 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 5/2 \end{pmatrix}$$

$$b = 10 = \frac{1}{16} \begin{pmatrix} 2 \\ 2 \\ 2 \\ 1 \end{pmatrix}$$

2004-04-13:

⑥ Give a variational formulation for the BVP: $\begin{cases} -u'' + \alpha u = f, 0 < x < 1 \\ \text{with periodic} \quad u(0) = u(1) \& u'(0) = u'(1) \end{cases}$

where α is a constant, $f \in L^2(0,1)$

conditions in

Show that with approximation on α , the Lax-Milgram theorem are fulfilled

Solution: Let $V = \{v \in H^1(0,1) : v(0) = v(1) = 0\}$ with

$$\|v\|_V = \|v\|_{H^1} = \sqrt{\|v\|^2 + \|v'\|^2}$$

$$\|v\| = \left(\int_0^1 (v')^2 dx \right)^{1/2}$$

Multiply the (DE) by $v \in V$ & $\int_0^1 \dots dx \Rightarrow a(u, v) = L(v)$

$$a(u, v) = \int_0^1 [uv' - u'(x)v(x)] dx + \alpha \int_0^1 uv dx \Leftrightarrow a(u, v) = (u, v') + \alpha(u, v)$$

$$L(v) = \int_0^1 fv dx = (f, v)$$

$$(VF): a(u, v) = L(v) \quad \forall v \in V$$

$$\text{If } \underline{\alpha > 0} \text{ then } a(u, u) = (u', u') + \alpha(u, u) = \|u'\|^2 + \alpha \|u\|^2 \geq \min_{i=1,2} \frac{\|u_i\|^2}{c_i} \|u\|^2,$$

$$a(u, v) = (u', v') + \alpha(u, v) \leq \underline{2(c-s)} + \|u'\|\|v'\| + \alpha\|u\|\|v\| \leq \frac{(1+\alpha)}{c_2} \|u\|_{H^1} \|v\|_{H^1}$$

$$\|L(v)\| = \left| \int_0^1 fv dx \right| \leq \underline{2(c-s)} \leq \frac{\|f\|}{c_3} \|v\|$$

④ Prove an a priori and an a posteriori error estimate for the FEM for the problem:

$$(DE) -u' + \alpha u = f \quad \text{in } I = (0,1) \quad \text{with } 0 \leq \alpha(x) \leq M$$

$$\text{BC: } u(0) = u(1) = 0$$

Solution:

$$\text{Multiply (DE) by } v \in H^1(I) \text{ & } \int_I \dots dx \Rightarrow \int_I (u'v - \alpha uv) dx = \int_I fv dx \quad (VF)$$

$$\text{FEM for CGCII: Find } u \in V_h \quad \int_I (u'v - \alpha uv) dx = \int_I fv dx \quad \forall v \in V_h$$

$$V_h = \left\{ v : v \text{ is piecewise continuous on a partition } \right. \\ \left. \text{In of } I, v \text{ is cont } v(l) = v(1) = 0 \right\}$$

$$\text{OBS! } V_h \subset H_0$$

$$(VF) - (\text{FEM}) \Rightarrow (G^+), \quad \int_I (e'v - \alpha ev) = 0 \quad \forall v \in V_h \text{ where } e := u - u_h$$

Define energy norm

$$(v, w)_E = \int_I (v'w' + \alpha v w) dx \Rightarrow \|v\|_E^2 = (v, v)_E = \int_I (v')^2 + \alpha v^2 dx$$

$$\Rightarrow \boxed{(e, v)_E = 0 \quad \forall v \in V_h} \Leftrightarrow (G^+)$$

A posteriori error estimate:

$$\begin{aligned} \|e\|_E^2 &= \int_I e'e' + \alpha ee = \int_I (u - \bar{u})'e' + \alpha(u - \bar{u})e \{ dx = \{v = e \text{ in (VFI)}\} = \\ &= \int_I fe - \int_I (u'e' + \alpha ue) = \{v = \pi_h e \text{ in (FEM)}\} = \int_I f(e - \pi_h e) - \int_I (u'(e - \pi_h e)) \\ &\quad + \alpha u(e - \pi_h e) dx = [\text{PI}] \cdot \int_I R(u) (e - \pi_h e) \end{aligned}$$

where $R(u) = f + u'' - \alpha u = f - \alpha u$ is the residual
 (approximately piecewise)

$$\Rightarrow \|e\|_E^2 = \int_I f(u)(e - \pi_h e) \stackrel{\text{C-S}}{\leq} \|hR(u)\|_{L_2(I)} \|h'(e - \pi_h e)\|_{L_2(I)}$$

$$\leq c_i \|hR(u)\|_{L_2(I)} \|e\|_{L_2(I)} \leq c_i \|R(u)\|_{L_2(I)} \|e\|_E$$

$$\Rightarrow \|e\|_E \leq c_i \|R(u)\|_{L_2(I)}$$

A priori error estimate:

$$\begin{aligned} \|e\|_E^2 &= (e, e)_E = (e, u - \bar{u})_E = \{v = \bar{u} - \pi_h u\} = (e, u - \pi_h u)_E + (e, \pi_h u - \bar{u})_E \\ &\leq 3(C-s) \leq 10 \|u - \pi_h u\|_E \end{aligned}$$

$$\|u - \pi_h u\|_E^2 = \|(u - \pi_h u)'\|_{L_2(I)}^2 + \|\sqrt{\alpha}(u - \pi_h u)\|_{L_2(I)}^2 \leq c_i^2 \|hu'\|_{L_2(I)}^2 + c$$

$$+ c_i^2 M \|h^2 u''\|_{L_2(I)}^2 \Rightarrow \|e\|_E \leq c_i (\|hu'\|_{L_2(I)} + \sqrt{M} \|h^2 u''\|_{L_2(I)})$$

Föreläsning 4/3:

2000-03-13:

$$\begin{cases} \dot{u} - u'' = f(x), & 0 < x < 1, t > 0 \\ u(0, t) = u'(1, t) = 0, & t > 0 \\ u(x, 0) = u_0(x), & 0 < x < 1 \end{cases}$$

a) Show that if $f=0$, then

$$(E1) \frac{d}{dt} \|u\|^2 + 2\|u'\|^2 = 0 \quad \left. \right\} \text{theorem}$$

$$(E2) \|u(\cdot, t)\| \leq e^{-t} \|u_0\|$$

b) Let $u = u_s$ be the solution for $f=0$ & stationary problem:

$$\begin{cases} -u_s'' = f & 0 < x < 1 \\ u_s(0) = u_s'(0) = 0 \end{cases}$$

Show that $\|u - u_s\| \rightarrow 0$ as $t \rightarrow \infty$

Solution:

a) Multiply (DE) by u & $\int_0^1 \dots dx$

$$0 = \int_0^1 fu dx = \int_0^1 (\dot{u} - u'') u dx = \underbrace{\int_0^1 \dot{u} u dx}_{\stackrel{\text{int by parts}}{\Rightarrow}} + \underbrace{\int_0^1 u' u dx}_{\|u'\|^2} - \underbrace{[u u']_0^1}_{\cancel{0}}$$

$$\Leftrightarrow \frac{1}{2} \frac{d}{dt} \int_0^1 u^2 dx + \|u'\|^2 = 0 \Rightarrow \left\{ \begin{array}{l} \text{Poincaré} \\ \|u\| \leq \|u'\| \end{array} \right\} \Rightarrow \frac{1}{2} \frac{d}{dt} \|u\|^2 + \|u'\|^2 = 0 \Rightarrow (E2)$$

$$\overrightarrow{\int_0^t \dots ds} \left(\int_0^t \frac{d}{ds} \|u(\cdot, s)\|^2 ds + 2 \int_0^t \|u'\|^2 ds \leq 0 \right) e^{2s}$$

$$\Rightarrow \int_0^t \frac{d}{ds} (\|u\|^2 e^{2s}) \leq 0 \quad \|u\|^2(t) e^{2t} \leq \|u_0\|^2 \Rightarrow (E2)$$

b) Let $w(x, t) = u(x, t) - u_s(x) \Rightarrow \dot{w} - w'' = \dot{u} - u'' + u_s'' = f - f = 0$

$$\Rightarrow \|w(\cdot, t)\|^2 \leq e^{-t} \|w_0\|^2 = e^{-t} \|u_0 - u_s\|^2 \rightarrow 0 \text{ as } t \rightarrow \infty$$

$$\Rightarrow \|u - u_s\| \rightarrow 0 \text{ as } t \rightarrow \infty$$

2011-08-24:

⑤ Consider 2D Poisson equation with the Neumann BC.

$$\begin{cases} -\Delta u = f \text{ in } \Omega \subset \mathbb{R}^2 \\ -\nabla \cdot \mathbf{n} u = \kappa u \text{ on } \partial \Omega \end{cases}$$

$\kappa > 0$, \mathbf{n} - outward unit normal

Show that:

a) $\|u\|_{L_2(\Omega)} \stackrel{\text{Poincaré}}{\leq} C_{\Omega} (\|u\|_{L_2(\Omega)} + \|\nabla u\|_{L_2(\Omega)})$

b) $\|\nabla u\|_{L_2(\partial\Omega)} \rightarrow 0 \text{ as } \kappa \rightarrow \infty$

Solution:

a) $\exists \phi \text{ such that } \Delta \phi = 1 \leftarrow$

$$\Rightarrow \|u\|_{L_2(\Omega)}^2 = \int_{\Omega} u^2 \cdot \Delta \phi dx \quad \text{Greens:}$$

$$\Rightarrow \int_{\Omega} u^2 (\mathbf{n} \cdot \nabla \phi) ds - \int_{\Omega} 2u \nabla u \nabla \phi dx$$

$$\leq [C_1] \|u\|_{L_2(\Omega)}^2 + [C_2] \|u\|_{L_2(\Omega)} \|\nabla \phi\| \leq C_1 \|u\|_{L_2(\Omega)}^2 + \frac{1}{2} \|u\|^2 + \frac{1}{2} \|u\|^2 + \frac{1}{2} C_2^2 \|\nabla u\|^2$$

$$ab \leq \frac{1}{2} a^2 + \frac{1}{2} b^2$$

$$\Rightarrow \|u\|_{L_2(\Omega)}^2 \leq 2C_1 \|u\|_{L_2(\Omega)}^2 + C_2^2 \|\nabla u\|^2 \leq C^2 (\|\nabla u\|^2 + \|u\|_{L_2(\Omega)}^2)$$

$$C^2 = 2 \max(2C_1, C_2^2)$$

$$\Rightarrow \|\nabla u\|_{L_2(\Omega)} \leq \textcircled{A} + 2C \|\nabla u\| \|u\|_{L_2(\Omega)} \leq \{C^2 (\|\nabla u\| + \|u\|_{L_2(\Omega)})\}^{1/2}$$

b) $\int_{\Omega} (-\Delta u - f) u = \|\nabla u\|_{L_2}^2 - \int_{\Omega} (\nabla u \cdot \mathbf{n}) u ds = \int_{\Omega} fu$

$$\Rightarrow \|\nabla u\|_{L_2}^2 + \kappa \|u\|_{L_2(\partial\Omega)}^2 \leq \|u\| \|f\| \leq C_{\Omega} (\|u\|_{L_2} + \|\nabla u\|_{L_2}) \|f\|$$

$$\Rightarrow \|u\|_{L_2(\partial\Omega)} (C_{\Omega} \|f\|) + \|\nabla u\|_{L_2} (C_{\Omega} \|f\|) \leq \underbrace{\frac{1}{2} \|u\|_{L_2}^2 + \frac{1}{2} \|\nabla u\|_{L_2}^2}_{=(*)} + C_{\Omega}^2 \|f\|^2$$

$$(x) = \frac{1}{2} \|\nabla u\|_{L^2}^2 + (\kappa - \frac{1}{2}) \|u\|_{L^2}^2 \leq C_{\kappa}^2 \|f\|^2$$

$$\Rightarrow (\kappa - \frac{1}{2}) \|u\|_{L^2}^2 \leq C_{\kappa}^2 \|f\|^2 \Rightarrow u \Big|_{L^2} \rightarrow 0 \text{ as } t \rightarrow \infty$$

More on Poincaré:

2019-06-11:

① V continuously differentiable on $(0, b)$. Then show the Poincaré:

$$\|V\|^2 \leq b(v(0)^2 + v(b)^2 + b \|V'\|^2)$$

Solution:

$$\begin{aligned} \|V\|_{L_2(0,b)}^2 &= \int_0^b V^2(x) dx = \int_0^{b/2} v^2(x) dx + \int_{b/2}^b V^2(x) dx \stackrel{\text{PI}}{\leq} \left[(x - b/2) v^2(x) \right]_0^{b/2} \\ &+ \left[(x - b/2) V^2(x) \right]_{b/2}^b - \int_0^b (x - \frac{b}{2}) \cdot 2v(x) V'(x) dx \\ \Rightarrow \|V\|_{L_2(0,b)}^2 &\leq \frac{b}{2} v^2(0) + \frac{b}{2} v(b)^2 + b \|V'\| \|V'\| \\ &\leq \frac{b}{2} \|V'\|^2 + \frac{b}{2} \|V\|^2 \end{aligned}$$

$$\frac{1}{2} \|V\|^2 \leq \frac{b}{2} [v(0)^2 + v(b)^2 + b \|V'\|^2]$$

→ \blacksquare

2003-04-22:

⑥ Consider (BVP): $-\operatorname{div}(2\nabla u + \beta u) = f$ in Ω , $u = 0$, on $\partial\Omega$

Ω bdd, polyg., $\Omega \subset \mathbb{R}^2$, $\varepsilon > 0$ constant, $\beta = (\beta_1(x), \beta_2(x), \beta_3(x))$, $f \in L_2(\Omega)$

b) Derive stability estimate in terms of $\varepsilon \leq k \|f\|_{L_2(\Omega)}$ & domain Ω

c) Give conditions for $\exists !$ à la Lax-Milgram

Solution:

a) Multiply (DE) by $v \in H_0^1(\Omega)$ & $\int \dots dx \Rightarrow$ {Green's thm.}

$$\begin{aligned} -\int_{\Omega} \operatorname{div}(2\nabla u + \beta u) v dx &= \int_{\Omega} (2\nabla u + \beta u) \cdot \nabla v dx = \int_{\Omega} f v dx \\ \Rightarrow (\text{VF}): \text{Find } u \in H_0^1(\Omega) \Rightarrow \boxed{L(u,v) = \int_{\Omega} f v dx} \quad &\begin{cases} \nabla u = \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \\ \operatorname{div}(w) = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \\ \operatorname{div}(uv) = u \operatorname{div}(v) + v \operatorname{div}(u) \end{cases} \\ &\Rightarrow \operatorname{div}(\nabla u) = \Delta u \end{aligned}$$

where $a(v, w) = \int_{\Omega} (\epsilon \nabla v \cdot \nabla w + \beta v \cdot \nabla w) \, dx$ & $L(v) = \int_{\Omega} fv \, dx$

To get a unique solution using "L-M" need verify these conditions:

$$(ii) |a(v, w)| \leq \gamma \|v\|_{H^1(\omega)} \|w\|_{H^1(\omega)} \quad \forall v, w \in H^1(\omega)$$

$$(iii) |a(v, w)| \geq \alpha \|v\|_{H^1(\omega)}^2$$

$$(iv) |L(v)| \leq \Lambda \|v\|_{H^1(\omega)} \text{ for some } \gamma, \alpha, \Lambda > 0$$

$$(ii) |L(v)| = \left| \int_{\Omega} fv \right| \leq \{\epsilon - \gamma\} \leq \|f\|_{L_2(\omega)} \|v\|_{H^1} \leq \|f\| \|v\|_{H^1}$$

$$\Rightarrow \text{Take } \Lambda = \|f\|_{L_2(\omega)} \Rightarrow (iv) \quad \square$$

$$(i) |a(v, w)| = \int_{\Omega} |\epsilon \nabla v + \beta v| |\nabla w| \, dx \leq \int_{\Omega} ((\epsilon |\nabla v| + |\beta| |v|) |\nabla w|) \, dx \leq$$

$$\leq \left(\int_{\Omega} (\epsilon |\nabla v| + |\beta| |v|)^2 \, dx \right)^{1/2} \left(\int_{\Omega} |\nabla w|^2 \, dx \right)^{1/2} \leq \sqrt{2} \max(\epsilon, |\beta|_{\infty}) \left(\int_{\Omega} (\epsilon |\nabla v|^2 + |v|^2) \, dx \right)^{1/2} \|w\|_{H^1(\omega)}^2$$

$$= \gamma \|v\|_{H^1(\omega)}^2 \|w\|_{H^1(\omega)}^2 \quad \text{with } \gamma = \sqrt{2} \max(\epsilon, |\beta|_{\infty})$$

$$\Rightarrow (i) \quad \square$$

Finally for (iii) we need to assume $\operatorname{div} \beta \geq 0$

$$a(v, w) = \int_{\Omega} (\epsilon |\nabla v|^2 + (\beta \cdot \nabla v) v) \, dx = \int_{\Omega} \left(\epsilon |\nabla v|^2 + \left(\beta_1 \frac{\partial v}{\partial x_1} + \beta_2 \frac{\partial v}{\partial x_2} \right) v \right) \, dx$$

$$= \int_{\Omega} \left[\epsilon |\nabla v|^2 + \frac{1}{2} \left(\underbrace{\beta_1 \frac{\partial}{\partial x_1} (v^2)}_{\text{ab}} + \underbrace{\beta_2 \frac{\partial}{\partial x_2} (v^2)}_{\text{ab}} \right) \right] \, dx = \{\text{Green's}\}$$

$$+ \int_{\Omega} \left(\epsilon |\nabla v|^2 - \frac{1}{2} (\operatorname{div} \beta) (v^2) \right) \, dx \geq \int_{\Omega} \epsilon |\nabla v|^2$$

$$\text{But } \int_{\Omega} |\nabla v|^2 = \frac{1}{2} \int_{\Omega} |\nabla v|^2 + \frac{1}{2} \int_{\Omega} |\nabla v|^2 = \frac{1}{2} \int_{\Omega} |\nabla v|^2 + \frac{1}{2} C_{\Omega} \int_{\Omega} |v|^2$$

$$|a(v, w)| \geq \frac{\epsilon}{2} \|v\|_{H^1(\omega)}^2 - \frac{1}{2} C_{\Omega} \|v\|_{H^1(\omega)}^2 \geq \frac{\epsilon}{4} \min(1, |C_{\Omega}|) \|v\|_{H^1(\omega)}^2 \Rightarrow (iii)$$

$$\alpha \|u\|_{H^1(\omega)}^2 \leq a(u, u) = L(v) \leq \Lambda \|u\|_{H^1(\omega)}$$

$$\Rightarrow \|u\|_{H^1(\omega)} \leq \frac{1}{\alpha} > \frac{1}{\alpha} \|f\|_{L_2(\omega)} \quad \xrightarrow{\text{Stability}} (b)$$

Föreläsning 5/6:

KF 2019-08-29:

- ① $\Pi_K \phi$, L_2 -projection of ϕ into piecewise constants

$$\int_{I_j} \Pi_K \epsilon ds = \int_{I_j} \epsilon ds \Leftrightarrow \int_{I_j} c(\Pi_K \epsilon - \epsilon) ds = 0$$

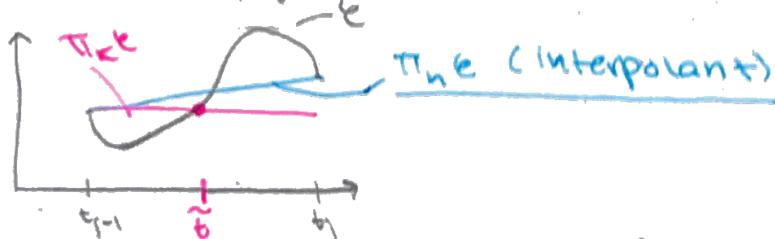
f was L_2 -proj perf if
 $(\Pi_K f - f, g) = 0 \forall g \in L_2(\omega)$

$$\Rightarrow (\Pi_K \epsilon - \epsilon, c) = 0, \forall c \text{ constant}$$

Show that for a subinterval $I_j = (t_{j-1}, t_j)$, $t_j = jk$, k = timestep positive k constant

$$\int_{I_j} |\epsilon - \Pi_K \epsilon| ds \leq k \int_{I_j} |\dot{\epsilon}| ds \quad \left(\dot{\epsilon} = \frac{d\epsilon}{dt} \right)$$

$$\Leftrightarrow \|\epsilon - \Pi_K \epsilon\|_{L_1(I_j)} = k \|\dot{\epsilon}\|_{L_1(I_j)}$$



Case 1: $\tilde{t} \leq t \leq t_j \Rightarrow |\epsilon(t) - \Pi_K \epsilon| = \left| \int_{\tilde{t}}^t \dot{\epsilon}(s) ds \right| \leq \int_{\tilde{t}}^{t_j} |\dot{\epsilon}(s)| ds =$

$$\int_{t_{j-1}}^{t_j} |\dot{\epsilon}(s)| ds \Rightarrow \int_{\tilde{t}}^{t_j} |\epsilon(t) - \Pi_K \epsilon| dt \leq \int_{\tilde{t}}^{t_j} \left(\int_{t_{j-1}}^{t_j} |\dot{\epsilon}(s)| ds \right) dt$$

\Rightarrow for $\tilde{t} \leq t \leq t_j$

$$\int_{\tilde{t}}^{t_j} |\epsilon(t) - \Pi_K \epsilon| dt \leq (t_j - \tilde{t}) \int_{I_j} |\dot{\epsilon}| dt \quad \textcircled{I}$$

Similarly for case 2: $t_{j+1} \leq t \leq \tilde{t}$

We get $\int_{t_{j+1}}^{\tilde{t}} |\epsilon(t) - \Pi_K \epsilon| dt \leq (\tilde{t} - t_{j+1}) \int_{I_j} |\dot{\epsilon}| dt \quad \textcircled{II}$

$$\textcircled{I} + \textcircled{II} \rightarrow \int_{t_{j+1}}^{t_j} |\epsilon(t) - \Pi_K \epsilon| dt \leq k \int_{I_j} |\dot{\epsilon}| dt \quad \textcircled{III}$$

$$(2) \quad \begin{cases} u_t - \Delta u = f & \text{in } \Omega \quad 0 < t \leq T \\ u = 0 & \text{at } \partial\Omega \\ u(x, 0) = u_0(x), \quad x \in \Omega \subset \mathbb{R}^2 \end{cases}$$

Let \tilde{u} be the sol of (1) with modified initial data $\tilde{u}_0(x) = u_0(x) + \varepsilon(x)$

a) Show that $w = u - \tilde{u}$ solves (1) with data $w_0(x) = \varepsilon(x)$ (and $t \leq 0$)

Derive the following stability estimate for w , i.e.

bound $\|w(T)\|^2 + 2 \int_0^T \|\nabla w\|^2 dt$ by $\|w_0\|^2$

Solution: we have:

$$(2) \quad \begin{cases} \tilde{u}_t - \Delta \tilde{u} = f & \text{in } \Omega \\ \tilde{u} = 0 & \text{in } \partial\Omega \\ \tilde{u}(x, 0) = u_0(x) + \varepsilon(x), \quad x \in \Omega \end{cases}$$

Then (1)-(2) \Rightarrow

$$(3) \quad \begin{cases} w_t - \Delta w = 0 & \text{in } \Omega \\ w = 0 & \text{in } \partial\Omega \\ w(x, 0) = \varepsilon(x) & \text{in } \Omega \end{cases}$$

You get this via stability
Ehm for the heat equation

By stability for the heat equation: $\|w(T)\|^2 + 2 \int_0^T \|\nabla w\|^2 dt \leq \|\varepsilon\|^2$ (**)

b) Use the stability estimate for w to show that the solution to problem (1) is unique.

Solution:

$$\text{Take } \varepsilon = 0 \stackrel{(a)}{\Rightarrow} \|w(T)\|^2 + 2 \int_0^T \|\nabla w\|^2 dt \leq 0 \quad \left\{ \begin{array}{l} \nabla w = 0 \Rightarrow w = \text{constant} \\ w(T) = 0 \\ \Leftrightarrow w = 0 \end{array} \right\}$$

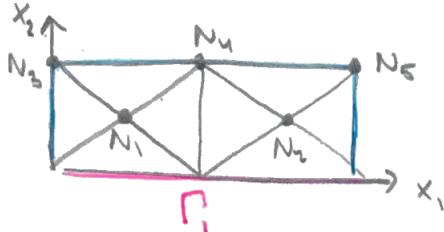
But $\|w(T)\|^2 \geq 0$ & $\|w\|^2 \geq 0$

⇒ Uniqueness

2019-08-29:

③ Formulate CG(1) for the problem:

$$\begin{cases} -\Delta u(x) + u(x) = 1, & x \in \Omega \\ u=0 \quad \text{on } \Gamma_1 \\ \nabla u \cdot \mathbf{n} = f \quad \text{for } x \in \partial\Omega \setminus \{\Gamma_1\} \end{cases}$$



$$\begin{cases} -\Delta u = 1, & x \in \Omega \\ u=0, & x \in \Gamma_1 \\ \nabla u \cdot \mathbf{n} = 1 & x \in \partial\Omega \setminus \{\Gamma_1\} \end{cases}$$

Solution:

$$\text{Let } V = \{v: \int_{\Omega} (v^2 + |\nabla v|^2) dx < \infty, v=0 \text{ on } \Gamma_1\}$$

$$\text{Multiply (DE) by } v \text{ & } \int_{\Omega} \dots \Rightarrow -(\Delta u, v) + (u, v) = (1, v) \quad \forall v \in V$$

$$\Rightarrow \{\text{Green's}\} \Rightarrow (\nabla u, \nabla v) + (u, v) = (1, v) + \int_{\partial\Omega \setminus \{\Gamma_1\}} (\mathbf{n} \cdot \nabla u) v ds = \int_{\Omega} v dx + \int_{\partial\Omega \setminus \{\Gamma_1\}} v ds$$

$$\text{Let now } V_h = \{v: v \text{ continuous piecewise linear } v=0 \text{ on } \Gamma_1\}$$

FEM { Find $u \in V_h$ such that

$$\int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\Omega} uv dx = \int_{\Omega} v dx + \int_{\partial\Omega \setminus \{\Gamma_1\}} v ds$$

$$\Rightarrow \text{Ansatz } u(x) = \sum_{j=1}^5 z_j e_j(x) \& v = e_i(x), i=1, 2, 3, 4, 5 \quad \forall v \in V_h$$

$$\Rightarrow \sum_{j=1}^5 z_j \left(\int_{\Omega} \nabla e_i \cdot \nabla e_j dx + \int_{\Omega} e_i(x) e_j(x) dx \right) = \int_{\Omega} e_i(x) dx + \int_{\partial\Omega \setminus \{\Gamma_1\}} e_i(x) ds$$

$$\Leftrightarrow (S+M)z = b, \quad S = (S_{ij}) = \left(\int_{\Omega} e_i \cdot \nabla e_j \right)$$

$$M = (M_{ij}) = \left(\int_{\Omega} e_i e_j \right)$$

$$b = b_1 + b_2$$

$$b_1 = (b_1); \quad b_{1,i} = \int_{\Omega} e_i dx$$

$$\text{likewise } b_{2,i} = \int_{\partial\Omega \setminus \{\Gamma_1\}} e_i ds$$

For the standard element:

$$\left. \begin{array}{l} \Phi_1(x,y) = 1 - \frac{x}{h} - \frac{y}{h} \Rightarrow \nabla \Phi_1 = -\frac{1}{h} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \Phi_2(x,y) = x/h \Rightarrow \nabla \Phi_2 = \frac{1}{h} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \Phi_3(x,y) = y/h \Rightarrow \nabla \Phi_3 = \frac{1}{h} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{array} \right\} \Rightarrow \text{Local stiffness matrix}$$

$$S = (S_{ij}) = (\nabla \Phi_i, \nabla \Phi_j)_{i,j=1}^3 \Rightarrow S_{11} = \int_T \nabla \Phi_1 \cdot \nabla \Phi_1 = \frac{h^2}{2} \left(\frac{1}{h^2} + \frac{1}{h^2} \right) = 1$$

$$\Rightarrow S = \frac{1}{2} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

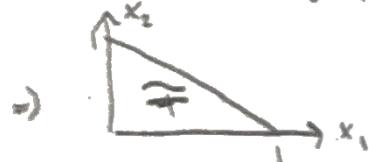
$$S_{22} = \int_T \nabla \Phi_2 \cdot \nabla \Phi_2 = \frac{1}{2} = S_{33}$$

$$S_{12} = \int_T \nabla \Phi_1 \cdot \nabla \Phi_2 = -\frac{h^2}{2} \left(\frac{1}{h^2} \right) = -\frac{1}{2} = S_{13}$$

$$S_{23} = 0$$

Local mass matrix:

$$m = (m_{ij}) = (e_i, e_j)^T_{i,j=1} \rightarrow$$



$$\text{OBJ: } m_{11} = m_{22} = m_{33} = \frac{h^2}{12}$$

$$\begin{aligned} m_{11} &= \int_T \Phi_1 \Phi_1 = \int_T \left(\frac{x_1}{h}, \frac{x_2}{h} \right) \left(\frac{x_1}{h}, \frac{x_2}{h} \right)^T dA = h^2 \int_0^1 \left(\int_0^{1-x_1} (1-x_1-x_2)^2 dx_2 \right) dx_1 = \\ &= h^2 \left(\int_0^1 \left. \frac{(1-x_1-x_2)^3}{-3} \right|_{x_2=0}^{1-x_1} dx_1 \right) \\ &= h^2 \int_0^1 \frac{(1-x)^3}{3} dx = -\frac{h^2}{12} (1-x)^4 = \frac{h^2}{12} \end{aligned}$$

$$m_{12} = h^2 \int_0^1 \left(\int_0^{1-x_1} (1-x_1-x_2)x_1 dx_2 \right) dx_1 = h^2 \int_0^1 \left(\int_0^{1-x} (1-x-y)x dy \right) dx = \dots = \frac{h^2}{24}$$

===== Standard triangle

$$S: \quad S_{11} = S_{22} = 4S_{11} = 4$$

$$S_{12} = 0, \quad S_{13} = S_{14} = 2S_{12} = 2 \cdot (-\frac{1}{2}) = -1$$

$$S_{33} = 2S_{22} = 2 \cdot \frac{1}{2} = 1$$

$$S_{34} = S_{23} = 0$$

$$S_{24} = S_{25} = 2S_{12} = -1$$

$$S_{44} = 4S_{22} = 4 \cdot \frac{1}{2} = 2$$

$$M_{24} = M_{25} = 2m_{12} = \frac{2h^2}{24}$$

$$M_{44} = 4m_{22} = 4 \cdot \frac{h^2}{12} = \frac{h^2}{3}$$

$$m = \frac{h^2}{24} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$M_{11} = M_{22} = 4m_{11} = \frac{4h^2}{12}$$

$$M_{12} = 0$$

$$M_{13} = M_{14} = 2m_{12} = \frac{2 \cdot h^2}{24}$$

$$M_{23} = 2m_{22} = 2 \cdot \frac{h^2}{12}$$

$$S_{15} = 0$$

$$S_{45} = S_{23} = 0$$

$$S_{55} = 2S_{22} = 2 \cdot \frac{1}{2} = 1$$

Standardtangential

$$\Theta = \begin{bmatrix} 4 & 0 & -1 & -1 & 0 \\ 0 & 4 & 0 & -1 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{15} = 0$$

$$M_{45} = m_{23} = \frac{h^2}{24}$$

$$M_{55} = 2m_{22} = \frac{2h^2}{12}$$

$$M_{34} = m_{23} = \frac{h^2}{24}$$

$$M = \frac{h^2}{24} \begin{bmatrix} 8 & 0 & 2 & 2 & 0 \\ 0 & 8 & 0 & 2 & 2 \\ 2 & 0 & 4 & 1 & 0 \\ 2 & 2 & 1 & 8 & 1 \\ 0 & 2 & 0 & 1 & 4 \end{bmatrix}$$