Mathematics Chalmers & GU

TMA372/MMG800: Partial Differential Equations, 2017–03–15, 14:00-18:00

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Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 5p. Valid bonus poits will be added to the scores.
Breakings from total of 36 points: Exam(30)+Bonus(6). **3**: 15-20p, **4**: 21-27p och **5**: 28p-
For solutions see couse diary: http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1617/

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1. Consider the problem: $\quad -\varepsilon u'' + xu' + u = f \quad \text{in } I = (0,1), \qquad u(0) = u'(1) = 0,$
where ε is a positive constant, and $f \in L_2(I)$. Prove that

$$||\varepsilon u''|| \le ||f||, \qquad (||\cdot|| \text{ is the } L_2(I) - \text{norm}).$$

2. Show that the solution of the wave equation with homogeneous Dirichlet data and $f = 0$, conserves the quantity

$$\|\nabla \dot{u}\|^2 + \|\Delta u\|^2.$$

3. Derive an a priori and an a posteriori error estimate in the energy norm:
$\|u\|_E^2 = \|u\|^2_{L_2(0,1)} + \|u'\|^2_{L_2(0,1)}$, for the cG(1) finite element method for the problem

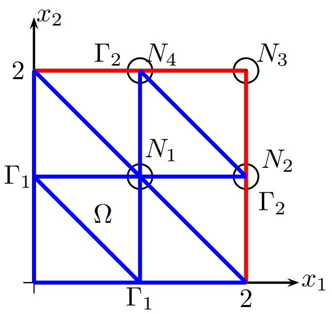
$$-u'' + 2xu' + 2u = f, \quad 0 < x < 1, \qquad u(0) = u(1) = 0.$$

4. In the square domain $\Omega := (0,2)^2$, with the boundary $\Gamma := \partial\Omega$, consider the problem of solving

$$(1) \quad \begin{cases} -\Delta u + u = 1, & \text{in} \quad \Omega = \{x = (x_1, x_2) : 0 < x_1 < 2,\ 0 < x_2 < 2\}, \\ u = 0, \quad \text{on} \quad \Gamma_1 := \Gamma \setminus \Gamma_2, & \frac{\partial u}{\partial x_1}|_{x_1=2} = \frac{\partial u}{\partial x_2}|_{x_2=2} = 1, \quad \text{on} \quad \Gamma_2 := \{x_1 = 2\} \cup \{x_2 = 2\}. \end{cases}$$

Determine the stiffness- and mass-matrics (local matrices are given) and the load vector if the cG(1) finite element method is applied to the equation (1) above and on the following triangulation:

$$m = \frac{h^2}{24}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$



$$s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

5. Consider the following problem for the Klein-Gordon equation of quantum field theory:

$$\begin{cases} \ddot{u} - \Delta u + u = 0, & x \in \Omega \quad t > 0, \\ u = 0, & x \in \partial\Omega \quad t > 0, \\ u(x,0) = u_0(x), & \dot{u}(x,0) = u_1(x), \quad x \in \Omega. \end{cases}$$

(a) Define a suitable energy for this problem and show that the energy is conserved.
(b) Rewrite the equation as a system of two equations with time derivatives of order at most one, both in scalar and matrix form. Why is this reformulation needed?

6. Formulate and prove the Lax-Milgram theorem.

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TMA372/MMG800: Partial Differential Equations, 2017–03–15, 14:00-18:00. Solutions.

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1. Multiply the equation by $-\varepsilon u''$ and integrate over I to get:

$$||\varepsilon u''||^2_{L_2(I)} - \varepsilon \int_0^1 x u' u''\, dx - \varepsilon \int_0^1 u u''\, dx = \int_0^1 (-\varepsilon u'') f\, dx. \tag{2}$$

But using the boundary condition we have

$$\begin{aligned}\int_0^1 x u' u''\, dx &= [PI] = [x u'^2]_0^1 - \int_0^1 (u' + x u'') u'\, dx = \{u'(1) = 0\}\\ &= -\int_0^1 u'^2\, dx - \int_0^1 x u' u''\, dx.\end{aligned}$$

which implies that

$$\int_0^1 x u' u''\, dx = -\frac{1}{2}\int_0^1 u'^2\, dx. \tag{3}$$

Further

$$\int_0^1 u u''\, dx = [u u']_0^1 - \int_0^1 u'^2\, dx = -\int_0^1 u'^2\, dx. \tag{4}$$

Inserting (2) and (3) in (1) we get

$$\begin{aligned}&||\varepsilon u''||^2_{L_2(I)} + \frac{\varepsilon}{2}\int_0^1 u'^2\, dx + \varepsilon \int_0^1 u'^2\, dx = \int_0^1 (-\varepsilon u'') f\, dx\\ &\Longrightarrow ||\varepsilon u''||^2_{L_2(I)} \le \int_0^1 (-\varepsilon u'') f\, dx \le \{\text{Cauchy-Schwartz}\}\\ &\qquad\qquad \le ||\varepsilon u''||_{L_2(I)} ||f||_{L_2(I)}.\end{aligned} \tag{5}$$

Thus we have

$$||\varepsilon u''||_{L_2(I)} \le ||f||_{L_2(I)}.$$

2. Multiply the equation by $\Delta \dot u$ and integrate over Ω to get

$$\begin{aligned}(\ddot u\,, \Delta \dot u) - (\Delta u,\, \Delta \dot u) &= -(\nabla \ddot u\,, \nabla \dot u) - (\Delta u,\, \Delta \dot u)\\ &= -\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega |\nabla \dot u|^2\, dx + \int_\Omega |\Delta u|^2\, dx\Big) = 0,\end{aligned}$$

where in the first equality we used Green's formula and the vanishing boundary data. Relabeling t to s and integrating over $(0, t)$ we get the desired result.

3. <u>The Variational formulation:</u> Let $V^0 := H_0^1(0, 1)$,
Multiply the equation by $v \in V^0$, integrate by parts over $(0, 1)$ and use the boundary conditions to obtain

$$\text{Find } u \in V^0:\ \int_0^1 u' v'\, dx + 2\int_0^1 x u' v\, dx + 2\int_0^1 u v\, dx = \int_0^1 f v\, dx, \quad \forall v \in V^0. \tag{6}$$

<u>cG(1):</u> Let $V_n^0 := \{w \in V^0 : w \text{ is cont., p.l. on a partition of } I, w(0) = w(1) = 0\}$

$$\text{Find } U \in V_h^0:\ \int_0^1 U' v'\, dx + 2\int_0^1 x U' v\, dx + 2\int_0^1 U v\, dx = \int_0^1 f v\, dx, \quad \forall v \in V_h^0. \tag{7}$$

From (1)-(2), we find The Galerkin orthogonality:

$$\int_0^1 \Big((u-U)'v' + 2x(u-U)'v + 2(u-U)v \Big)\, dx = 0, \quad \forall v \in V_h^0. \tag{8}$$

We define the inner product $(\cdot,\cdot)_E$ associated to the energy norm to be

$$(v,w)_E = \int_0^1 (v'w' + vw)\, dx, \qquad \forall v, w \in V^0.$$

Note that

$$2\int_0^1 xe'e\, dx = \int_0^1 x\frac{d}{dx}\Big(e^2\Big)\, dx = [xe^2]_0^1 - \int_0^1 e^2\, dx \tag{9}$$

Thus using (9) we have

$$||e||_E^2 = \int_0^1 (e'e' + ee)\, dx = \int_0^1 (e'e' + 2e'e + 2ee)\, dx. \tag{10}$$

We split the second factor e as $e = u - U = u - v + v - U$, with $v \in V_h$ and write

$$\begin{aligned}
||e||_E^2 &= \int_0^1 \Big(e'(u-U)' + 2xe'(u-U) + 2e(u-U) \Big)\, dx = \Big\{ v \in V_h^0 \Big\} \\
&= \int_0^1 \Big(e'(u-v)' + 2xe'(u-v) + 2e(u-v) \Big)\, dx \\
&+ \int_0^1 \Big(e'(v-U)' + 2xe'(v-U) + 2e(v-U) \Big)\, dx \\
&= \int_0^1 \Big(e'(u-v)' + 2xe'(u-v) + 2e(u-v) \Big)\, dx,
\end{aligned}$$

where, in the last step, we have used the Galerkin orthogonality to eliminate terms involving U. Now we can write

$$\begin{aligned}
||e||_E^2 &= \int_0^1 \Big(e'(u-v)' + 2xe'(u-v) + 2e(u-v) \Big)\, dx \\
&\le 2||e'|| \cdot ||u-v||_E + 2||e|| \cdot ||u-v|| \\
&\le 2||e||_E \cdot ||u-v||_E
\end{aligned}$$

and derive the a priori error estimate:

$$||e||_E \le ||u-v||_E(1+\alpha), \quad \forall v \in V_h.$$

To obtain a posteriori error estimates the idea is to eliminate u-terms, by using the differential equation, and replacing their contributions by the data f. Then this f combined with the remaining U-terms would yield to the residual error:

A posteriori error estimate:

$$\begin{aligned}
||e||_E^2 &= \int_0^1 (e'e' + ee)\, dx = \int_0^1 (e'e' + 2xe'e + 2ee)\, dx \\
&= \int_0^1 (u'e' + 2xu'e + 2ue)\, dx - \int_0^1 (U'e' + 2xU'e + 2Ue)\, dx.
\end{aligned} \tag{11}$$

Now using the variational formulation (6) we have that

$$\int_0^1 (u'e' + 2xu'e + 2ue)\, dx = \int_0^1 fe\, dx.$$

Inserting in (11) and using (7) with $v = \Pi_k e$ we get

(12)
$$\begin{aligned}||e||_E^2 = &\int_0^1 fe\,dx - \int_0^1 (U'e' + 2xU'e + 2Ue)\,dx \\ &+ \int_0^1 (U'\Pi_h e' + 2xU'\Pi_h e + 2U\Pi_h e)\,dx - \int_0^1 f\Pi_h e\,dx.\end{aligned}$$

Thus

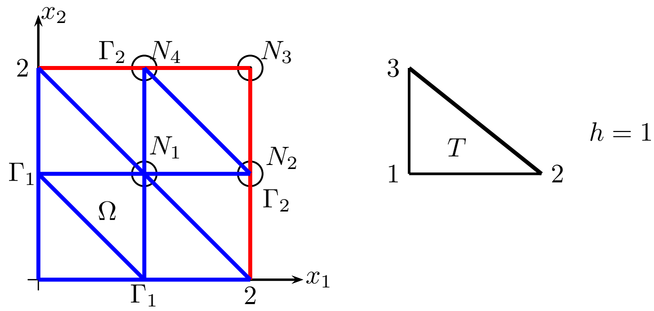
$$\begin{aligned}||e||_E^2 &= \int_0^1 f(e-\Pi_h e)\,dx - \int_0^1 \Big(U'(e-\Pi_h e)' + 2xU'(e-\Pi_h e) + 2U(e-\Pi_h e)\Big)\,dx \\ &= \int_0^1 f(e-\Pi_h e)\,dx - \int_0^1 (2xU'+2U)(e-\Pi_h e)\,dx - \sum_{j=1}^{M+1}\int_{I_j} U'(e-\Pi_h e)'\,dx \\ &= \{\text{partial integration}\} \\ &= \int_0^1 f(e-\Pi_h e)\,dx - \int_0^1 (2xU'+2U)(e-\Pi_h e)\,dx + \sum_{j=1}^{M+1}\int_{I_j} U''(e-\Pi_h e)\,dx \\ &= \int_0^1 (f+U''-2xU'-2U)(e-\Pi_h e)\,dx = \int_0^1 R(U)(e-\Pi_h e)\,dx \\ &= \int_0^1 hR(U)h^{-1}(e-\Pi_h e)\,dx \le ||hR(U)||_{L_2}||h^{-1}(e-\Pi_h e)||_{L_2} \\ &\quad \le C_i||hR(U)||_{L_2}\cdot||e'||_{L_2} \le ||hR(U)||_{L_2}\cdot||e||_E.\end{aligned}$$

This gives the a posteriori error estimate:

$$||e||_E \le C_i||hR(U)||_{L_2},$$

with $R(U) = f + U'' - 2xU' - 2U = f - 2xU' - 2U$ on (x_{i-1}, x_i), $i = 1, \dots, M+1$.

4. Recall that the mesh size is $h = 1$. Further, the first triangle (the triangle with nodes at $(0,0)$, $(1,0)$ and $(0,1)$) is not in the support of the test function of N_1, whereas the last triangle (the triangle with nodes at $(4,4)$, $(2,4)$ and $(4,2)$) is in the support of the test function for all other 3 nodes: $N_2, N3, N4$!. Thus, the nodal basis function φ_1 shares 2 triangles with φ_2 and 2 triangles with φ_4. Likewise, φ_2 and φ_3 are sharing 1 triangle, φ_2 and φ_4, 2 triangle, and finally φ_3 and φ_4 1 triangle. see figure below. We define the test function space



(13)
$$V = \{v : v \in H^1(\Omega), \quad v = 0 \ \text{ on } \Gamma_1\}.$$

Multiplying the differential equation by $v \in V$ and integrating over Ω we get that

$$-(\Delta u, v) + (u, v) = (1, v), \qquad \forall v \in V.$$

Now using Green's formula we have that

$$\begin{aligned}-(\Delta u, v) &= (\nabla u, \nabla v) - \int_{\partial\Omega} (n\cdot\nabla u) v\, ds \\ &= (\nabla u, \nabla v) - \int_{\Gamma_1} (n\cdot\nabla u) v\, ds - \int_{\Gamma_2} (n\cdot\nabla u) v\, ds \\ &= (\nabla u, \nabla v) - < 1, v >_{\Gamma_2}, \qquad \forall v \in V.\end{aligned}$$

Thus the variational formulation reads as

$$(\nabla u, \nabla v) + (u, v) = (1, v) + < 1, v >_{\Gamma_2}, \qquad \forall v \in V.$$

The corresponding cG(1) finite element is: Find $u_h \in V_h^0$ such that

$$(\nabla u_h, \nabla v) + (u_h, v) = (1, v) + < 1, v >_{\Gamma_2}, \qquad \forall v \in V_h^0,$$

where

$$V_h^0 := \{v : v \text{ is continuous, piecewise linear on the above partition and } \ v = 0, \ \text{ on } \Gamma_1\}.$$

Making the "Ansatz" $U(x) = \sum_{j=1}^4 \xi_j \varphi_j(x)$, where φ_i are the standard basis functions, we obtain the system of equations

$$\sum_{j=1}^4 \xi_j \Big(\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx + \int_\Omega \varphi_i \cdot \varphi_j \, dx \Big) = \int_\Omega \varphi_i \, dx + \int_{\Gamma_2} \varphi_i \, d\sigma, \quad i = 1, 2, 3, 4.$$

or, in matrix form,

$$(S + M)\xi = F,$$

where $S_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix,. $M_{ij} = (\varphi_i, \varphi_j)$ is the mass matrix and and $F_i = (1, \varphi_i) + < 1, \varphi_i >_{\Gamma_2}$ is the load vector. We first compute the stiffness matrix for the reference triangle T. The local basis functions are

$$\begin{aligned}\phi_1(x_1, x_2) &= 1 - \frac{x_1}{h} - \frac{x_2}{h}, & \nabla\phi_1(x_1, x_2) &= -\frac{1}{h}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ \phi_2(x_1, x_2) &= \frac{x_1}{h}, & \nabla\phi_2(x_1, x_2) &= \frac{1}{h}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \phi_3(x_1, x_2) &= \frac{x_2}{h}, & \nabla\phi_3(x_1, x_2) &= \frac{1}{h}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.\end{aligned}$$

Hence, with $|T| = \int_T dz = h^2/2$,

$$\begin{aligned}m_{11} &= (\phi_1, \phi_1) = \int_T \phi_1^2 \, dx = h^2 \int_0^1 \int_0^{1-x_2} (1 - x_1 - x_2)^2 \, dx_1 dx_2 = \frac{h^2}{12}, \\ s_{11} &= (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2 \, dx = \frac{2}{h^2}|T| = 1.\end{aligned}$$

Alternatively, we can use the midpoint rule, which is exact for polynomials of degree 2 (precision 3):

$$m_{11} = (\phi_1, \phi_1) = \int_T \phi_1^2 \, dx = \frac{|T|}{3} \sum_{j=1}^3 \phi_1(\hat{x}_j)^2 = \frac{h^2}{6}\Big(0 + \frac{1}{4} + \frac{1}{4}\Big) = \frac{h^2}{12},$$

where $\hat{x}_j$ are the midpoints of the edges. Similarly we can compute the other elements and obtain

$$m = \frac{h^2}{24}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrices M and S from the local ones m and s:

$$
\begin{aligned}
M_{11} &= 2m_{11} + 4m_{22} = \frac{6}{12}h^2, & S_{11} &= 2s_{11} + 4s_{22} = 4, \\
M_{12} &= M_{14} = 2m_{12} = \frac{1}{12}h^2, & S_{12} &= S_{14} = 2s_{12} = -1, \\
M_{13} &= 0, & S_{13} &= 0, \\
M_{22} &= M_{44} = m_{11} + 2m_{22} = \frac{3}{12}h^2, & S_{22} &= S_{44} = s_{11} + 2s_{22} = 2, \\
M_{23} &= M_{34} = m_{12} = \frac{1}{24}h^2, & S_{23} &= S_{34} = s_{12} = -1/2, \\
M_{24} &= 2m_{23} = \frac{1}{12}h^2, & S_{24} &= 2s_{23} = 0, \\
M_{33} &= m_{11} = \frac{1}{12}h^2, & S_{33} &= s_{11} = 1,
\end{aligned}
$$

The remaining matrix elements are obtained by symmetry $M_{ij} = M_{ji}$, $S_{ij} = S_{ji}$. Hence,

$$
M = \frac{h^2}{24}\begin{bmatrix} 12 & 2 & 0 & 2 \\ 2 & 6 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 2 & 2 & 1 & 6 \end{bmatrix}, \quad S = \begin{bmatrix} 4 & -1 & 0 & -1 \\ -1 & 2 & -1/2 & 0 \\ 0 & -1/2 & 1 & -1/2 \\ -1 & 0 & -1/2 & 2 \end{bmatrix},
$$

and

$$
\mathbf{b} = \begin{bmatrix} (1,\varphi_1) + < 1, \varphi_1 >_{\Gamma_2} \\ (1,\varphi_2) + < 1, \varphi_2 >_{\Gamma_2} \\ (1,\varphi_3) + < 1, \varphi_3 >_{\Gamma_2} \\ (1,\varphi_4) + < 1, \varphi_4 >_{\Gamma_2} \end{bmatrix} = \begin{bmatrix} 6 \cdot \frac{1}{3} \cdot \frac{1}{2} + 0 = 1 \\ 3 \cdot \frac{1}{3} \cdot \frac{1}{2} + 2 \cdot 1 \cdot 1 \cdot 1/2 = \frac{3}{2} \\ \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6} + 2 \cdot 1 \cdot 1 \cdot 1/2 = \frac{7}{6} \\ 3 \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{2} + 2 \cdot 1 \cdot 1 \cdot 1/2 = \frac{3}{2} \end{bmatrix}.
$$

5. a) Multiply the equation by $\dot{u}$ and integrate to obtain

$$
\begin{aligned}
&(\ddot{u}, \dot{u}) - (\Delta u, \dot{u}) + (u, \dot{u}) = 0, \\
&(\ddot{u}, \dot{u}) + (\nabla u, \nabla \dot{u}) + (u, \dot{u}) = 0, \\
&\frac{1}{2}\frac{d}{dt}(||\dot{u}||^2 + ||\nabla u||^2 + ||u||^2) = 0, \\
&\frac{1}{2}(||\dot{u}(t)||^2 + ||\nabla u(t)||^2 + ||u(t)||^2) = \frac{1}{2}(||u_1||^2 + ||\nabla u_0||^2 + ||u_0||^2).
\end{aligned}
$$

This means that the energy $E = \frac{1}{2}(||\dot{u}(t)||^2 + ||\nabla u(t)||^2 + ||u(t)||^2)$ is conserved.

b) Set $v_1 = \dot{u}$, $v_2 = u$. Then

$$
\begin{aligned}
&\dot{v}_1 - \Delta v_2 + v_2 = 0, \\
&\dot{v}_2 - v_1 = 0.
\end{aligned}
$$

Now we have a system $\dot{v} + Av = 0$ of first order in t and we can use various techniques developed for such systems, for example, we can apply standard time-discretization methods such as $dG(0)$ or $cG(1)$.

6. See the lecture notes.

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