Mathematics Chalmers & GU

TMA372/MMG800: Partial Differential Equations , 2016–03–16, 14:00-16:00

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Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 6p. Valid bonus poits will be added to the scores.
Breakings: **3**: 15-21p, **4**: 22-28p och **5**: 29p- For GU students **G:** 15-26p, **VG:** 27p-
For solutions see the couse diary: http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1516/

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1. Prove an *a posteriori* error estimate for piecewise linear finite element method for the boundary value problem, (the required interpolation estimates can be used without proofs):

$$-u_{xx} + u_x = f, \quad x \in (0,1); \qquad u(0) = u(1) = 0.$$

2. Consider the Dirichlet problem

$$-\nabla \cdot (a(x)\nabla u) = f(x), \quad x \in \Omega \subset \mathbb{R}^2, \qquad u = 0, \text{ for } x \in \partial\Omega.$$

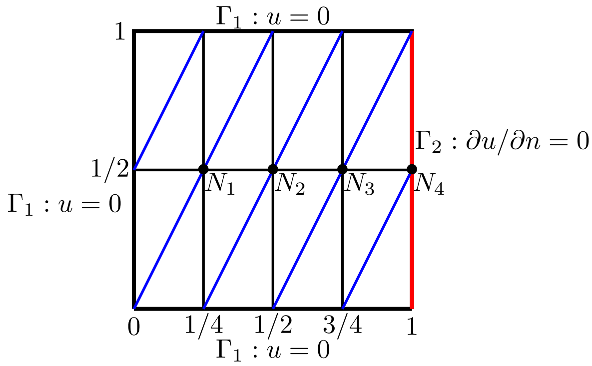
Assume that c_0 and c_1 are constants such that $c_0 \le a(x) \le c_1,\ \forall x \in \Omega$ and let $U = \sum_{j=1}^N \alpha_j w_j(x)$ be a Galerkin approximation of u in a finite dimensional subspace M of $H_0^1(\Omega)$. Prove the there is a consctant C depending on c_0 and c_1 such that we have the *a priori* error estimate

$$||u - U||_{H_0^1(\Omega)} \le C \inf_{\chi \in M} ||u - \chi||_{H_0^1(\Omega)},$$

3. Determine the stiffness matrix and load vector if the $cG(1)$ finite element method approximation is applied to the following Poisson's equation with mixed boundary conditions:

$$\begin{cases} -\Delta u = 1, & \text{on} \quad \Omega = (0,1)\times(0,1), \\ \frac{\partial u}{\partial n} = 0, & \text{for} \quad x_1 = 1, (x \in \Gamma_2) \\ u = 0, & \text{for} \quad x \in \partial\Omega \setminus \{x_1 = 1\} = \partial\Omega \setminus \Gamma_2, \end{cases} \qquad \text{verifying the local stiffness:} \quad s = \begin{pmatrix} 5/4 & -1 & -1/4 \\ -1 & 1 & 0 \\ -1/4 & 0 & 1/4 \end{pmatrix}$$

on a triangulation with triangles of side length 1/4 in the x_1-direction and 1/2 in the x_2-direction.



4. Let $\varepsilon > 0$ be a constant, $a(x) \ge 0$ and $a_x(x) \ge 0$. Consider the boundary value problem

$$u + a(x)u_x - \varepsilon u_{xx} = f, \quad x \in (0,1); \qquad u(0) = u_x(1) = 0.$$

Let $||\cdot||$ denotes the $L_2(I)$-norm, $I = (0,1)$. Prove the following stability estimate:

$$||\sqrt{\varepsilon} u_x|| + ||\sqrt{\varepsilon a_x} u_x|| + ||\varepsilon u_{xx}|| \le C||f||,$$

5. Consider the Dirichlet boundary value problem:

$$(\text{BVP}) - (a(x)u'(x))' = f(x), \quad \text{for} \quad 0 < x < 1, \qquad u(0) = 0, \quad u(1) = 0.$$

where $a(x) > 0$ (*the modulus of elasticity*). Formulate the corresponding variational formulation (VF), the minimization problem (MP) and prove that $\quad (VF) \Longleftrightarrow (MP)$.

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Solutions.

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1. We multiply the differential equation by a test function $v \in H_0^1 = \{v : ||v|| + ||v'|| < \infty,\ v(0) = v(1) = 0\}$ and integrate over I. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u \in H_0^1(I)$ such that

$$\int_I (u'v' + u'v) = \int_I fv, \quad \forall v \in H_0^1(I). \tag{1}$$

Or equivalently, find $u \in H_0^1(I)$ such that

$$(u_x, v_x) + (u_x, v) = (f, v), \quad \forall v \in H_0^1(I), \tag{2}$$

with $(\cdot,\cdot)$ denoting the $L_2(I)$ scalar product: $(u, v) = \int_I u(x)v(x)\,dx$. A *Finite Element Method* with $cG(1)$ reads as follows: Find $u_h \in V_h^0$ such that

$$\int_I (u_h'v' + u_h'v) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I), \tag{3}$$

where

$$V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition of } I,\ v(0) = v(1) = 0\}.$$

Or equivalently, find $u_h \in V_h^0$ such that

$$(u_{h,x}, v_x) + (u_{h,x}, v) = (f, v), \quad \forall v \in V_h^0. \tag{4}$$

Let now

$$a(u, v) = (u_x, v_x) + (u_x, v).$$

We want to show that $a(\cdot,\cdot)$ is both elliptic and continuous:
ellipticity

$$a(u, u) = (u_x, u_x) + (u_x, u) = ||u_x||^2, \tag{5}$$

where we have used the boundary data, viz,

$$\int_0^1 u_x u\,dx = \Big[\frac{u^2}{2}\Big]_0^1 = 0.$$

continuity

$$a(u, v) = (u_x, v_x) + (u_x, v) \le ||u_x||||v_x|| + ||u_x||||v|| \le 2||u_x||||v_x||, \tag{6}$$

where we used the Poincare inequality $||v|| \le ||v_x||$.
Let now $e = u - u_h$, then (2)- (4) gives that

$$a(u - u_h, v) = (u_x - u_{h,x}, v_x) + (u_x - u_{h,x}, v) = 0, \quad \forall v \in V_h^0, \text{(Galerkin Orthogonality)}. \tag{7}$$

A posteriori error estimate: We use again ellipticity (5), Galerkin orthogonality (7), and the variational formulation (1) to get

$$\begin{aligned} \|e_x\|^2 &= a(e, e) = a(e, e - \pi e) = a(u, e - \pi e) - a(u_h, e - \pi e) \\ &= (f, e - \pi e) - a(u_h, e - \pi e) = (f, e - \pi e) - (u_{h,x}, e_x - (\pi e)_x) - (u_{h,x}, e - \pi e) \\ &= (f - u_{h,x}, e - \pi e) \le C\|h(f - u_{h,x})\|\|e_x\|, \end{aligned} \tag{8}$$

where in the last equality we use the fact that $e(x_j) = (\pi e)(x_j)$, for j:s being the node points, also $u_{h,xx} \equiv 0$ on each $I_j := (x_{j-1}, x_j)$. Thus

$$(u_{h,x}, e_x - (\pi e)_x) = -\sum_j \int_{I_j} u_{h,xx}(e - \pi e) + \sum_j \Big(u_{h,x}(e - \pi e)\Big)\Big|_{I_j} = 0.$$

Hence, (8) yields:

$$\|e_x\| \le C\|h(f - u_{h,x})\|. \tag{9}$$

2. Solution: Recall the continuous and approximate weak formulations:

$$(a\nabla u, \nabla v) = (f, v), \qquad \forall v \in H_0^1(\Omega), \tag{10}$$

and

$$(a\nabla U, \nabla v) = (f, v), \qquad \forall v \in M, \tag{11}$$

respectively, so that

$$(a\nabla(u - U), \nabla v) = 0, \qquad \forall v \in M. \tag{12}$$

We may write

$$u - U = u - \chi + \chi - U,$$

where χ is an arbitrary element of M, it follows that

$$\begin{aligned}(a\nabla(u-U), \nabla(u-U)) =& (a\nabla(u-U), \nabla(u-\chi)) \\ &\le ||a\nabla(u-U)|| \cdot ||u-\chi||_{H_0^1(\Omega)} \\ &\le c_1||u-U||_{H_0^1(\Omega)}||u-\chi||_{H_0^1(\Omega)},\end{aligned} \tag{13}$$

on using (3), Schwarz's inequality and the boundedness of a. Also, from the boundedness condition on a, we have that

$$(a\nabla(u-U), \nabla(u-U)) \ge c_0||u-U||^2_{H_0^1(\Omega)}. \tag{14}$$

Combining (4) and (5) gives

$$||u-U||_{H_0^1(\Omega)} \le \frac{c_1}{c_0}||u-\chi||_{H_0^1(\Omega)}.$$

Since χ is an arbitrary element of M, we obtain the result.

3. Solution: Let $\Gamma_1 := \partial\Omega \setminus \Gamma_2$ where $\Gamma_2 := \{(1, x_2) : 0 \le x_2 \le 1\}$. Define

$$V = \{v : v \in H^1(\Omega),\ v = 0 \text{ on } \Gamma_1\}.$$

Multiply the equation by $v \in V$ and integrate over Ω; using Green's formula

$$\int_\Omega \nabla u \cdot \nabla v - \int_\Gamma \frac{\partial u}{\partial n} v = \int_\Omega \nabla u \cdot \nabla v = \int_\Omega v,$$

where we have used $\Gamma = \Gamma_1 \cup \Gamma_2$ and the fact that $v = 0$ on Γ_1 and $\frac{\partial u}{\partial n} = 0$ on Γ_2.

<u>Variational formulation</u>:

Find $u \in V$ such that

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega v, \qquad \forall v \in V.$$

<u>FEM: cG(1)</u>:

Find $U \in V_h$ such that

$$\int_\Omega \nabla U \cdot \nabla v = \int_\Omega v, \qquad \forall v \in V_h \subset V, \tag{15}$$

where

$$V_h = \{v : v \text{ is piecewise linear and continuous in } \Omega,\ v = 0 \text{ on } \Gamma_1, \text{ on above mesh }\}.$$

A set of bases functions for the finite dimensional space V_h can be written as $\{\varphi_i\}_{i=1}^4$, where

$$\begin{cases} \varphi_i \in V_h, & i = 1, 2, 3, 4 \\ \varphi_i(N_j) = \delta_{ij}, & i, j = 1, 2, 3, 4. \end{cases}$$

Then the equation (2) is equivalent to: Find $U \in V_h$ such that

$$\int_\Omega \nabla U \cdot \nabla \varphi_i = \int_\Omega \varphi_i, \qquad i = 1, 2, 3, 4. \tag{16}$$

Set $U = \sum_{j=1}^{4} \xi_j \varphi_j$. Invoking in the relation (3) above we get

$$\sum_{j=1}^{4} \xi_j \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i = \int_\Omega \varphi_i, \qquad i = 1, 2, 3, 4.$$

Now let $a_{ij} = \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i$ and $b_i = \int_\Omega \varphi_i$, then we have that

$$A\xi = b, \quad A \text{ is the stiffness matrix } \; b \text{ is the load vector.}$$

Below we compute a_{ij} and b_i

$$b_i = \int_\Omega \varphi_i = \begin{cases} 6 \cdot \frac{1}{3} \cdot \frac{1/4 \cdot 1/2}{2} \cdot 1 = 1/8, & i = 1, 2, 3 \\ 3 \cdot \frac{1}{3} \cdot \frac{1/4 \cdot 1/2}{2} \cdot 1 = 1/16, & i = 4 \end{cases}$$

and

$$a_{ii} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_i = \begin{cases} 2 \cdot (\frac{5}{4} + 1 + \frac{1}{4}) = 5, & i = 1, 2, 3 \\ \frac{5}{4} + 1 + \frac{1}{4} = 5/2, & i = 4 \end{cases}$$

Further

$$a_{i,i+1} = \int_\Omega \nabla\varphi_{i+1} \cdot \nabla\varphi_i = 2 \cdot (-1) = -2 = a_{i+1,i}, \quad i = 1, 2, 3,$$

and

$$a_{ij} = 0, \quad |i - j| > 1.$$

Thus we have

$$A = \begin{pmatrix} 5 & -2 & 0 & 0 \\ -2 & 5 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 5/2 \end{pmatrix} \qquad b = \frac{1}{16} \begin{pmatrix} 2 \\ 2 \\ 2 \\ 1 \end{pmatrix}.$$

4. Multiply the equation by $-\varepsilon u_{xx}$ and integrate over $I = (0, 1)$:

$$\int_0^1 -\varepsilon u u_{xx} + \int_0^1 -\varepsilon a(x) u_x u_{xx} + \int_0^1 \varepsilon^2 u_{xx}^2 = -\int_0^1 \varepsilon f u_{xx}. \tag{17}$$

We calculate the first two integral on the left hand side of (17)as:

$$\int_0^1 -\varepsilon u u_{xx} = -\Big[\varepsilon u u_x\Big]_0^1 + \int_0^1 \varepsilon u_x^2 = \int_0^1 \varepsilon u_x^2. \tag{18}$$

$$\int_0^1 -\varepsilon a(x) u_x u_{xx} = \Big[-\varepsilon a(x)\frac{u_x^2}{2}\Big] + \frac{1}{2}\int_0^1 \varepsilon a_x u_x^2 = \varepsilon a(0)\frac{u_x^2(0)}{2} + \frac{1}{2}\int_0^1 \varepsilon a_x u_x^2. \tag{19}$$

Inserting (18) and (19) in (18) yields

$$\begin{aligned} &\int_0^1 \varepsilon u_x^2 + \varepsilon a(0)\frac{u_x^2(0)}{2} + \frac{1}{2}\int_0^1 \varepsilon a_x u_x^2 + \int_0^1 \varepsilon^2 u_{xx}^2 \\ &\qquad = -\int_0^1 \varepsilon f u_{xx} \le \|f\|\|\varepsilon u_{xx}\| \le \|f\|^2 + \frac{1}{4}\|\varepsilon u_{xx}\|^2. \end{aligned} \tag{20}$$

Thus

$$\|\sqrt{\varepsilon} u_x\|^2 + \frac{1}{2}\|\sqrt{\varepsilon a_x} u_x\|^2 + \frac{3}{4}\|\varepsilon u_{xx}\|^2 \le \|f\|^2. \tag{21}$$

Hence

$$\|\sqrt{\varepsilon} u_x\| + \|\sqrt{\varepsilon a_x} u_x\| + \|\varepsilon u_{xx}\| \le C\|f\|. \tag{22}$$

5. See the lecture notes.

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