Mathematics Chalmers & GU

TMA372/MMG800: Partial Differential Equations, 2015–03–18, 14:00-18:00

Telephone: Mohammad Asadzadeh: 0703-088304
Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 6p. Valid bonus poits will be added to the scores.
Breakings: **3**: 15-21p, **4**: 22-28p och **5**: 29p- For GU students**G**:15-25p, **VG**: 26p-
For solutions and information about gradings see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1415/index.html

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1. Let $\pi_k\varphi$ be the L_2-projection of φ into piecewise constants, i.e. $\int_{I_j}\pi_k\varphi\,ds=\int_{I_j}\varphi\,ds$. Show that for a subinterval $I_j=(t_{j-1},t_j)$, with $t_j=jk$ and k being a positive constant

$$\int_{I_j}|\varphi-\pi_k\varphi|\,ds\le k\int_{I_j}|\dot\varphi|\,ds,\qquad \text{with}\quad \dot\varphi=\frac{d\varphi}{dt}.$$

2. Consider the following general form of the heat equation for $\Omega\subset\mathbb{R}^2$ with boundary $\partial\Omega=\Gamma$,

$$(1)\qquad \begin{cases} u_t(x,t)-\Delta u(x,t)=f(x,t), & \text{for } x\in\Omega,\ \ 0<t\le T,\\ u(x,t)=0, & \text{for } x\in\Gamma,\ \ 0<t\le T,\\ u(x,0)=u_0(x), & \text{for } x\in\Omega,\end{cases}$$

Let $\tilde u$ be the solution of (1) with a modified initial data $\tilde u_0(x)=u_0(x)+\varepsilon(x)$.

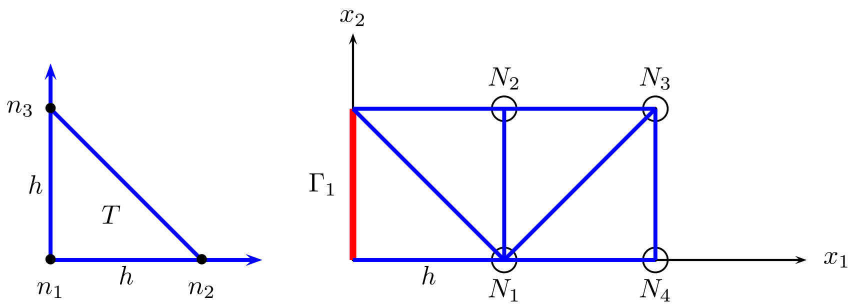
a) Show that $w:=\tilde u-u$ solves (1) with data $w_0(x)=\varepsilon(x)$ (and $f=0$). Derive stability estimates for w, i.e. estimate $\|w(T)\|^2+2\int_0^T\|\nabla w\|^2dt$ by $\|w_0\|^2$.

b) Use stability estimate for w to prove that the solution of (1) is unique.

3. Formulate the cG(1) piecewise continuous Galerkin method in Ω (see fig. below) for the problem

$$-\Delta u(x)=1,\quad \text{for } x\in\Omega,\qquad u(x)=0,\quad \text{for } x\in\Gamma_1,\quad \text{and } \nabla u(x)\cdot\mathbf{n}(x)=1\quad \text{for } x\in\partial\Omega\setminus\Gamma_1,$$

where $\mathbf{n}(x)$ is the outward unit normal to $\partial\Omega$ at $x\in\partial\Omega$. Determine the coefficient matrix and load vector for the resulting equation system using the mesh as in the fig. with nodes at $N_1,\ N_2,\ N_3$ and N_4 and a uniform mesh size h. Hint: First compute the matrix for the standard element T.



4. a) Let p be a positive constant. Prove an a priori and an a posteriori error estimate (in the H^1-norm: $||e||_{H^1}^2=||e'||_{L_2}^2+||e||_{L_2}^2$) for a finite element method for problem

$$-u''+pxu'+(1+\frac{p}{2})u=f,\quad \text{in } (0,1),\qquad u(0)=u(1)=0.$$

b) For which value of p the a priori error estimate is optimal?

5. Consider the heat equation (1) in problem 2 above, with $f\equiv 0$. Prove the following stability estimates

$$i)\quad \|\nabla u\|(t)\le\frac{1}{\sqrt{2t}}\|u_0\|\qquad \text{and}\qquad ii)\quad \Big(\int_0^t s\|\Delta u\|^2(s)\,ds\Big)^{1/2}\le\frac12\|u_0\|.$$

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void!

TMA372/MMG800: Partial Differential Equations, 2015–03–18, 14:00-18:00. Solutions.

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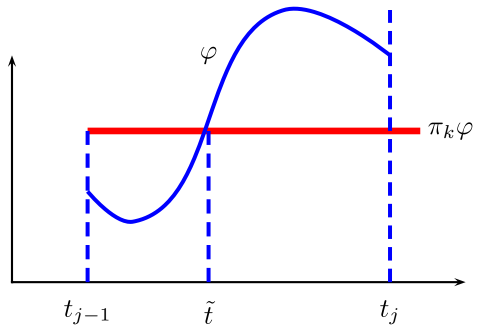
1. We may assume that $\varphi - \pi_k\varphi = 0$ only in one point , $t = \tilde{t}$.



For $\tilde{t} \le t \le t_j$, we have

$$\varphi(t) - \pi_k\varphi = \int_{\tilde{t}}^{t} \dot{\varphi}(s)\, ds$$

This implies that

$$|\varphi(t) - \pi_k\varphi| = |\int_{\tilde{t}}^{t} \dot{\varphi}(s)\, ds| \le \int_{\tilde{t}}^{t} |\dot{\varphi}(s)|\, ds \le \int_{t_{j-1}}^{t_j} |\dot{\varphi}(s)|\, ds.$$

Integrating over $(\tilde{t}, t_j)$ we get

$$\int_{\tilde{t}}^{t_j} |\varphi(t) - \pi_k\varphi|\, ds \le \int_{\tilde{t}}^{t_j} \int_{t_{j-1}}^{t_j} |\dot{\varphi}(s)|\, dt\, ds \le (t_j - \tilde{t}) \int_{t_{j-1}}^{t_j} |\dot{\varphi}|\, dt. \tag{2}$$

Similarly for $t_{j-1} \le t \le \tilde{t}$

$$\int_{t-j-1}^{\tilde{t}} |\varphi(t) - \pi_k\varphi|\, ds \le (\tilde{t} - t_{j-1}) \int_{t_{j-1}}^{t_j} |\dot{\varphi}|\, dt. \tag{3}$$

Combining (2) and (3) yields the desired result.

2. We have that

$$\begin{cases} u_t - \Delta u = f, & \text{in } \Omega, \ \ 0 < t \le T, \\ u(x,t) = 0, & \text{on } \Gamma, \ \ 0 < t \le T, \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{4}$$

and

$$\begin{cases} \widetilde{u}_t - \Delta \widetilde{u} = f, & \text{in } \Omega, \ \ 0 < t \le T, \\ \widetilde{u}(x,t) = 0, & \text{on } \Gamma, \ \ 0 < t \le T, \\ \widetilde{u}(x,0) = u_0(x) + \varepsilon(x), & \text{in } \Omega, \end{cases} \tag{5}$$

Now we study $w = \widetilde{u} - u$. (Propagation of disturbance).
a) Through subtracting (4) from (5) we get the differential equation for w:

$$\begin{cases} w_t - \Delta w = f, & \text{in } \Omega, \ \ 0 < t \le T, \\ w(x,t) = 0, & \text{on } \Gamma, \ \ 0 < t \le T, \\ w(x,0) = \varepsilon(x), & \text{in } \Omega, \end{cases} \tag{6}$$

By the stability estimates for the heat equation we have that

$$\|w(T)\| + 2\int_0^T \|\nabla w\|^2\, dt \le \|\varepsilon\|^2. \qquad \text{(No growth of disturbance).} \tag{7}$$

b) To prove uniqueness for (4), take $\varepsilon = 0$ in (6) and prove that $w \equiv 0$. This is obvious from (7):

$$\|w(T)\| + 2\int_0^T \|\nabla w\|^2 \, dt \leq 0,$$

where both $\|w(T)\| \geq 0$ and $\|\nabla w\|^2 \geq 0$. Thus $w \equiv 0$, so the uniqueness is proved.

3. Let V be the linear function space defined by

$$V := \{v : \int_\Omega \Big(v^2 + |\nabla v|^2\Big)\, dx < \infty, \quad v = 0, \ \text{ on } \Gamma_1\}.$$

Multiplying the differential equation by $v \in V$ and integrating over Ω we get that

$$-(\Delta u, v) = (1, v), \qquad \forall v \in V.$$

Now using Green's formula and the boundary conditions we have that

$$-(\Delta u, \nabla v) = (\nabla u, \nabla v) - \int_{\partial\Omega} (n \cdot \nabla u) v \, ds = (\nabla u, \nabla v) - \int_{\partial\Omega\setminus\Gamma_1} v \, ds, \qquad \forall v \in V.$$

Thus the variational formulation is:

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega v \, dx + \int_{\partial\Omega\setminus\Gamma_1} v \, ds, \qquad \forall v \in V.$$

Let V_h be the usual finite element space consisting of continuous piecewise linear functions satisfying the boundary condition $v = 0$ on Γ_1:

$$V_h := \{v : v \ \text{ is continuous piecewise linear in } \Omega, \ v = 0, \ \text{ on } \Gamma_1\}.$$

The $cG(1)$ method is: Find $U \in V_h$ such that

$$\int_\Omega \nabla U \cdot \nabla v \, dx = \int_\Omega v \, dx + \int_{\partial\Omega\setminus\Gamma_1} v \, ds, \qquad \forall v \in V_h$$

Making the "Ansatz" $U(x) = \sum_{j=1}^4 \xi_j \varphi_j(x)$, where φ_i are the standard basis functions, we obtain the system of equations

$$\sum_{j=1}^4 \xi_j \Big(\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx \Big) = \int_\Omega \varphi_i \, dx + \int_{\partial\Omega\setminus\Gamma_1} \varphi_i \, ds, \quad i = 1, 2, 3, 4$$

or, in matrix form,

$$S\xi = \mathbf{b}, \qquad S_{ij} = (\nabla\varphi_i, \nabla\varphi_j$$

where S is the stiffness matrix, and $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2$ is the load vector with components

$$\mathbf{b}_{1,i} = \int_\Omega \varphi_i \, dx, \quad \text{and} \quad \mathbf{b}_{2,i} = \int_{\partial\Omega\setminus\Gamma_1} \varphi_i \, ds.$$

We first compute stiffness matrix for the reference triangle T. The local basis functions are

$$\phi_1(x_1, x_2) = 1 - \frac{x_1}{h} - \frac{x_2}{h}, \qquad \nabla\phi_1(x_1, x_2) = -\frac{1}{h}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\phi_2(x_1, x_2) = \frac{x_1}{h}, \qquad \nabla\phi_2(x_1, x_2) = \frac{1}{h}\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\phi_3(x_1, x_2) = \frac{x_2}{h}, \qquad \nabla\phi_3(x_1, x_2) = \frac{1}{h}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence, with $|T| = \int_T dz = h^2/2$,

$$
\begin{aligned}
s_{11} &= (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2\,dx = \frac{2}{h^2}|T| = 1, \\
s_{12} &= (\nabla\phi_1, \nabla\phi_2) = \int_T |\nabla\phi_1|^2\,dx = -\frac{1}{h^2}|T| = -1/2, && s_{13} = -1/2 \\
s_{22} &= (\nabla\phi_2, \nabla\phi_2) = \int_T |\nabla\phi_2|^2\,dx = \frac{1}{h^2}|T| = 1/2, && s_{23} = (\nabla\phi_2, \nabla\phi_3) = 0, \\
s_{33} &= (\nabla\phi_3, \nabla\phi_3) = \int_T |\nabla\phi_3|^2\,dx = \frac{1}{h^2}|T| = 1/2,
\end{aligned}
$$

Thus using the symmetry we have the local stiffness matrix as

$$
s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.
$$

We can now assemble the global matrix S from the local s, using the character of our mesh, viz:

$$
\begin{aligned}
&S_{11} = 4s_{22} = 2, \quad S_{12} = 2s_{12} = -1 \quad S_{13} = 2s_{23} = 0 \quad S_{14} = s_{12} = -1/2 \\
&S_{22} = 2s_{11} = 2, \quad S_{23} = s_{12} = -1/2 \quad S_{24} = 0 \\
&S_{33} = 2s_{22} = 1, \quad S_{34} = s_{12} = -1/2 \\
&S_{44} = s_{11} = 1
\end{aligned}
$$

The remaining matrix elements are obtained by symmetry $S_{ij} = S_{ji}$. Hence,

$$
S = \frac{1}{2}\begin{bmatrix} 4 & -2 & 0 & -1 \\ -2 & 4 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.
$$

As for the load vector we note that

$$
\begin{aligned}
\mathbf{b}_{1,1} &= \int_\Omega \varphi_1 = 4 \cdot \frac{1}{3} \cdot \frac{h^2}{2} \cdot 1 = 4\frac{h^2}{6}, \\
\mathbf{b}_{1,2} &= \mathbf{b}_{1,2} = 2 \cdot \frac{1}{3} \cdot \frac{h^2}{2} \cdot 1 = 2\frac{h^2}{6}, \\
\mathbf{b}_{1,4} &= 1 \cdot \frac{1}{3} \cdot \frac{h^2}{2} \cdot 1 = \frac{h^2}{6},
\end{aligned} \tag{8}
$$

$$
\mathbf{b}_{2,i} = \int_{\partial\Omega} \varphi_i = 2 \cdot \frac{1}{2}(h \cdot 1) = h, \quad i = 1, 2, 3, 4. \tag{9}
$$

Hence the load vector $\mathbf{b}$ is:

$$
\mathbf{b} = \frac{h^2}{6}\begin{bmatrix} 4 \\ 2 \\ 2 \\ 1 \end{bmatrix} + h\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
$$

4. We multiply the differential equation by a test function $v \in H_0^1(I)$, $I = (0,1)$ and integrate over I. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u \in H_0^1(I)$ such that

$$
\int_I \Big(u'v' + pxu'v + (1 + \frac{p}{2})uv\Big) = \int_I fv, \quad \forall v \in H_0^1(I). \tag{10}
$$

A *Finite Element Method* with *cG*(1) reads as follows: Find $U \in V_h^0$ such that

$$
\int_I \Big(U'v' + pxU'v + (1 + \frac{p}{2})Uv\Big) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I), \tag{11}
$$

where

$$V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition of } I,\ v(0) = v(1) = 0\}.$$

Now let $e = u - U$, then (10)-(11) gives that

$$\int_I \Big(e'v' + pxe'v + (1 + \frac{p}{2})ev\Big) = 0, \quad \forall v \in V_h^0. \tag{12}$$

A posteriori error estimate: We note that using $e(0) = e(1) = 0$, we get

$$\int_I pxe'e = \frac{p}{2}\int_I x\frac{d}{dx}(e^2) = \frac{p}{2}(xe^2)|_0^1 - \frac{p}{2}\int_I e^2 = -\frac{p}{2}\int_I e^2, \tag{13}$$

so that

$$\begin{aligned}
\|e\|_{H^1}^2 &= \int_I (e'e' + ee) = \int_I \Big(e'e' + pxe'e + (1 + \frac{p}{2})ee\Big) \\
&= \int_I \Big((u-U)'e' + px(u-U)'e + (1 + \frac{p}{2})(u-U)e\Big) = \{v = e \text{ in}(1)\} \\
&= \int_I fe - \int_I \Big(U'e' + pxU'e + (1 + \frac{p}{2})Ue\Big) = \{v = \pi_h e \text{ in}(2)\} \qquad (14)\\
&= \int_I f(e - \pi_h e) - \int_I \Big(U'(e-\pi_h e)' + pxU'(e - \pi_h e) + (1 + \frac{p}{2})U(e - \pi_h e)\Big) \\
&= \{P.I. \text{ on each subinterval}\} = \int_I \mathcal{R}(U)(e - \pi_h e),
\end{aligned}$$

where $\mathcal{R}(U) := f + U'' - pxU' - (1+\frac{p}{2})U = f - pxU' - (1+\frac{p}{2})U$, (for approximation with piecewise linears, $U \equiv 0$, on each subinterval). Thus (14) implies that

$$\begin{aligned}
\|e\|_{H^1}^2 &\le \|h\mathcal{R}(U)\|\|h^{-1}(e - \pi_h e)\| \\
&\le C_i\|h\mathcal{R}(U)\|\|e'\| \le C_i\|h\mathcal{R}(U)\|\|e\|_{H^1},
\end{aligned}$$

where C_i is an interpolation constant, and hence we have with $\|\cdot\| = \|\cdot\|_{L_2(I)}$ that

$$\|e\|_{H^1} \le C_i\|h\mathcal{R}(U)\|.$$

A priori error estimate: We use (13) and write

$$\begin{aligned}
\|e\|_{H^1}^2 &= \int_I (e'e' + ee) = \int_I (e'e' + pxe'e + (1 + \frac{p}{2})ee) \\
&= \int_I \Big(e'(u-U)' + pxe'(u-U) + (1 + \frac{p}{2})e(u-U)\Big) = \{v = U - \pi_h u \text{ in}(3)\} \\
&= \int_I \Big(e'(u - \pi_h u)' + pxe'(u - \pi_h u) + (1 + \frac{p}{2})e(u - \pi_h u)\Big) \\
&\le \|(u - \pi_h u)'\|\|e'\| + p\|u - \pi_h u\|\|e'\| + (1 + \frac{p}{2})\|u - \pi_h u\|\|e\| \\
&\le \{\|(u - \pi_h u)'\| + (1+p)\|u - \pi_h u\|\}\|e\|_{H^1} \\
&\le C_i\{\|hu''\| + (1+p)\|h^2u''\|\}\|e\|_{H^1},
\end{aligned}$$

this gives that

$$\|e\|_{H^1} \le C_i\{\|hu''\| + (1+p)\|h^2u''\|\},$$

which is the a priori error estimate.

b) As seen $p = 0$ (corresponding to zero convection) yields optimal a priori error estimate.

5. See the Lecture Notes.

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