Mathematics Chalmers & GU

TMA372/MMG800: Partial Differential Equations, 2014–06–10, 8:30-12:30

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Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 6p. Valid bonus poits will be added to the scores.
Breakings: **3**: 15-20p, **4**: 21-27p och **5**: 28p- For GU students**G**:15-24p, **VG**: 25p-
For solutions and information about gradings see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1314/index.html

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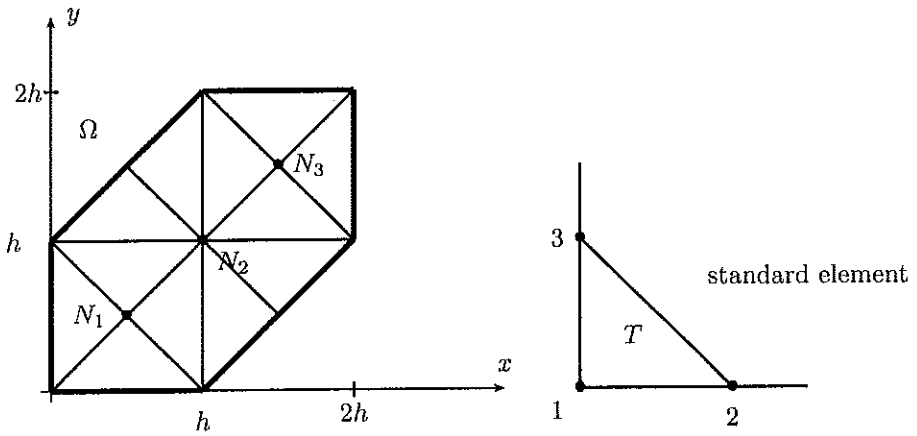
1. Prove the following error estimate for the linear interpolation for a function $f \in C^2(0,1)$,

$$||\pi_1 f - f||_{L_2(a,b)} \le (b-a)^2 ||f''||_{L_2(a,b)}.$$

2. Prove an a priori and an a posteriori error estimate for the cG(1) finite element method for

$$-u''(x) + u'(x) = f, \quad 0 < x < 1; \qquad u(0) = u(1) = 0.$$

3. Let Ω be the hexagonal domain with the uniform triangulation as in the figure below. Compute



the stiffness matrix and the load vector for the cG(1) approximate solution for the problem:

$$(1) \qquad \begin{cases} -\Delta u = 1, & \text{in } \Omega, \quad \text{(the mesh size is } h) \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

4. Consider the convection-diffusion problem

$$-\operatorname{div}(\varepsilon \nabla u + \beta u) = f, \text{ in } \Omega \subset \mathbb{R}^2, \quad u = 0, \text{ on } \partial\Omega,$$

where Ω is a bounded convex polygonal domain, $\varepsilon > 0$ is constant, $\beta = (\beta_1(x), \beta_2(x))$ and $f = f(x)$. Determine the conditions in the Lax-Milgram theorem that would guarantee existence of a unique solution for this problem. Prove a stability estimate for u i terms of $||f||_{L_2(\Omega)}$, ε and diam (Ω), and under the conditions that you derived.

5. Consider the boundary value problem

$$(BVP) \qquad -(a(x)u'(x))' = f(x), \quad 0 < x < 1, \qquad u(0) = u(1) = 0, \qquad a(x) > 0.$$

Show that the variational formulation and minimization problem for BVP are equivalent.

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Solutions.

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1. Let $\lambda_0(x) = \frac{\xi_1 - x}{\xi_1 - x_0}$ and $\lambda_1(x) = \frac{x - \xi_0}{\xi_1 - x_0}$ be two linear base functions, where $\xi_0 \neq \xi_1$, ξ_0, $\xi_1 \in [a, b]$, can be taken as arbitrary interpolation points or just $\xi_0 = a$, $\xi_1 = b$. Then by the integral form of the Taylor formula we may write

$$\begin{cases} f(\xi_0) &= f(x) + f'(x)(\xi_0 - x) + \int_x^{\xi_0} (\xi_0 - y) f''(y)\, dy, \\ f(\xi_1) &= f(x) + f'(x)(\xi_1 - x) + \int_x^{\xi_1} (\xi_1 - y) f''(y)\, dy, \end{cases}$$

Therefore, the linear function interpolating f in the points ξ_0, $\xi_1 \in [a, b]$, can be written as

$$\begin{aligned} \Pi_1 f(x) &= f(\xi_0)\lambda_0(x) + f(\xi_1)\lambda_1(x) \\ &= f(x) + \lambda_0(x) \int_x^{\xi_0} (\xi_0 - y) f''(y)\, dy + \lambda_1(x) \int_x^{\xi_1} (\xi_1 - y) f''(y)\, dy \end{aligned}$$

and by the triangle inequality we get

$$\begin{aligned} |f(x) - \Pi_1 f(x)| &= \Big|\lambda_0(x) \int_x^{\xi_0} (\xi_0 - y) f''(y)\, dy + \lambda_1(x) \int_x^{\xi_1} (\xi_1 - y) f''(y)\, dy\Big| \\ &\le |\lambda_0(x)| \Big| \int_x^{\xi_0} (\xi_0 - y) f''(y)\, dy\Big| + |\lambda_1(x)| \Big| \int_x^{\xi_1} (\xi_1 - y) f''(y)\, dy\Big| \\ &\le |\lambda_0(x)| \int_x^{\xi_0} |\xi_0 - y||f''(y)|\, dy + |\lambda_1(x)| \int_x^{\xi_1} |\xi_1 - y||f''(y)|\, dy \\ &\le |\lambda_0(x)| \int_x^{\xi_0} (b-a)|f''(y)|\, dy + |\lambda_1(x)| \int_x^{\xi_1} (b-a)|f''(y)|\, dy \\ &\le (b-a)\Big(|\lambda_0(x)| + |\lambda_1(x)|\Big) \int_a^b |f''(y)|\, dy \\ &= (b-a)\Big(\lambda_0(x) + \lambda_1(x)\Big) \int_a^b |f''(y)|\, dy = (b-a) \int_a^b |f''(y)|\, dy. \end{aligned}$$

Through repeated use of the Cauchy's inequality it follows that

$$\begin{aligned} \int_a^b |f(x) - \Pi_1 f(x)|^2 dx &\le \int_a^b (b-a)^2 \Big(\int_a^b |f''(y)|\, dy\Big)^2 dx \\ &= (b-a)^3 \Big(\int_a^b |f''(y)|\, dy\Big)^2 = (b-a)^3 \Big(\int_a^b 1 \cdot |f''(y)|\, dy\Big)^2 \\ &\le (b-a)^3 \int_a^b 1^2\, dy \cdot \int_a^b |f''(y)|^2\, dy \\ &= (b-a)^4 \int_a^b |f''(y)|^2\, dy. \end{aligned}$$

Consequently

$$\Big(\int_a^b |f(x) - \Pi_1 f(x)|^2 dx\Big)^{1/2} \le (b-a)^2 \Big(\int_a^b |f''(y)|^2\, dy\Big)^{1/2},$$

and we have the desired result viz,

$$||f - \Pi_1 f||_{L_2(a,b)} \le (b-a)^2 ||f''||_{L_2(a,b)}.$$

2. We multiply the differential equation by a test function $v \in H_0^1(I)$, $I = (0,1)$ and integrate over I. Using integration by parts integration and the boundary conditions we get the following *variational problem*: Find $u \in H_0^1(I)$ such that

$$\int_I (u'v' + u'v) = \int_I fv, \quad \forall v \in H_0^1(I). \tag{2}$$

A *Finite Element Method* with $cG(1)$ reads as follows: Find $U \in V_h^0$ such that

$$\int_I (U'v' + U'v) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I), \tag{3}$$

where $V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition } \mathcal{T}_h \text{ of } I,\ v(0) = v(1) = 0\}$. Now let $e = u - U$, then (2)−(3) gives that

$$\int_I (e'v' + e'v) = 0, \quad \forall v \in V_h^0. \tag{4}$$

We note that using $e(0) = e(1) = 0$, we get

$$\int_I e'e = \int_I \frac{1}{2}\frac{d}{dx}\left(e^2\right) = \frac{1}{2}(e^2)|_0^1 = 0. \tag{5}$$

Further, using Poincare inequality we have

$$\|e\|^2 \le \|e'\|^2.$$

A priori error estimate: We use Poincare inequality, Galerkin orthogonality (4), (5) and standard interpolation estimates to get

$$\begin{aligned}
\|e\|_{H^1}^2 &= \int_I (e'e' + ee) \le 2\int_I e'e' = 2\int_I (e'e' + e'e) = 2\int_I \Big(e'(u-U)' + e'(u-U)\Big) \\
&= 2\int_I \Big(e'(u-\pi_h u)' + e'(u-\pi_h u)\Big) + 2\int_I \Big(e'(\pi_h u - U)' + e'(\pi_h u - U)\Big) \\
&= \{v = U - \pi_h u \ \text{ in } (4)\} = 2\int_I \Big(e'(u-\pi_h u)' + e'(u-\pi_h u)\Big) \\
&\le 2\|(u-\pi_h u)'\|\|e'\| + 2\|u-\pi_h u\|\|e'\| \\
&\le 2C_i\{\|hu''\| + \|h^2u''\|\}\|e\|_{H^1},
\end{aligned}$$

this gives that

$$\|e\|_{H^1} \le C_i\{\|hu''\| + \|h^2u''\|\},$$

which is the a priori error estimate.

A posteriori error estimate:

$$\begin{aligned}
\|e\|_{H^1}^2 &= \int_I (e'e' + ee) \le 2\int_I e'e' = 2\int_I (e'e' + e'e) \\
&= 2\int_I \big((u-U)'e' + (u-U)'e\big) = \{v = e \ \text{ in } (4)\} \\
&= 2\int_I fe - \int_I (U'e' + U'e) = \{v = \pi_h e \ \text{ in } (5)\} \\
&= \int_I f(e - \pi_h e) - \int_I \Big(U'(e-\pi_h e)' + U'(e - \pi_h e)\Big) \\
&= \{P.I. \text{ on each subinterval}\} = \int_I \mathcal{R}(U)(e - \pi_h e),
\end{aligned} \tag{6}$$

where $\mathcal{R}(U) := f + U'' - U' = f - U'$, (for approximation with piecewise linears, $U'' \equiv 0$, on each subinterval). Thus Cauchy Schwars and standard interpolation estimates implies that

$$\|e\|_{H^1}^2 \le \|h\mathcal{R}(U)\|\|h^{-1}(e - \pi_h e)\| \le C_i\|h\mathcal{R}(U)\|\|e'\| \le C_i\|h\mathcal{R}(U)\|\|e\|_{H^1},$$

where C_i is an interpolation constant, and hence we have with $\|\cdot\| = \|\cdot\|_{L_2(I)}$ that

$$\|e\|_{H^1} \le C_i\|h\mathcal{R}(U)\|.$$

3. Let V be the linear function space defined by

$$V := \{v : v \in H^1(\Omega),\ \ v = 0,\ \text{ on } \partial\Omega\}.$$

Multiplying the differential equation by $v \in V$ and integrating over Ω we get that

$$-(\Delta u, v) = (1, v), \qquad \forall v \in V.$$

Now using Green's formula and the fact that $v = 0$ on $\partial\Omega \setminus \Gamma_1$, we have that

$$-(\Delta u, \nabla v) == (\nabla u, \nabla v) - \int_{\partial\Omega} (n \cdot \nabla u) v\, ds = (\nabla u, \nabla v), \qquad \forall v \in V,$$

Hence, the variational formulation is:

$$(\nabla u, \nabla v) = (1, v), \qquad \forall v \in V.$$

Let V_h be the usual finite element space consisting of continuous piecewise linear functions satisfying the boundary condition $v = 0$ on $\partial\Omega$: Then, the $cG(1)$ method is: Find $U \in V_h$ such that

$$(\nabla U, \nabla v) = (1, v) \qquad \forall v \in V_h$$

Making the "Ansatz" $U(x) = \sum_{j=1}^{3} \xi_j \varphi_j(x)$, where $\varphi_j, j = 1,2,3$ are the standard basis functions corresponding to the interior nodes N_1, N_2 and N_3, we obtain the system of equations

$$\sum_{j=1}^{3} \xi_j \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j\, dx = \int_\Omega f \varphi_i\, dx, \quad i = 1,2,3.$$

In matrix form this can be written as $S\xi = F$, where $S_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix, and $F_i = (f, \varphi_i)$ is the load vector.

We first compute the stiffness matrix for the reference triangle T. The local basis functions are

$$\phi_1(x_1,x_2) = 1 - \frac{x_1}{h} - \frac{x_2}{h}, \qquad \nabla\phi_1(x_1,x_2) = -\frac{1}{h}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\phi_2(x_1,x_2) = \frac{x_1}{h}, \qquad \nabla\phi_2(x_1,x_2) = \frac{1}{h}\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\phi_3(x_1,x_2) = \frac{x_2}{h}, \qquad \nabla\phi_3(x_1,x_2) = \frac{1}{h}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence, with $|T| = \int_T dz = h^2/2$, we can easily compute

$$s_{11} = (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2\, dx = \frac{2}{h^2}|T| = 1,$$

$$s_{12} = s_{21} = (\nabla\phi_1, \nabla\phi_2) = \int_T \frac{-1}{h^2}|T| = -1/2,$$

$$s_{23} = s_{32} = (\nabla\phi_2, \nabla\phi_3) = 0,$$

$$s_{22} = s_{33} = \ldots = \frac{1}{h^2}|T| = 1/2.$$

Thus by symmetry we get that the local stiffness matrix for the standard element is:

$$s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global stiffness matrix S from the local stiffness matrix s:

$$S_{11} = 4s_{11} = 4, \qquad S_{12} = S_{21} = 2s_{12} = -1 \qquad S_{13} = s_{31} = 0$$

$$S_{22} = 8s_{22} = 4 \qquad S_{23} = 2s_{12} = -1, \qquad S_{33} = 4s_{11} = 4.$$

The remaining matrix elements are obtained by symmetry $S_{ij} = S_{ji}$. Hence,

$$S = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}.$$

As for the load vector we have that

$$\int_\Omega \varphi_1 = \int_\Omega \varphi_3 = 4\frac{1}{3}\frac{h^2}{2}.1 = \frac{2}{3}h^2, \quad \int_\Omega \varphi_2 = 8\frac{1}{3}\frac{h^2}{2}.1 = \frac{4}{3}h^2.$$

This the load vector is given by $b = \frac{h^2}{3}(2,4,2)^t$. Observe that, here S has become independent of h.

4. Consider

$$-\operatorname{div}(\varepsilon\nabla u + \beta u) = f, \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma = \partial\Omega. \tag{7}$$

a) Multiply the equation (7) by $v \in H_0^1(\Omega)$ and integrate over Ω to obtain the Green's formula

$$-\int_\Omega \operatorname{div}(\varepsilon\nabla u + \beta u)v\,dx = \int_\Omega (\varepsilon\nabla u + \beta u)\cdot\nabla v\,dx = \int_\Omega fv\,dx.$$

The variational formulation for our problem is now: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = L(v), \qquad \forall v \in H_0^1(\Omega), \tag{8}$$

where

$$a(u,v) = \int_\Omega (\varepsilon\nabla u + \beta u)\cdot\nabla v\,dx,$$

and

$$L(v) = \int_\Omega fv\,dx.$$

According to the Lax-Milgram theorem, for a unique solution for (7) we need to verify that the following relations are valid:

i)

$$|a(v,w)| \le \gamma||u||_{H^1(\Omega)}||w||_{H^1(\Omega)}, \qquad \forall v,\, w \in H_0^1(\Omega),$$

ii)

$$a(v,v) \ge \alpha||v||^2_{H^1(\Omega)}, \qquad \forall v \in H_0^1(\Omega),$$

iii)

$$|L(v)| \le \Lambda||v||_{H^1(\Omega)}, \qquad \forall v \in H_0^1(\Omega),$$

for some $\gamma,\ \alpha,\ \Lambda > 0$.

Now since

$$|L(v)| = |\int_\Omega fv\,dx| \le ||f||_{L_2(\Omega)}||v||_{L_2(\Omega)} \le ||f||_{L_2(\Omega)}||v||_{H^1(\Omega)},$$

thus iii) follows with $\Lambda = ||f||_{L_2(\Omega)}$.

Further we have that

$$\begin{aligned}
|a(v,w)| &\le \int_\Omega |\varepsilon\nabla v + \beta v||\nabla w|\,dx \le \int_\Omega (\varepsilon|\nabla v| + |\beta||v|)|\nabla w|\,dx \\
&\le \Big(\int_\Omega (\varepsilon|\nabla v| + |\beta||v|)^2\,dx\Big)^{1/2}\Big(\int_\Omega |\nabla w|^2\,dx\Big)^{1/2} \\
&\le \sqrt{2}\max(\varepsilon, ||\beta||_\infty)\Big(\int_\Omega (|\nabla v|^2 + v^2)\,dx\Big)^{1/2}||w||_{H^1(\Omega)} \\
&= \gamma||v||_{H^1(\Omega)}||w||_{H^1(\Omega)},
\end{aligned}$$

which, with $\gamma = \sqrt{2}\max(\varepsilon, ||\beta||_\infty)$, gives i).

Finally, if $\operatorname{div}\beta \le 0$, then

$$\begin{aligned}
a(v,v) &= \int_\Omega \Big(\varepsilon|\nabla v|^2 + (\beta\cdot\nabla v)v\Big)\,dx = \int_\Omega \Big(\varepsilon|\nabla v|^2 + (\beta_1\frac{\partial v}{\partial x_1} + \beta_2\frac{\partial v}{\partial x_2})v\Big)\,dx \\
&= \int_\Omega \Big(\varepsilon|\nabla v|^2 + \frac{1}{2}(\beta_1\frac{\partial}{\partial x_1}(v)^2 + \beta_2\frac{\partial}{\partial x_2}(v)^2)\Big)\,dx = \text{Green's formula} \\
&= \int_\Omega \Big(\varepsilon|\nabla v|^2 - \frac{1}{2}(\operatorname{div}\beta)v^2\Big)\,dx \ge \int_\Omega \varepsilon|\nabla v|^2\,dx.
\end{aligned}$$

Now by the Poincare's inequality

$$\int_\Omega |\nabla v|^2\,dx \geq C\int_\Omega (|\nabla v|^2 + v^2)\,dx = C||v||^2_{H^1(\Omega)},$$

for some constant $C = C(\Omega)$, we have

$$a(v,v) \geq \alpha||v||^2_{H^1(\Omega)}, \qquad \text{with } \alpha = C\varepsilon,$$

thus ii) is valid under the *condition that* $\text{div}\beta \leq 0$.
From ii), (8) (with $v = u$) and iii) we get that

$$\alpha||u||^2_{H^1(\Omega)} \leq a(u,u) = L(u) \leq \Lambda||u||_{H^1(\Omega)},$$

which gives the *stability estimate*

$$||u||_{H^1(\Omega)} \leq \frac{\Lambda}{\alpha},$$

with $\Lambda = ||f||_{L_2(\Omega)}$ and $\alpha = C\varepsilon$ defined above.
To summarize: The conditions for the Lax-Milgram theorem are:
$f \in L_2(\Omega)$, $\beta \in L_\infty(\Omega)$ and $\nabla\cdot\beta \leq 0$ a.e.

5. See the Book and/or Lecture Notes.

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